Guiding Mathematical Discovery

How We Started a Math Circle

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About the Authors

Jackie Chan

Jackie is a senior computer science and mathematics major at Carleton College who has had an interest in teaching mathematics since an early age. Jackie's interest in mathematics education stems from his enjoyment of revealing the intuition behind mathematical concepts. In high school, Jackie volunteered in his community by tutoring at homework help sessions at his local library and grading during summer school as a teacher's assistant. This background in helping others understand mathematics continued all four years at Carleton where Jackie volunteered with Northfield Middle School's TORCH (Tackling Obstacles & Raising College Hopes) program helping low-income, first-generation students prepare for college. Outside of teaching, Jackie is the computer science department's student departmental advisor (SDA) and a Page Scholar. After Carleton, Jackie will be continuing his education at the University of Illinois at Urbana–Champaign as a computer science PhD student.

Tenzin Kunsang

Tenzin is a mathematics major at Carleton College. She came into Carleton set on majoring in physics or computer science but was drawn by the versatility of mathematics. Her love for math began when she was in middle school. She was humbled by the passion and love for mathematics shared by some of her favorite teachers and began relating to those experiences. She started enjoying mathematics even more when her classmates entrusted her to help them understand the subtle aspects of mathematical concepts; she soon noticed that more than half of her learning came from teaching the material. She believes that mathematics educators, in addition to the learner's mindset and curiosity, have huge impacts on one's relationship with the subject. It is a field that could easily be seen negatively, and running a Math Circle has made her realize that math can be taught and learnt in many fun ways. Beyond math, she sings in Carleton's choir and dances in Synchrony, a dance club at Carleton. During her leisure hours, she enjoys playing chess, writing poetry, watching films, doing yoga, and playing around on digital audio and visual workstations!

Elisa Loy

Elisa is a mathematics major at Carleton College, but despised the subject until her junior year of high school. Having had a wide range of bizarre math teachers, including a woman who punished herself in class for mistakes and a teacher who only answered the questions of male students, she found that positive impressions of mathematics are crucial at early stages of a student's educational career. Without the support of peers, it can be easy for students to fear math. Having had a negative experience with it herself, Elisa focuses on being a champion for the subject and encouraging others to find ways to turn problem sets into puzzles. When she is not running a Math Circle, Elisa is president of Carleton's Alpine Ski and Snowboard Club, a leader of the Society of Women and Non-Binary Students in Math and Stats (SWiMS+), and volunteers with Art Sprouts at the elementary school to teach Northfield youth about the environment through art.

Fares Soufan

Fares is a mathematics major at Carleton College. He became interested in mathematics his first year in college, as he did not know that anything beyond calculus existed prior to coming to Carleton. Now he is obsessed, and takes as many math classes as he can. What he enjoys and appreciates the most about mathematics is the rigor and attention to detail, as well as the importance of math and its applications in the sciences. He also enjoys talking to one of his professors/best friends about math, food, and music. Aside from mathematics, he is interested in education and history, and enjoys basketball and tennis.

Taylor Yeracaris

Taylor is a mathematics major and Japanese minor at Carleton College. He has been a math lover from an early age, a love allowed to flourish by homeschooling and fueled in part by participation in the Kaplans' Math Circle in the Boston area as a kid. He is now interested in applied math and, most prominently, math education. He is planning to pursue math teaching professionally, and hopes to increase inclusivity in mathematics, help students discover the joy and beauty of the subject, and generally teach it in a humanizing way. He has varied interests beyond math and math education as well, including linguistics, religion, computer science, folk dance, and music. At Carleton, he leads the Folk Music Society and the Folk Dance Club, arranging regular participatory dance events and concerts by folk musicians.

The authors: *(top row from left to right)* Jackie Chan, Tenzin Kunsang, Fares Soufan *(bottom row from left to right)* Elisa Loy and Taylor Yeracaris

Acknowledgments

Being able to conduct a Math Circle and write about it has been an amazing experience with which to conclude our undergraduate mathematics career. Before getting into the details of our experiences, we would like to acknowledge some of the individuals who made this project possible.

First of all, the Mathematics and Statistics Department at Carleton College has given us a wonderful environment in which to explore our academic interests in mathematics. Not only have the courses been fulfilling, but the relationships we have formed have been indispensable to our education. Within the faculty, we would like to specially thank our project advisor, Dr. Deanna Haunsperger, for offering this opportunity and guiding us to make our Math Circle the best it could possibly be. Before our senior year we all had a special interest in mathematics education, and Dr. Haunsperger has allowed us to express and explore that interest. We would also like to thank Dr. Stephen Kennedy for advising us on how to run Circles and giving us several topic ideas.

Along with the faculty at Carleton College, we would also like to thank the teachers and staff at Northfield Middle School. In particular, we would like to thank Anne Jarvis, Angela Schock, and Amy McBroom for being so hospitable to us and giving us a time and space to conduct our Circles. We thank them for opening up their classrooms to us, and for gathering the wonderful group of students with whom we got to run our Circles.

We would also like to acknowledge our friends who volunteered their time to allow us to pilot our plans for Circles before reaching the students. With their help, we were able to improve the way we approach students and provide a better environment for them to explore the areas of mathematics that interested us the most. Their active participation and feedback also allowed us to prepare our lesson plans for unforeseen questions and avenues of exploration.

We also appreciate the wonderful work of previous Math Circles, including the National Association of Math Circles, whose representatives we met at the Joint Mathematics Meetings 2020 in Denver, Colorado. Their previous work has allowed us to construct a strong foundation for our students. Among the other Math Circles, we would like to specially thank Robert and Ellen Kaplan, whose vision of Math Circles, as presented in their book *Out of the Labyrinth: Setting Mathematics Free*, inspired much of our approach.

Finally, we would like to express our deepest gratitude to our students at Northfield Middle School. We know that all students have an innate mathematical curiosity within them, but the students who attended our sessions took the extra step to explore that curiosity and share their interest in the subject with us. We hope that each one of our students maintains their interest in mathematics and shares that with their peers. With their help, we move one step closer to ending the stigma against mathematics.

Preface

Introduction

Before the start of their senior year, Carleton College students pick a project in their department to work on as their final comprehensive project, better known as “comps.” Because we are all mathematics majors and share an interest in mathematics education, we chose to lead a Math Circle for our comps project. We were guided by our advisor, Dr. Deanna Haunsperger, who is a mathematics professor at Carleton. Dr. Haunsperger was also the president of the Mathematical Association of America (MAA) from 2017 to 2018 and co-created the Carleton College Summer Mathematics Program for Women, which was held every summer from 1995 to 2014. We began our project in the Fall of 2019, and over the course of around twenty weeks held two Math Circle sessions every week for sixth graders at Northfield Middle School.

We wrote this book to be a guide for anybody who wishes to start their own Math Circle, with students of any age and any sort. The book is meant to be accessible to anyone; we aimed to write it in such a way that high school mathematics will be enough to make sense of everything contained within. We hope that it will be a helpful resource for running Math Circles, or for math-educational pursuits of any sort.

What are Math Circles?

Many people mean many different things when they talk about “Math Circles.” Approaches to leading them vary widely, from classrooms based on lectures and worksheets to free-form groups where the leader does little more than write students’ ideas on the board and prod the discussion in a fruitful direction. Whatever their form, Math Circles share a common goal: to set students free within math; to let them explore its beauty, make it their own, and let their mathematical intuition flourish.

Math Circles began in Russia and Bulgaria in the early twentieth century, as informal gatherings centered around mathematical exploration. This tradition made its way to the United States in the late twentieth century, and had many descendants there. One offshoot of Math Circles was started by Robert and Ellen Kaplan, from whose books and other materials we have drawn much of our inspiration. However, over the course of our Math Circle experience, we have taken inspiration from a variety of sources, including our college coursework and books of math problems. These sources are listed in the “Further Reading” section at the end of this book.

For us, a Math Circle in its ideal form is an egalitarian discussion between students and leaders. It is a space where mathematics is approached the way mathematicians approach it — as an exciting puzzle, a window into the structure of the universe, an ever-branching journey where each so-called solution inevitably leads to several new questions. It is a space where it is okay — in fact, encouraged — to make mistakes, where wild creativity is the name of the game, and where the “right answer” is just another stop along the way. Math Circles challenge the usual conception of mathematics as centered around tests, mindless repetition, and memorization. Instead, Math Circle leaders use guided discovery to present math as an innate and instinctive human art.

These Circles are not just for prodigies or hardened math veterans. They are for everyone. Similarly, leading Math Circles requires no mystical technique that only mathematicians possess; one only needs excitement for their subject, the ability to manage their group, and faith in their students. That being said, although the ideal Math Circle is certainly attainable with any group of students, continual challenges should be expected. We hope that the experiences and suggestions contained within this book will be of some help in navigating those challenges as they arise.

Why Math Circles?

Mathematics is something we are all born able to do; we all have what Bob and Ellen Kaplan, founders of the Global Math Circle, call the "architectural instinct." At its core, math is an exciting and creative site for problem-solving and discovery. However, most people seem to have a different image of mathematics. It is often seen as boring, not very applicable to everyday life, and something you cannot be good at unless you are "smart." Culturally, we have positioned math as the gatekeeper subject — that which separates the "smart" from the "dumb," and raises up only those who thrive on its very particular appeal. From the perspective of the Math Circle, this is like judging artistic ability based on who can create the best copy of the Mona Lisa — it misses the creativity and humanity involved.

With these ideas in mind, we envision Math Circles as working towards spreading a more inclusive and engaging perception of mathematics. By unlocking students' innate ability and love for math, and showing them what the subject really consists of — especially at the college level and beyond — we aim to do several things: show students the wide variety and beauty of mathematics; give students ownership over the mathematics they discover and create; let students see the fun of math; build students' confidence with math; and include those who do not feel as though they have a meaningful place in the world of mathematics.

We ran our Math Circle with sixth graders because we believe that this is the best time to work towards these goals. At this age, students are still learning foundational material in all subjects, and we wanted to teach students the foundational lesson that math is not just what you learn in middle school or high school — there is a lot more to it. Many students are turned off of math over the course of their schooling, which, beyond just affecting the student's relationship to the subject, can also narrow future job prospects. The Math Circle is an opportunity for students to see mathematics for what it is, find a place for themselves within it, and learn to approach it through a new framework. By running Math Circles at the beginning of Middle School, we hope to reach these students while they are still willing to give math a chance.

Designing Circles

During the early stages of the project, we experimented with different topics and ways to present the material to see which methods benefited students the most. We came up with the following suggestions that we think are important to keep in mind when designing Circles. You should think of these as recommendations that you can use when you start running a Math Circle. However, these recommendations were tailored for the students we were working with. Your students might be different, and different strategies might work better. These are general suggestions to start out with, which you can then change over time to better fit your students' needs.

- Use activities-based lessons to help the students create their own mathematics.

When leading a Circle, allow students to do most of the work. The person leading the Circle should hold more of an advising role than a teaching one. The Circle leader should ideally only have to explain the problem and check the students' progress while they work on it, perhaps giving the occasional nudge in a productive direction. It is also always better to have a student answer another student's question than to have the leader answer. This will help build a more collaborative environment.

To assist in student exploration, manipulatives (tangible objects that are used to explore ideas) and activities should be prepared for students to better engage the topics.The students should spend most of their time doing things and talking to each other, rather than sitting around and thinking in silence; a large part of the point is to learn through activities and engagement with peers. The leader should also encourage the students to take ownership of their findings. For example, if the students come across the idea of seven times itself ten times and do not immediately slap the label "seven to the power of ten" on it, the students should give the concept a name and notation of their own creation. This allows students to worry less about formality and focus more on exploring, which is part of the point of the Math Circle: to be an opportunity to learn mathematics without worrying about formality or prerequisites.

- Focus on “low-threshold, high ceiling” problems.

Low-threshold" means the leader should pick a topic that is easy to grasp at first and does not require a lot of time or mathematical background to get started with. This allows the students to spend most of the time engaging with the material. The point of this is to have an active learning environment, and having the leader lecture takes time away from that. However, the fact that a problem is easy to explain does not necessarily mean that it is shallow. The “high ceiling” aspect of the problem is what allows students to explore the problem for as long as they want during the session. The students should tackle problems that have clear directions, but do not end at just finding solutions. The Circle leader should choose topics that have more than one direction, or maybe one direction, but with multiple checkpoints. The leader should strive for problems that are both accessible and have enough depth to make for an engaging Circle. This will require the leader to be familiar with the potential avenues a Circle could lead to. Please refer to the “Taking It Further” sections in the lesson plans to understand what we mean by depth in a more concrete way.

- Emphasizing the learning process. The Circle is not a test, so the goal is not to get the right answer, but to learn and enjoy mathematics.

Our goal for each Circle was not to have the students become experts on the topics covered or to solve all the problems presented. Instead, we wanted them to just try things out and explore concepts on their own, so it was almost always the case that the students would struggle with the problem. This struggle is expected and is positive for their mathematical experience. The students should not get an easy answer from the Circle leader. The more they work, the more they can claim ownership of the material and feel more involved.

Leading Circles

How a Math Circle is led is inevitably going to vary from person to person, and true masters are hard to find. We started out as total novices when it came to leading Math Circles, and had significant difficulties with every part of the process, from managing student behavior to choosing topics to posing questions in an accessible way. We never fully attained our “ideal” Math Circle; leading a Math Circle is a challenge, and one inevitably makes mistakes. Just as in mathematics, however, that is all the more reason to keep trying.

Despite the challenges, we found that our students kept coming back and seemed to be enjoying the Circles. We firmly believe that what makes a Math Circle is the intention behind it; there is no magic formula you must know or training you must undergo to make it happen. One need only explore their topic beforehand, so that they can guide the students out if they find themselves in a rut; come to the classroom with an accessible mystery in hand; and let the students run, gently encouraging their romp through the world of mathematical structure.

In this section, we attempt to pass on some of what we learned about leading Math Circles during our twenty-week experience. We will focus less on general classroom management here, and more on the art of leading a Math Circle in particular.

The first thing to attend to when leading a Math Circle is preparation, which is discussed in more detail in the previous section. The important thing is to come up with a topic that you are passionate about and that students will be able to make progress with. The world of mathematics — loosely defined — is your oyster. Before the Circle, be sure to explore your topic, acquainting yourself with its branches and connections, treasure troves and pitfalls. What other preparation you will need depends on your topic — it may vary from a quick opening speech to carefully thought-out worksheets or game boards. The important thing is to set your students up for success.

With preparation done, perhaps the most important part of any Circle, we found, was the first couple of minutes. This is your chance to grab the students’ attention and interest, which are vital for a smoothly-running Circle. Circles should be started with a question. It should be simply stated, perhaps with some explanation of the rules of a game or puzzle. The question should be easy to understand and intriguing. It should be accessible enough that the students feel they have a chance of grappling with it, but not so easy that they are insulted you asked — unless it is a ruse to draw them in through their outrage at being asked something so pedestrian, only to find a much more worthy problem lurking behind. Wording, too, is especially vital. Initial questions can be stated very informally; part of a Math Circle is deciding what the question is, and an excess of mathematical specificity from the start will only serve to turn students off. The first couple of minutes are your chance to get the students on your side; nothing is more important than this.

Another important part of leading a Circle is showing the students that you are on their side. Show that you are having trouble with the problem too, that you have to think about it, that you are confused — that it is okay to be wrong. A healthy degree of “playing dumb” goes a long way. As well as normalizing struggle, by showing your slight incompetence you are putting responsibility on the students — putting them in charge of their own mathematics. If they are taking the lead because they have had enough of your antics, your Circle is going well. Finally, the most important part of making friends with the students is to be friendly. You may be the one up at the board, but you are their ally, and an equal participant with them on this mathematical adventure.

Part of putting the students in charge is letting them talk. The ideal Circle, in our view, is one where the students are bouncing ideas off of each other and doing math amongst themselves. In general, the more the students are talking and the leader is silent, the better the Circle is going. As part of this, encourage students to invent their own terms, visualizations, and notation, and use those things when talking with them. We feel there is no need, at any point during a Math Circle, to tell students what something is "actually called" or how things are "actually written." We believe that this generally does more harm than good. Use their notation freely; name theorems after them; let them be the creators and the owners.

The principle behind how students learn in a Math Circle is what we call "guided discovery." Rather than being taught mathematics, in a Math Circle students are guided by the instructor to discover mathematics for themselves. This central philosophy is why we recommend letting the students do as much of the talking and creating of terminology as possible. They should not just be learning math — they should be doing math.

Important here is what it looks like to "do math" — that is, what we want to see our students doing during a Math Circle. In our view, perhaps the majority of doing mathematics is what at first glance looks like "making mistakes" — going down a winding road of logic only to find that it leads to a dead end. In many cases, barking up the wrong tree for a good long while is the key to knowing the correct tree when you come to it. That is to say, let mistakes be made; run with them, or better yet, let the students run with them, until their inevitable contradictions lead their ideas to implode. Your role as the leader is not to keep students on the straight-and-narrow path to the solution, but to keep the cogs turning, help them out when they are stuck and demoralized, and keep them from running too far down roads where there is truly no productive struggle to be had. The Kaplans call leading a Math Circle "a high wire act without a safety net," and for good reason. It is a constant dance between confusion and inspiration, excitement and demoralization. Every Circle is unique, and there is no formula for how to navigate their shifting landscapes.

Another very important aspect of leading a Circle is picking up those who are being left behind. Along with gently guiding the conversation, gently evening out which voices are heard is vital. If you find that a student is shy or quiet, find a way to draw them in. Throw them a relatively straightforward question, ask them to draw something on the board, ask what they think about what is going on. Even the smallest bit of participation can be invaluable for making them feel as though they belong in the Circle.

We found that group work can also be a good way to get students engaged. If the discussion seems to be lagging, and if the group is on the larger side, splitting students up into smaller pods to have them discuss some issue, try some problem, or play some game can help break the intellectual ice, and get more voices heard. We also often found, however, that too much group work broke down the cohesion of the Circle as a whole, making it difficult to bring the students back together and let them learn from what other groups had done. Group work is a useful tool, especially in larger Circles, but one to be used carefully.

The above paragraphs are attempts to express just some of the many lessons we have learned about how to lead Math Circles. In the end, the only way to really learn is to try it, and we recommend doing just that. We wish you the best of luck — may you run many joyous Math Circles!

Logistics & Classroom Management

Support from the School and Promotion

With the help of Northfield Middle School's Volunteer Coordinator Amy McBroom and teachers Anne Jarvis and Angela Schock, we were able to create and promote our Math Circle session to all sixth graders. Their enthusiasm for the project was an essential bridge between our group and the students and without them we would not have been able to create a successful community.

The club's initial promotion was done through the efforts of both Amy and Anne. They sent out emails to parents and asked study hall teachers to introduce the opportunity to students. The following was sent to parents at the start of the school year:

> "The Northfield Middle School in conjunction with Carleton College Math Department is excited to bring a new Math opportunity to interested 6th graders.
>
> **WHAT**: A Math Circle starts with an easily-stated, interesting question posed by the College Math leaders - something like: "What if my bicycle had wheels that were shapes other than circles? How could I make the bike roll smoothly along?" The group embarks on an encouraging sense of discovery and excitement in the problem where the leaders guide the students to find mathematical truths.
>
> **WHEN**: Math Circle will meet on Mondays and Tuesdays at NMS during 7th hour. Math Circle will start Sept. 30th and will run throughout the year.
>
> **WHERE:** The Math Circle will meet at Northfield Middle School 2nd floor 6th grade break out area.
>
> **WHO:** Do I have to be a math whiz? Absolutely not, Math Circle is built for all levels of math. It is not about getting the right answer but thinking about math questions in a different way. "

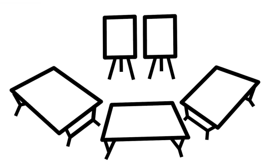
Half way through our time at Northfield Middle School, we did a second round of promotion. We cancelled Math Circle for that week and instead visited all the sixth grade classrooms to introduce ourselves. We covered the time and location of our meetings, as well as how students can get permission to come. We then explained the goal of Math Circle and had some students in the classrooms who had attended sessions before share some of their experiences. Following our introduction, we presented each of the classes with the Monty Hall problem. The lesson plan for this problem appears later in this book. We started by playing *Let's Make a Deal* as described in the lesson plan and promised to cover the winning strategies the following week at the next Math Circle. The presentation took about five minutes per classroom, and was meant to give them a taste of the kinds of problems we solve. After this promotion, the size of our Circle doubled, but the students themselves varied from week to week.

Classroom Setup and Materials

Our Math Circle met during the students' study hall at the end of the day. We met for about 40 minutes of their hour-long period so students could have about 10 minutes to come from study hall and enough time to go back and pack up their bags. As the Circle leaders, we taught two sessions a week, on Mondays and on Tuesdays, to accommodate students who could only come on one day or the other. At first, the topics on the two days were the same, so students could only participate once a week. Eventually, our Monday and Tuesday topics diverged, and we allowed students without prior commitments on either day to come to all Circles offered.

To come to Math Circle, students had to get permission slips from their study hall teachers. The slip included the time students left study hall, arrived at Math Circle, and left Math Circle. It was meant to make sure students leaving would actually come to our sessions rather than explore the hallways with friends. At first, this was the only requirement. Eventually, one of the school administrators noticed that some students came to Math Circle only to hang out with friends and not to participate, so she implemented a late homework rule. If a student had more than five late homeworks, they were required to stay in study hall to finish them rather than come to Math Circle. We were disappointed that some students would no longer be able to attend, but that being said, the smaller group was much easier to handle.

Our Math Circle was set up so that students could face each other while working together in smaller groups, but also easily face the whiteboards. We had three large tables organized in a semi circle facing the whiteboards as pictured below. We met in the cafeteria, which provided us with a lot of space to do interactive activities and let the students move around. Many other students would walk by the cafeteria and distract our students, but because of the size and personality of our Circle, an actual classroom was too restrictive and volume control too difficult.



We used the whiteboards to write the question, lay out rules, or keep track of different ideas that floated around. The goal of the whiteboards was to serve as a way to collaborate as an entire group and share ideas. We also allowed students to come up and present to each other. We found it more valuable for them to lecture the group rather than for the Circle leaders to do so.

In addition to the whiteboards, we highly encourage having multiple marker colors available as they were helpful in distinguishing different ideas. For example, if the Circle is the Four Islands Problem, then the four islands can be drawn in black, but paths drawn by different students can be color coded so they are not confused with each other. Make sure your students also have a good supply of scrap paper and pencils for every Circle, as they are essential.

Classroom Size

Our Math Circle reached about 65 unique students over the span of five months. We had a broad range of class sizes; on some days we had fifteen to nineteen students, while on others we would have four students. There was even one day when not a single student came. In general, however, our classroom size averaged around nine students. We found that smaller groups of five to seven made it easier to keep track of the progress each student was making. When the group was about this size, we focused on working all together as a way to emphasize collaborative thinking. We usually broke off into smaller groups of two or three students each before coming together to share ideas. This way, each student was able to have a voice and have time to think through the problem before collaborating with the group as a whole.

Classroom Management

There are many ways to organize a Math Circle. In general, we chose to explain the rules and then set up the problem by asking some sort of leading question that hinted at where we hoped our Circles would go. After the initial problem statement, we refrained from posing any new leading questions. We wanted students to explore the mathematics and the puzzles on their own. This meant letting them travel down paths that may lead them to a wrong answer, but they are given the opportunity to reason out their mistakes. We also decided early on that we would not force students to solve the problems within the time frame we gave them. We prioritized their ability to learn rather than arriving at an answer. Whether or not you choose to follow the same format, it is essential that all Circle leaders agree on how they will run a session.

During our Circles, there was always more than one Circle leader present. To divide up the responsibilities, we had one designated leader who explained the problem rules and posed the problem statement. They were also responsible for bringing the group together for times of collaboration. Any additional leaders were responsible for handling the noise by making sure everyone was listening and participating and maintaining the standards of respect discussed above. When students broke out into subgroups, all Circle leaders rotated around and helped students interact with the material. They reminded them of rules, gave extra supplies, and answered any clarifying questions. We found this was the easiest way to manage the classroom and make certain that the opinions of all students were heard. Based on our own experiences, this division of responsibility worked best with Circles of six to ten students. With a group this size, we were always able to check in with students on more than one occasion and make sure all voices of the group were heard.

Occasionally, there would be students who were already familiar with the problem. In those cases, we had them try to explain their reasoning to other students so they could work it out together. This would help them gain critical thinking skills as they practiced communication and analyzed their answer. We would do the same when students solved puzzles faster than we anticipated or faster than other students in the group. Because most of them had not had experience communicating mathematics, it was not only a good way to fill our time, but also a great way for them to practice.

Sometimes our group had up to twelve students participating and they all varied in their classroom confidence. Some students were very shy and we would need to quiet others down so that they could be heard. Other times we would have students answer at the same time so that no one student was always answering the questions. A version of this was done with the Counting Cups Circle. Sometimes the students would naturally split off into groups; energetic students would try the activity in an interactive way, while shy students participated by creating a paper and pencil version of the interactive activity. This version was done for the Cup Flips Circle.

We also found that it was common for students to call each other stupid, which, of course, we did not allow during our session. When it did occur, we made sure to discourage that behavior and acknowledge the original points that the student was making. If a student was being particularly troublesome — for example, one of our students spoke in an offensive accent — we would pull them aside after the lesson to address their behavior and make sure they understood that they would be sent back to their study hall if they continued.

All in all, students can be quite unpredictable. Especially when they are put in an environment like this where it is academic but not in a traditional way. However, we had a lot of positive experiences with the students in terms of classroom management. Sometimes when a student was stuck on a problem, we witnessed their peers come to their aid to work on clarifying the confusion. We have seen students upset because they had to miss Math Circle for the week, and we have also seen a lot of students laughing and enjoying mathematics with their friends. Furthermore, some students met at Math Circle for the first time, and later became close friends, and it was a privilege watching these relationships of love and respect form in front of us.

The Circles

Beginning after this section, we include descriptions of all the Circles we ran at the Northfield Middle School, in addition to some that we did as online videos, and some that we came up with but did not get to run. The ladder two arose as alternatives to in person Circles because of the COVID-19 pandemic. We wanted to be sure students still had access to our Circles, even if they were learning from home. These descriptions are meant as guides for those who may wish to do these topics as part of their own Math Circle. Most of these topics were intended to be one-hour Circles, but many could have benefited from more time. We found that 40 minutes was a good length for a single session, but we encourage you to extend topics beyond one session if there is interest and you are able to gather the same group of students more than once.

The topics vary widely in subject matter, from shapes to algorithms, from counting to game strategies, from free-form and creative to carefully structured. If you are looking for something in particular, or looking to get an overview of what is contained within this book, we recommend a quick glance at each Circle's "Keywords" section, which appears at the top of each Circle's description. These brief keywords should give a basic idea of the mathematics contained within. For a somewhat more detailed description of a specific Circle, look to the "Problem Statement" section, which is meant to give a concise explanation of the problem that the Circle centers around.

Each topic in the book is introduced as follows. The section begins with a title and an illustration, followed by the aforementioned "Keywords," which give a brief taste of the topics and ideas that the Circle covers. Below the Keywords section, we list the "Materials" needed to run the Circle. In all cases, the availability of pencils and plenty of paper is assumed. Having a whiteboard or blackboard as well is ideal; we never did a Circle without one, but we think it would be difficult.

Following the Materials section is the "Problem Statement." This describes the problem that the Circle begins with or centers around, and gives you a relatively quick reference for what the Circle is all about and how you might introduce the topic to students.

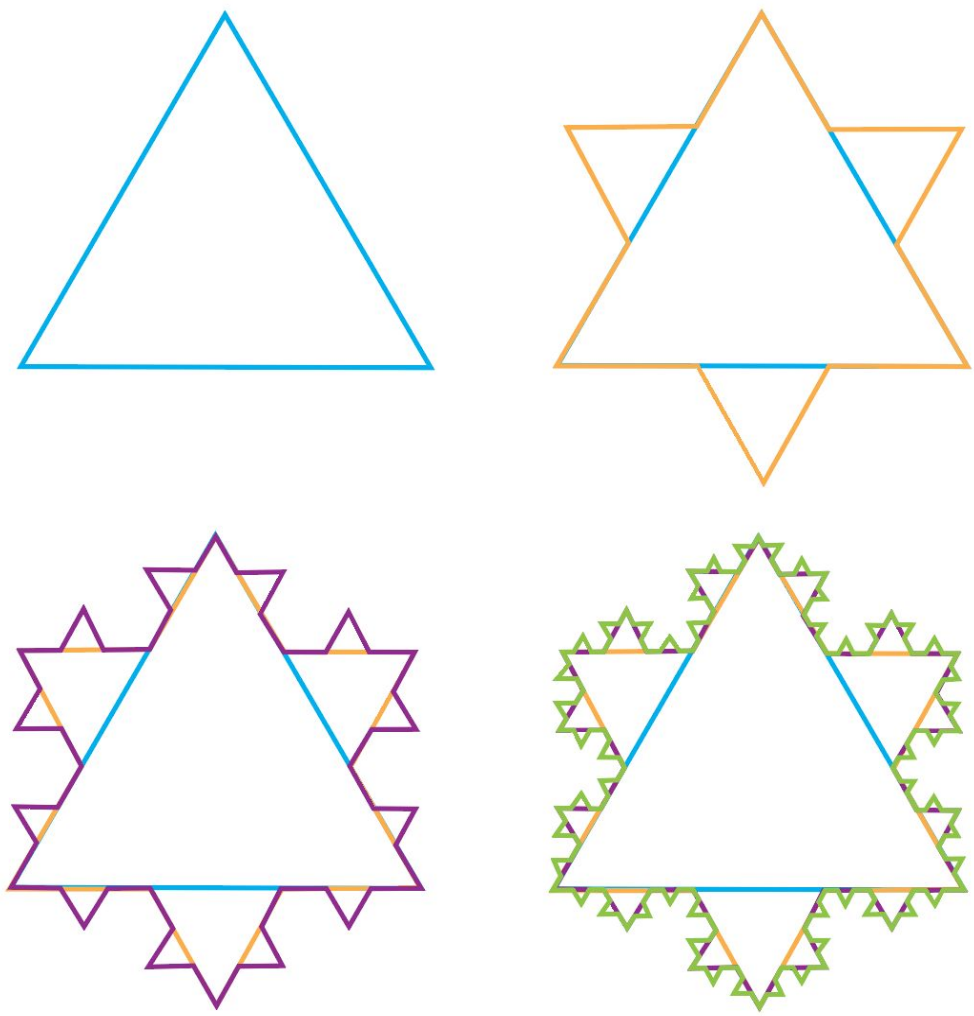
After the Problem Statement, we include a section on "Math Background." This outlines the mathematics you will want to know going into the Circle, so that you are able to lead effectively and confidently, knowing how best to guide students towards productive struggle and eventually the satisfaction of answers (and more questions). All Math Background sections are written with little or no math knowledge assumed.

The next section of each Circle's explanation, "Our Experiences & Suggestions," gives some of our experiences leading the Circle and advice for those who wish to lead one with the same topic. For Circles we were not able to lead in person or at all, we give suggestions based on our other experiences as best we can.

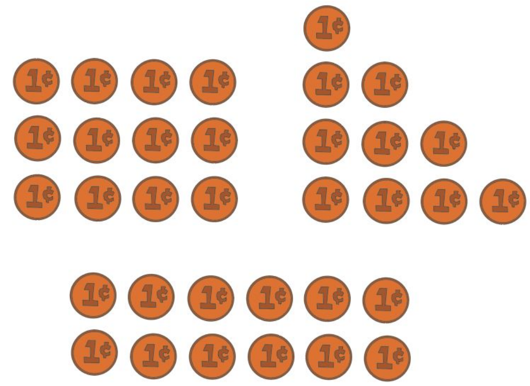
The next section, "Taking It Further," is meant as a springboard for further exploration in the given topic. In general, there are countless new questions to be asked about each topic, and this section is our attempt to describe some of those in case you have the chance to take the Circle further. These lists of ideas are by no means exhaustive, and if a student comes up with their own new question, we encourage you to explore that one rather than any of ours.

Finally, at the end of each Circle description we have a section on “Sources & Additional Resources,” in which we describe where we got our ideas, give thank-yous to anybody who may have inspired or helped to create the Circle, and give materials you can explore if you wish to round out your understanding of the topic, take it further, or look at where our ideas came from. We have also included an additional section on “Further Reading” at the end of the book to highlight some of the works that have had the greatest influence on our Circles. All of these resources have contributed to the success of our Circles.

Shapes and Patterns



Penny Shapes



Keywords
Prime Numbers, Triangular Numbers, Factors, Geometry

Materials
Pennies or any kind of small object that is easy to move around and has a uniform shape. Make sure that the object will not distract the students.

Problem Statement
To start, give the students any number of pennies and ask them to make filled rectangles (see image for two examples using twelve pennies). Keep increasing the number of pennies and see if they notice any patterns or identify any numbers that will not work. We recommend starting with eight, since the number is small. It also has interesting numbers that follow; nine is a square number, ten is divisible by primes, eleven is prime, and twelve is divisible by two, three, four, and six. What numbers of pennies can form a rectangle without any left over? What numbers cannot?

After sharing strategies for rectangles, the Circle can move on to triangles. To keep things consistent with the rectangular set up, have them build right triangles like in the image above. What quantities of pennies can create triangles? How can we guess which numbers will work without constructing the triangle?

Math Background
<u>Rectangles</u>
Rectangles with dimension $x \times 1$ can be made with any number x of pennies. If there are a prime number p of pennies, then the only rectangles that can be made have dimension $p \times 1$ because primes are only divisible by one and themselves. So if we exclude any "straight line" rectangles, we can only make rectangles from a non-prime number of pennies. Composite numbers (non-prime numbers) can be used to make rectangles with dimensions of their factor pairs. A factor pair for fifteen, for example, is three and five, since $3 \times 5 = 15$. Therefore we can make a 3×5 rectangle out of fifteen pennies.

<u>Triangles</u>
There are a couple of ways to understand the number of pennies that are necessary to create a triangle. The smallest "triangle" that can be made consists of only one penny. There is only one penny in the bottom-most row of the triangle. If another row is added, our total quantity of pennies increases to three, with two pennies in the bottom-most row. We can continue adding new rows and get totals of six pennies in a triangle with three pennies on the bottom-most row, ten total pennies for a bottom row of four, fifteen for five, 21 for six, etc. One way to compute the number of total pennies is to sum up $1 + 2 + 3 + 4 + 5 + \ldots$ until we reach the desired number of pennies in the bottom row. Another way to calculate the number is to use the following equation: $\frac{n(n+1)}{2}$. In the equation, n

refers to the number of pennies in the bottom-most row. For example, if we want to compute the number of pennies used in the triangle formed above, we let $n = 4$ and compute $\frac{n(n+1)}{2} = \frac{4(4+1)}{2} = \frac{4(5)}{2} = \frac{20}{2} = 10$.

Our Experiences & Suggestions

- The sixth-grade students at Northfield Middle School had already developed an understanding of factors by studying fractions, so some may have been subconsciously thinking about the factors of the number of pennies we give them to create the rectangles.
- Some students would make straight line rectangles when given a prime number of pennies. That is okay! Although the purpose of the exercise is to show that it is impossible to make a rectangle using a prime number of pennies, it is a valuable way to understand primes are only divisible by one and themselves. If they start to make straight line rectangles for all numbers, however, then encourage them to look for other ways to create rectangles so that they can better explore the possibilities of both prime numbers and other non-prime numbers.
- Many understood that primes did not work, but could not come up with a reason as to why the others worked. The pattern may become more clear if students drew their rectangles for other students to better understand the dimensions they have naturally created.
- The students enjoyed exploring triangles more than rectangles. Some students quickly picked up on the pattern for rectangles, while others solved the tasks quickly but were not motivated to explore why. They did, however, enjoy the triangles. Many built their triangles by adding new rows to the bottom each time. This was one of the more common ways that students found a pattern.
- In general, the students were more engaged with the more challenging task of creating triangles. The Circle was a good way to gauge how difficult others should be.

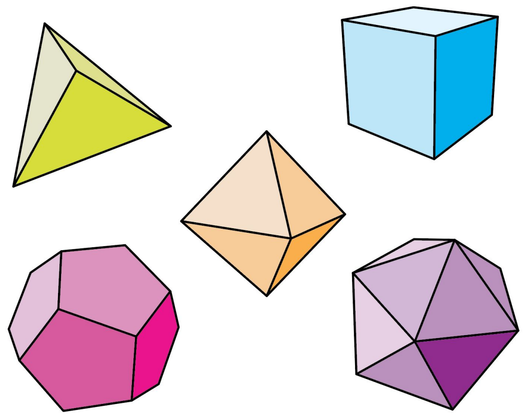
Taking It Further

- Ask the students to make other kinds of shapes: octagons, trapezoids, etc.
- How can we combine shapes to make others?
- How many pennies do we need to make the outline of a four by five rectangle (no fill)? What about other dimensions? What about other shapes?

Sources & Additional Resources

Thank you to Stephen Kennedy for showing us this problem.

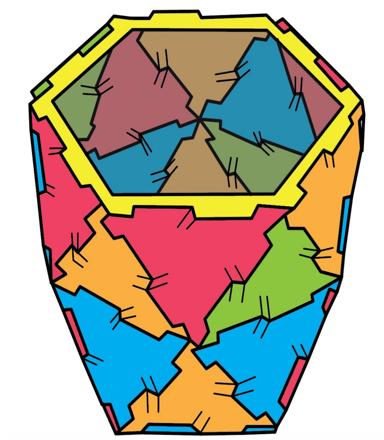
Polydrons



Keywords
Shapes, Symmetry, Closed Shapes
Geometry

Materials
Multi-colored Polydrons (as many as you can get!)

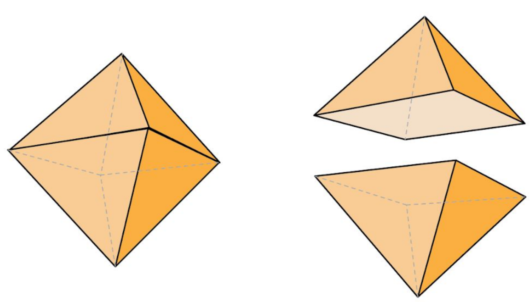
Problem Statement
The goal of this Circle is to allow students to explore various three dimensional shapes and structures with Polydrons. This is a great Circle to lead if you wish for students to have freedom with how they want to explore geometric shapes so that they are able to learn and discover new concepts.



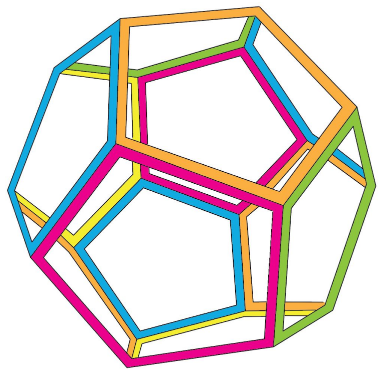
We began the Circle by allowing every student to play around with Polydrons' geometric shapes and build any shape and structure whatsoever for the first 25 minutes. In our Circle, students made various three-dimensional shapes, such as cubes, cylinders, cones, and spheres. We then let them explore symmetrical shapes and structures with the remaining time. Please refer to the math background section to learn more about different types of symmetry.

Math Background
Polydrons are usually 50-piece sets of interlocking shapes, e.g. squares, equilateral triangles, and pentagons, of different colors.

A 3-D structure that is made out of these pieces is said to have mirror symmetry if it can be divided in two such that the two halves are mirror images of each other. The figure below is symmetrical because it can be divided into two shapes that are exactly the same.



A 3-D structure is said to have rotational symmetry if you rotate it and it appears the same after some rotation. Below is an illustration of a dodecahedron (a solid shape with 12 flat surfaces). This structure has both mirror symmetry as well as rotational symmetry. If you divide this structure in two halves, they are both identical. If you rotate the structure, it will look the same as it did before the rotation after some time.



Our Experiences & Suggestions

- Since this Circle is more free-form, it has the potential of getting rowdy. We recommend working on this in small groups to allow students the opportunity to learn and discover while avoiding disorder.

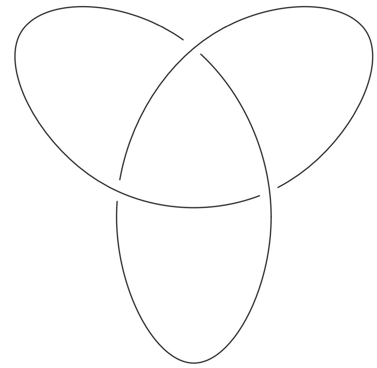
Taking It Further

- Circle leaders can take this Circle further by allowing the students to build structures where two different common shapes, like cubes and pyramids, come together to form one big structure
- You can also challenge the students further by making them guess what common structures can be formed by dismantling 3-D structures and laying their 2-D versions on a flat surface.

Sources & Additional Resources

Students can try to make more complicated three dimensional shapes and structures.

Knots and What-Not



Keywords
Topology, Knot Theory

Materials
An extension cord for each group, and a long rope such as a tug-of-war rope or similar object for the entire group.

Problem Statement
This activity was done in two parts on two different days. The first part is meant to serve as an introduction to knots and how to represent them on paper. The second part will give the students the opportunity to explore knot theory and try to use their intuition to understand what knots are, and how some complicated knots could be simplified to simpler knots that are the "same."

<u>Part One</u>
Start by having your students try making a human knot. This game is usually played at camps as an ice breaker, but for the purpose of this activity, it will be used to explain knots. Start first by forming a standing circle and ask each student to reach across the circle and grab the hands of two different students. Once the students are ready, ask them to untangle themselves by moving over, under, or around each other without letting go of each other's hands. Have them play a couple times to get the hang of the game before moving back to the whiteboard. Ask the students to form a new human knot, but ask for one student to step out and try to draw the knot on the whiteboard. When they have finished, ask the students to try for themselves on paper. Before they let go of each other, make sure to take a picture of the knot for later in the Circle, but without the faces of the students. With the photo as reference, have them all try to replicate the knot on the paper. This is tricky, so do not worry if they do not get it right away. Simplify the problem; ask the students to hold their own hands and try to draw it. What if they crossed their arms and held hands with their neighbor — what might that look like on paper? How can we tell the difference between someone with their left arm over their right arm versus someone with their right arm over their left arm? Try the original big knot again.

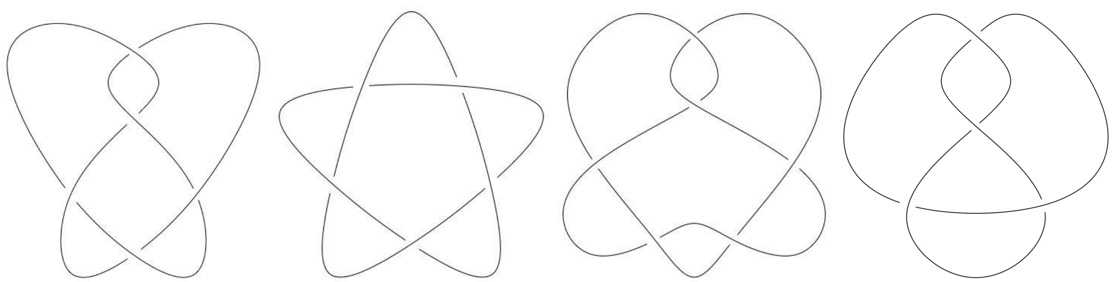
<u>Part Two</u>
To refresh their memories, have them play a round of the human knot again. After they are done with this activity, help them form a specific kind of (human) knot, the trefoil knot, pictured above. You can create this knot by having the students stand in a line. The Circle leader should hold the first student's hand and start going over and under arms the same way this knot goes over and under itself until you connect the first person in the line with the last person. Ask them to unknot themselves, just like in any other human knot. This puzzle is actually impossible, but it gives the students a chance to challenge themselves and consider the reasons why they are not able to untangle. After the students are done with this activity, have them look at some of the four knots pictured below, but using the extension cords. Let them think about some questions. Which ones can you untangle? Which ones can you not? Is there a way to know? After the students have worked with the

knots without touching them, and have developed the intuition, they can start trying to untangle them by moving the ropes around.

Math Background

First some quick terminology: a knot in mathematics is like an everyday knot, but with the ends glued together. An unknot is one that can be untangled into a circle. A couple of good knots to work with are the ones below. These are some of the simpler knots that are not unknots. The leader can form these knots by tracing the extension cords following the knot's path, and then connecting the two ends of the cord. You can then make it look more complicated by making twists and putting strands of the knot over other strands. Make sure that the ends always remain connected together.

Now, what about simplifying knots? We can simplify knots by reversing the process of making them look more complicated! For example, if there is a twist in the same strand without going over or under another strand, we can just untwist that strand, etc. There are three ways to make a knot more complicated, they are called Reidemeister moves. More on the Reidemeister moves is included in the *Sources & Additional Resources* section.



Our Experiences & Suggestions

- All of our students were comfortable holding hands with each other, but recognize that this might not always be the case. Make sure that your students are always comfortable participating in the activity, and if they are not, have some sort of alternative for them. For example, students not comfortable with this activity might help untangle from the sidelines. To make sure their voice is not lost, perhaps they and one other student could be in charge of telling the other students how to move to successfully untangle the group.
- We had more than one Circle leader for this lesson. Therefore, the leader that was not responsible for explaining the rules participated in the human knot. They moved only when the students told them to. We found that this helped everyone participate, even if for some reason they did not need to move to untangle the knot. It was also a fun way to build community in our group.
- For part one, the student who drew the human knot on the whiteboard struggled to see the knot that was made. We found it useful to have some of the students in the back lift their arms while the students in the front squatted or sat so that the knot was perpendicular to the floor rather than parallel.
- From our experience, it was more valuable to have the students just look at the knots formed by the chords and tell the leaders what to do to try to untangle them. This way the students would not touch the knot, but learn how to give good

instructions on how the process of unknotting happens. Of course the leaders would follow the students' orders and they would move the knot around.

Taking It Further

- You can ask your students how they can come up with knots if they cut two trefoil knots, and then glue their ends together. Is there a specific way to glue trefoil knots to still get a knot? Is there a way where you glue trefoil knots to get an unknot?
- Which knots are three colorable? More information on what that means is below.

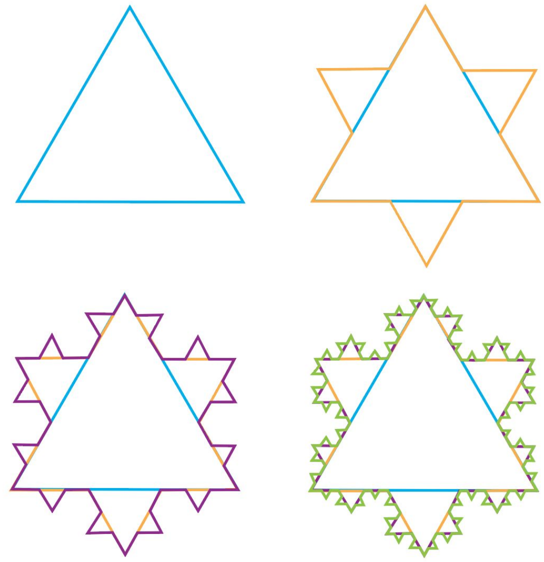
Sources & Additional Resources

There are a few ways to work through a knot, called the Reidemeister moves. This image untangles a seemingly complex knot into the unknot. Try to identify which Reidemeister moves are used at each step!

Want to learn about a three colorable knot? This website describes the rules of tricolorability and gives some examples.

The Knot Book by Colin Adams is also a friendly resource that you could use to learn about knots in general!

Fractals



Keywords
Fractals, Perimeter, Area, Infinity, Shapes, Creativity

Materials
None needed beyond scratch paper and pencils.

Problem Statement
The problem statement is brief: “Can you draw a shape that has finite area, but an infinitely long edge?” This can also be rephrased, with a circle drawn on the board: “Can you draw a shape inside this circle that has an infinitely long edge?” If the students need help, start drawing the first couple of iterations of the Koch snowflake (see “Math Background” for more).

Math Background
A fractal is a shape that is infinitely detailed, usually with patterns that repeat themselves at a smaller scale when you zoom in on part of the shape (that is, it is self-similar at different scales). Common examples are trees, coastlines, and snowflakes; for each of these shapes, if you zoom in on part of the shape, it looks similar to the shape as a whole.

For this Circle, we use the Koch snowflake (pictured at the top of the page) as a jumping-off point. The Koch snowflake is constructed as follows. Start with an equilateral triangle. Then, split each edge into thirds. Take the center third of each edge, and attach an equilateral triangle to it — so each edge grows a “bump”. Then repeat this process, adding a “bump” to each edge of the new shape (the edges should all be $\frac{1}{3}$ the size of the edges of the previous iteration), and repeating for, in theory, forever.

Notice that if you zoom in on part of the snowflake, the pattern you will see is the same as that of one of the sides of the full-size snowflake. This is what makes it a fractal.

What is the area of the snowflake? Let us call the area of the starting triangle 1. Each triangle we add has $\frac{1}{3}$ the side length of the original triangle, so the area decreases by the square of that — each new triangle has an area of $\frac{1}{9}$. We add 3 triangles, so we add a total area of $\frac{3}{9} = \frac{1}{3}$. How much area do we add in the next iteration? Note that from now on, every edge we had before has turned into four new edges of $\frac{1}{3}$ the length. So, we will add four times as many triangles as we did the previous time, but each will have $\frac{1}{9}$ the area. So the area will increase by $\frac{4}{9}$ of how much it increased in the previous iteration; so the second iteration adds $\frac{4}{9} \times \frac{1}{3} = \frac{4}{27}$ area.

How much area does this give us, in the end? Observing how the shape grows should be enough to convince students that the area will always be finite, no matter how many iterations you do (because it fits inside a circle). That is likely to be enough for this Circle. However, to get the exact final area, we can consider it to be one plus the sum of a geometric series of the form $a + ar + ar^2 + ar^3 + \ldots$, where a is $\frac{1}{3}$ and r is $\frac{4}{9}$. (We add one because the series only counts the added triangles; we need to add one for the original triangle.) The formula for the sum of such a series is $\frac{a}{1-r}$, which gives us $\frac{1/3}{1-4/9} = \frac{1/3}{5/9} = \frac{3}{5}$. Adding one gives us that the total area of the Koch snowflake is $\frac{8}{5}$ times the area of the starting triangle.

What is the perimeter of the snowflake? Let us decide that the starting edges are length one; so the starting perimeter is three. Then, at each iteration, we remove one third of each edge, but add a "bump". Since the "bumps" are equilateral triangles, both of their edges are equal to the side that was removed — so, in effect, each side loses $\frac{1}{3}$ of its original length, but gains back $\frac{2}{3}$. This gives us a final length of $1 - \frac{1}{3} + \frac{2}{3} = \frac{4}{3}$ of its original length.

This rate of increase applies not only to every edge of the original shape, but also to every edge of every iteration of the snowflake. Thus, at every iteration, the perimeter increases by a factor of $\frac{4}{3}$. Since the rate of increase is larger than one, this quantity will increase forever, without bound. That means that the perimeter of the infinitely-detailed snowflake is infinite. Thus, the Koch snowflake — in its final form — has finite area, but infinite perimeter. Similar processes to those we used here can be applied to other fractals, to determine whether their final perimeter is indeed infinite, and whether their final area is indeed finite (or perhaps even zero).

Another interesting notion related to fractals is that of "fractal dimension." For a normal 2D drawing, multiplying the dimensions (width and length) of the figure by some amount (n) will multiply the area by the square of that amount (n^2). For a 1D drawing (a line or a curve), multiplying the dimensions by n will multiply the length by n as well (n^1). In general, when you scale up a d-dimensional object by a factor of n, its d-dimensional area (length for 1D, area for 2D, volume for 3D, etc.) increases by a factor of n^d.

Fractals, however, do not always fit neatly into a dimensional category. For example, doubling the dimensions of Sierpinski's triangle (a relatively well-known fractal) multiplies its area by a factor of three. This is neither $2^1 = 2$, as it would be for a 1D shape, nor $2^2 = 4$, as it would be for a 2D shape. Thus it is said to have a "fractal dimension" somewhere in between one and two — specifically, its dimension is the power to which you raise two in order to get three.

Our Experiences & Suggestions

Note: this problem was not conducted in person due to the COVID-19 pandemic.

- We suggest that the leader of this Circle try creating several fractals of their own, or look up fractals before leading the Circle (we suggest some interesting ones in the Sources & Additional Resources section below). This should give the leader some ideas as to where the students could be nudged in order to make interesting discoveries.
- We also suggest that the leader allow the students to be creative, nudging them in a productive direction when necessary, but only when necessary. However, be sure

that mathematical rigor is maintained; simply stating that "this shape has infinite perimeter" is not enough — it can and should be shown mathematically.

Taking It Further

There are many, many questions that can be asked, and avenues that can be explored. One is taking a more numerical approach to the snowflake, or to any fractal that comes out of the Circle: What is the area of this shape? What is the perimeter? How do these quantities change when you change the size of the fractal? What does this mean about the dimension of the fractal? (See the last two paragraphs of the Math Background section for more on "fractal dimension.") The results will not be too counter-intuitive for the Koch Snowflake, but for other fractals, like Sierpinski's Triangle, they will be.

The Circle can also be taken in more creative and exploratory directions: Are there other shapes like this that you can create? Can you create other self-repeating shapes of different sorts? (You may nudge them towards Sierpinski's Triangle, or other fractals; see the resources below for some ideas.)

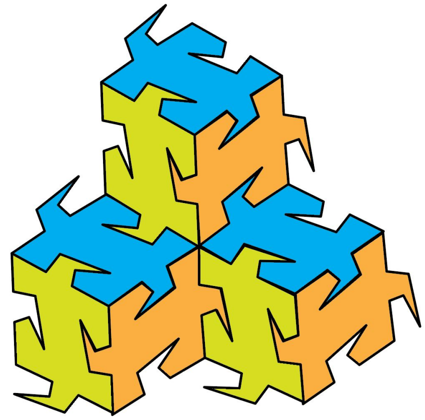
Another interesting question is this: can you create a shape with infinite perimeter, but zero area? It cannot just be a line — it has to be a two-dimensional shape, or a collection of shapes. A fractal called the "Cantor Dust" (linked below) is a good example of how this can be done.

Another question, for a curve-ball, is this: can you draw a shape with finite perimeter but infinite area? (Answer: no, but this can lead to interesting places. For example, what is the largest area you can fit inside a perimeter of a certain length? A circle is your best bet here, although proving this will be quite difficult.)

Sources & Additional Resources

This Circle was an original idea by us, inspired by a suggestion from Robert Kaplan. For more on the Koch Snowflake, Wikipedia is an excellent source. Some other fractals that were mentioned in this lesson plan include Sierpinski's Triangle and the Cantor Dust. Other interesting fractals include the Dragon Curve, space-filling curves such as the Hilbert Curve (about which there is also an excellent video), and the Mandelbrot Set (this one is likely too hard for students to fully understand, but an excellent illustration of the beauty of fractals).

Tilings and Tessellations



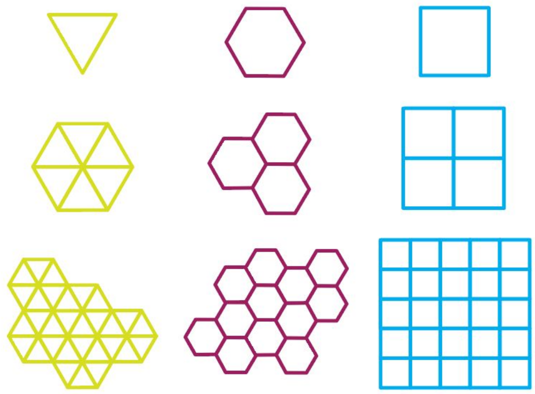
Keywords
Tiling, Tessellation, Shapes, Patterns, Creativity, Art

Materials
Graph paper is recommended, possibly including hexagonal or triangular graph paper, if available. Enough for each student to have at least two or three pages to themselves.

Problem Statement
What tiling means is to take a shape (or multiple shapes) — "tiles" — and repeat them, possibly changing their orientation, such that they can fill an infinite flat surface with no gaps or holes. This can be introduced to the students in relation to tiling garden patios, bathroom floors, etc. We say that a shape (or collection of shapes) "tiles" if it can be used to tile a flat surface.
Given this definition, there are three basic questions:

1) What regular shapes can be used to tile a garden patio? By regular shapes, we specifically mean regular polygons — shapes where all the sides are the same length and all the angles between the sides are the same. (For example, triangles, squares, pentagons, hexagons, and so on.)
2) What other shapes can be used to tile a garden patio, or any flat surface? Especially fun here is challenging students to turn their favorite animal or thing in general into a shape that tiles.
    a) A good follow-up question is: which regular polygon is this shape acting like fundamentally, if any? That is, could you change it to a regular polygon, and still have the underlying structure of the tiling be the same? (For example, even if you are tiling with a wacky shape, if the tiles are forming horizontal and vertical rows and columns, it is most likely "acting like" a square.)
3) What collections of multiple different regular polygons (or other shapes) can be used to tile a flat surface?

Math Background
There are three regular polygons that tile the plane (that is, fill a flat surface without holes): triangles, squares, and hexagons. The tilings look as follows (note that the tiling with triangles requires two different orientations of triangles):



These are the only three tilings with regular polygons because these are the only regular polygons that fit together nicely. What exactly do we mean by that? Well, think about the corner of one of the polygons in the tiling. Several copies of the same polygon must come together at that corner, leaving no gaps, so the angles at the corners of the polygons must be able to add up in such a way that they come out to 360 degrees. For equilateral triangles, the angle at each corner is 60 degrees, and six such angles add up to 360. For squares, the angles are 90 degrees, and four such angles add up to 360. For pentagons, however, the angles are 108 degrees, and three such angles add up to 324 degrees, while four add up to 432 — so pentagons do not work. But hexagons do, because the angle of a corner is 120 degrees, so three hexagons together make 360 degrees.

For shapes with more sides than hexagons, the angles are larger, so there must be fewer that come together at each corner; however, the only possible whole numbers fewer than three are two and one, and neither of those will work. So triangles, squares, and hexagons are the only regular polygons that work. Actually, this is not a full proof; this assumes that polygons meet in such a way that their corners all come together, and that there are no places where corners meet sides. But it does not take much more to show that this will not make pentagons or any regular polygons with more sides work, so we get the same result.

For tessellations with irregular shapes, a large part of the process is creativity and artistry. However, there are some techniques that can help to get started. A good place to begin is by modifying one of the three regular polygons that tile (triangles, squares, and hexagons). The polygon can be stretched, squished, or rotated, and it can have pieces of more or less any shape added to it, as long as the same piece is taken out of the other side of the shape, so that the tiling will still work. These shapes taken out or added can be straight, curvy, or wiggly. The shape can even be non-contiguous; perhaps a hole is taken out of the shape near one edge and added as a dot outside the other. As long as the tiling still works, anything goes. There are other ways to come up with tessellations, too; shapes could be allowed to rotate in interesting ways, or the tessellation could not be based on one of the three regular polygons that tile. However, the process described above is a great place to start.

As for how to tell what the underlying structure of a tessellation is, our recommendation is to do something like the opposite of the above process, simplifying shapes until they turn into something fairly simple — ideally, a triangle, square, or hexagon. Just be sure that the structure of the tiling — that is, which tiles are touching which other tiles — remains the same as you simplify. Another way to do this is to overlay the tessellation with each of the three basic regular tilings, and see which has a one-to-one correspondence between tiles. Note that the regular tiling may need to be rotated or stretched to make it match up.

To tile a flat surface with multiple different shapes, again the key is mostly creativity. Try putting shapes together and seeing what the gaps look like. Try adding up angles at corners to make sure tiles are fitting together right, or to see what shape might be added to fit in any extra space. For any tiling you create, it is a good idea to check to make sure the angles and lengths all work out; for some shapes, it can be easy to draw them in such a way that they seem to fit nicely, when in reality they do not.

Our Experiences & Suggestions

Note: this problem was not conducted in person due to the COVID-19 pandemic. Instead, we sent students a video summarizing the problem and let them go about it on their own, unsupervised.

That being said, we have some suggestions:

- The fun and creativity of the problem should be embraced as much as possible. At heart, this problem is very much about art, and we recommend that it be treated as such.
- The leader should connect the problem with real life; definitely ask the initial question in terms of a patio, a floor, or something of that sort.
- Be sure that the students are being conscientious about their shapes. It can be easy to take a shape that does not tile and "make it work" by drawing it in a slightly warped way. Using graph paper can help with this, as can thinking carefully about angles and lengths of lines.
- M.C. Escher is an artist who has created some fantastic tessellations. These would be a great thing to show the students at some point, for inspiration.

Taking It Further

There are many places to go with this problem, but here are just a few roads you could go down:

- Can you find a shape that tiles in such a way that the underlying structure is not like that of any regular polygon?
- Why are triangles, squares, and hexagons the only regular polygons that tile a flat surface?
- What if the surface need not be flat?
- What if you want to construct a three-dimensional object out of flat shapes (like a soccer ball, although we think it best not to tell them that)?
- What if the surface you are trying to fill is flat but has some sort of boundary — perhaps a rectangle, or even a circle?
- Speaking of which, can you fill space with circles? How many do you need?
- Can you fill a rectangle with square tiles? (Here you can let them decide the question of "what sort of rectangle?")
- Give the students a regular shape, perhaps a triangle, and ask: "Can you fill this shape with infinitely many regular polygons of different sizes? Can you do it in a

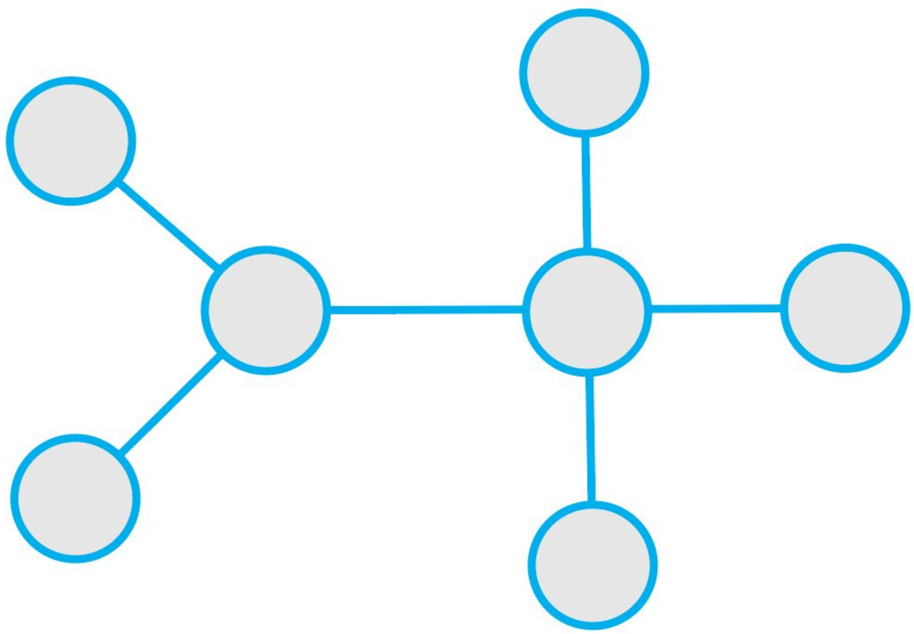
way that makes an interesting pattern?" This will potentially lead to fractals, which are discussed more in the "Fractals" Circle on page 28.

- Can you create a tiling, with a few different shapes, where the pattern does not repeat? That is, where there is no way you can shift the whole thing so that it matches up to a different part of the pattern? Penrose Tilings — an explanation of which is linked below — are a great source of inspiration for this.

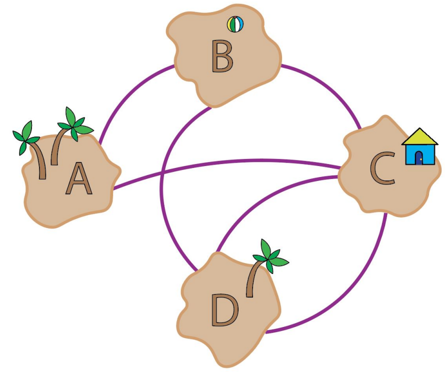
Sources & Additional Resources

Some of M.C. Escher's amazing tessellation artwork can be found here. Additionally, free printable hexagonal graph paper is available here, and free isometric graph paper (triangles) is available here. For more information about and examples of tessellations, Wikipedia is an excellent source. For tilings with regular polygons, see this article. For aperiodic tilings (tilings that do not repeat), there is an article here, and there is an article on Penrose tiling (a specific type of aperiodic tiling) here.

Graphs and Trees



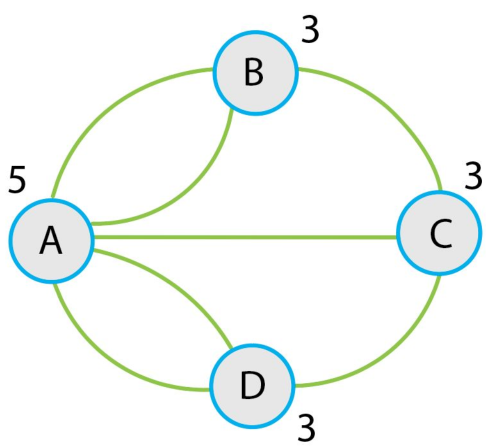
The Four Islands Problem (Königsberg Bridge Problem)



Keywords
Graph Theory, Eulerian Trails/Paths

Materials
A printout of the Königsberg bridge problem for each student.

Problem Statement
The goal is to find a route over a given number of bridges connecting four islands such that every bridge is crossed exactly once and we start and end on the same island.

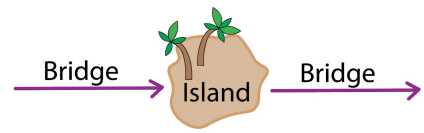


In the eighteenth century, Mathematician Carl Gottlieb Ehler grew obsessed with trying to figure out if the seven bridges of the quaint city of Königsberg could be traversed in a single trip without doubling back, such that the trip starts and ends on the same island. Carl wrote letters to a famous mathematician, Leonard Euler, asking if this problem could be solved, who in turn invented a field of mathematics called Graph Theory inspired by the problem that Carl presented. In our setup of the problem, we started off by showing the famous Königsberg's bridge problem to the students and asking them if there was a route that they could follow that solves the problem. After about ten minutes of attempting to find a route, we let them think about different alterations to the configuration — by getting rid of or adding bridges — in order to solve the same problem for about half an hour. For the remaining time, we had everyone show which alterations worked and which did not. We asked them if they could notice any pattern(s) that give a solvable version of the problem. Please refer to the Math Background to find out what pattern will give us routes that begin and end on the same island and that could be traversed in a single trip without doubling back.

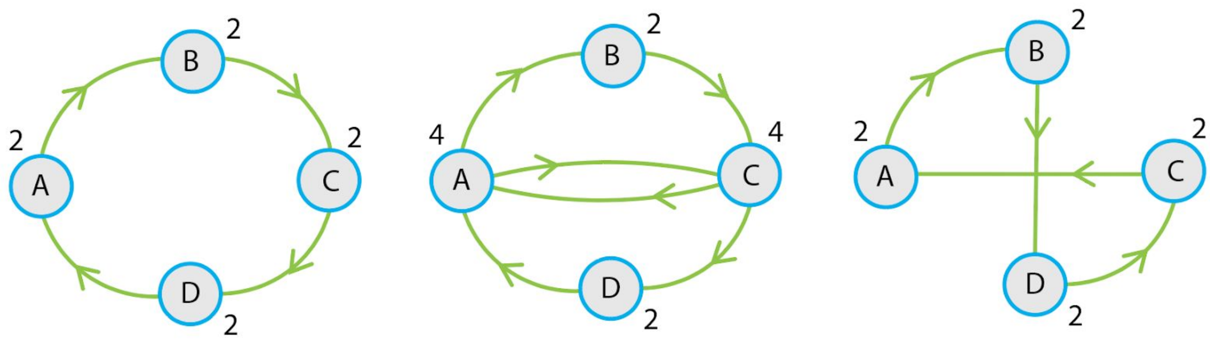
Math Background
The figure above is a way of visualizing the Königsberg bridge problem. Here the islands are drawn as circles and the bridges are drawn as lines or curves. This visualization is for ease of interpretability and how graphs are usually depicted in academic settings. However, if you or the students believe there are more fun ways of visualizing the islands and bridges, feel free to experiment!

The degree of a circle (known as a "node") is the number of lines (known as "edges") connected to the circle. In the figure above, the degree of each circle is the number noted next to it. Island A has degree five because there are five bridges connected to it. Island D has degree three because there are three bridges connected to it.

In 1741, Euler published the paper "Solutio problematis ad geometriam situs pertinentis," where he introduces the Königsberg bridge problem and proves that it is unsolvable. He goes on to generalize the problem by introducing the Eulerian path and circuit. In order to traverse through all the bridges only once such that the starting and ending locations are the same, every island (node) should have an even number of bridges (edges) connected to it because for every bridge taken to get to an island, there should be another bridge to leave that island. This path is called an Eulerian circuit.



It might be challenging for the students to notice the pattern that the degree of every circle must be even in order for it to be possible to traverse every bridge without doubling back, such that the starting and ending islands are the same. However, it might be easier to create variations of the Königsberg bridge problem and find routes on them. Below are some examples.



Our Experiences & Suggestions

- Understanding that the islands are nodes and bridges are edges might not come very easily to the students. Moreover, although it seems easier to find patterns in terms of nodes and edges (circles and lines), this might not be the case for the students. We recommend that the students decide on ways to visualize the islands and bridges. The remainder of the Circle can be led using the students' representations. This might mean not giving out printouts of Königsberg's bridges in the beginning of the Circle, which is fine. They could instead draw on scratch paper, as long as they have an understanding of what the initial Königsberg bridge problem looks like.
- We were pleasantly surprised when a student started visualizing and explaining routes by drawing arrows on the visuals. We always encourage introducing these "new" inventions to the rest of the students (crediting the inventor, of course).

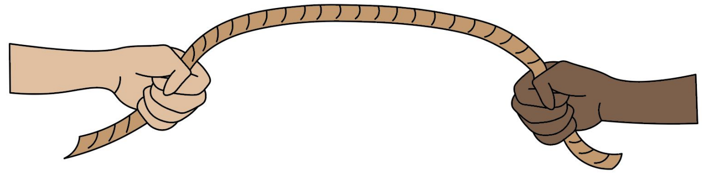
Taking It Further

Here are some questions that Circle leaders can explore further with the students. What happens if there is no need to start and end at the same location? Do all the nodes still have to have even degrees? If time permits, students can try to find a pattern in the case where the route does not have to begin and end at the same location. In the same paper as mentioned above, Euler introduces the concept of an Eulerian path — traversing through all the bridges only once without requiring that the route start and end on the same island. An Eulerian circuit is a particular kind of Eulerian path, where the starting and ending positions are the same. Additionally, in cases where the Eulerian path has different starting and ending nodes, there are exactly two nodes with odd degree.

Sources & Additional Resources

The paper that Euler published to introduce the Königsberg bridge problem can be found here. Here is a Ted-ed YouTube video that tells the history of the problem and the resolution that Euler discovered that started the field of Graph Theory.

Human Graphs



Keywords

Graph Theory, Weighted Graphs,
Shortest Paths

Materials

Strings of various whole-number lengths (in whatever units you choose; feet are good), labeled with their lengths (lengths ranging from one to six are good). There should be approximately two strings per student.

Problem Statement

This Circle is a free-form exploration of graphs, which are connected networks of people (explained further in the Math Background section). To start, pass a string to each student. Ask them to hold one end of the string, and have them give the other to somebody else. (One student may receive multiple strings, or none.) Then pick two students, A and B, and ask the question: “What is the shortest path from student A to student B, if you can only walk along the strings that connect people?” In other words, with the strings arranged how they are currently, how long is the shortest chain of strings from student A to student B?

Repeat this question for different A and B, with the same connections between students. Are there some people that all the paths seem to go through? What is the longest path between any two people in the network? It may also be worth dropping strings and reconnecting in a different way, with each student starting with a string of different length and finding different people to connect to.

The next layer of complexity is to give each student two strings to start with, and have them connect up to others as before. Pick another two students, and ask the same question: “What is the shortest path from student A to student B?” This question should be more difficult with twice as many strings in the network. The students may need to move around to make the network structure more clear. If the problem gets really complicated, propose trying to find a way to represent the problem on the whiteboard.

The ultimate question (at least, along this line of inquiry): “Given any network like the ones we just created, and any two people in that network, how can you be sure to find the shortest path between the two people?” If this problem begins to look insurmountable, a simpler — but still interesting — question is: “How might you do this if all the connections have the same length?”

Math Background

The networks that the students form with strings are representations of mathematical objects called “Graphs.” Graphs are collections of “nodes,” which are usually drawn as dots (or, in this case, represented by people). Some nodes are connected by “edges” (strings), and which nodes are connected to which other nodes defines a graph. For the purposes of this problem, we consider only *weighted* graphs, in which each edge has a *weight*. These weights indicate how “strong” or “long” each edge is. When traversing a graph — that is, walking along strings from one student to another — the length of the path you take is equal to the sum of the *weights* of the edges (lengths of the strings), not just the total number of edges you traverse. Thus a path between two students of several weight-one strings might be shorter than a single weight-ten string between two students.

The problem of finding the shortest path between two students, student A and student B, can be solved with various algorithms; most notable is Dijkstra's algorithm, which we will describe somewhat informally here. Throughout the algorithm, each student should keep track of the length of the shortest path that has been found to them from the starting student (student A). Call this length the student's "best guess distance." At the beginning, no paths have been found, so no students have a best guess distance.

The algorithm interacts with each student in turn, but the order matters. Start with student A, and then always go to the student with the smallest best guess distance next (except for ones who have been interacted with already; each student only goes once).

What does it mean to interact with a student? When the algorithm interacts with a student — call them student C — every student connected to that student updates their best guess distance. To clarify what this means, call a student connected to student C a "neighbor." If the neighbor's current best guess distance is shorter than student C's best guess distance plus the distance between student C and the neighbor, then the neighbor's best guess distance is still the shortest distance from student A to the neighbor that we know of.

However, if the distance through student C is shorter than the neighbor's current best guess distance, then we should update the neighbor's best guest distance to reflect the new information. If the neighbor does not yet have a best guest distance, they should take the distance through student C as their new best guest distance. Once this process has been repeated for each neighbor of student C, the algorithm is done interacting with student C, and moves on to the next student (the one with the smallest best guess distance out of those who have not yet been interacted with). The algorithm continues until it is done interacting with student B, the target student. Student B's best guess distance is then the length of the shortest path between student A and student B.

This algorithm is complex and can be difficult to understand intuitively. It is likely too much to expect that the students will come up with it on their own, and not much use to them if told to them outright. However, with some understanding of an algorithm that works consistently, the Circle leader may be better able to guide the discussion in more productive directions.

For a somewhat simpler discussion, a similar problem can be explored on *unweighted* graphs — that is, graphs where all the strings have length one. On such a graph, the problem can be solved by an algorithm called Breadth-First Search. For this algorithm, start with student A, and give them the number "zero." Take each of their neighbors, and give them each the number "one." Give a "two" to every neighbor of a student with a one who does not already have a number. Continue this process, increasing the number students are given, until student B is reached. The number student B has at the end of the algorithm is the distance between student A and student B.

Our Experiences & Suggestions

- Think through ways to make the standing activity engaging and fun. If it proves not to be, we suggest moving to the whiteboard quickly. When we ran this Circle, students were a little bit unfocused and confused by what was going on during the standing activity. We suggest making sure that as many students as possible are engaged with finding the shortest paths between people, and that groups of students prone to chatting are separated when doing the standing activity. Do not be afraid to move to the whiteboard earlier than expected; standing around waiting for one or two other students to figure everything out is not fun, and beyond setting up the problem, the standing activity provides little value if it is not fun.
- We suggest coming up with a convention for drawing graphs together at the whiteboard as a whole class, rather than individually or in small groups. Students were confused by what we were asking when we had them do it in small groups.

Taking It Further

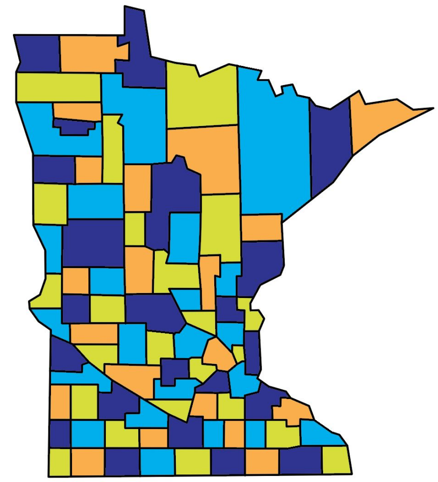
Another question, along similar lines to the shortest path question, is: ”If I choose a node in the graph, can you tell me the shortest distance from that node to every other node in the graph?” This can be accomplished with a modified version of Dijkstra’s algorithm (explained in Math Background), where the algorithm does not stop until every node has been visited.

Ask students to come up with other questions relating to graphs. Coloring nodes is one avenue. Specifically, how can you assign the people in a certain graph to teams (colors) so that no two people on the same team are connected? What is the smallest number of teams you can do this with? Does this number depend on the structure of the graph?

Sources & Additional Resources

For more Circles relating to Graphs and similar concepts, see the other Circles in this section (Graphs and Trees). For more on Dijkstra’s algorithm, Wikipedia has a detailed technical explanation, as well as one for Breadth-First Search. For more on graphs in general, see the article on graph theory, and for more on coloring graphs, see the article on graph coloring. There is also a Wikipedia article on the Shortest Path Problem.

Map Coloring



Keywords
Graph theory

Materials
At least three copies of maps of any country, in our case the United States, per student and at least ten unique color markers or pencils for students to share.

Problem Statement
In this Circle students will be placed in the shoes of nineteenth-century mapmakers who are tasked with coloring a map of a country so that no two adjacent states have the same color. This task would be trivial if we had unlimited colors, so the students are challenged to use as few colors as possible while also satisfying the constraint of no two adjacent states having the same color. To propose the problem, we would recommend a statement similar to the one below to set the scene:

> "Today we will put ourselves into the shoes of nineteenth century mapmakers. As a mapmaker, your friend comes by and hands you a map of the United States [or other country] and asks you to color it so no two neighboring states have the same color. You, being a smart mapmaker, know that it is hard to come by colors so you try to use a few colors as possible. How many colors do you need to get the job done?"

The statement above should give students a good jumping off point for the problem. Below will be a list of properties of the problem that should be written on the board.

1) All states must be colored.
2) No two neighboring states can share the same color.
3) Use as few colors as possible.

Math Background
Hidden underneath this mapmaker story is a complex graph theory problem that has a history that spans over a hundred years. For our case, the problem was to color a particular map with the fewest colors possible, but in actuality mathematicians were trying to figure out the fewest number of colors to color all maps following the constraints above. Additionally, instead of maps, mathematicians were using graphs, but in this section familiarity with graphs is not necessary.

Mathematicians have hypothesized that the minimum number of colors necessary to color all maps is four, so the goal now shifts to proving that the minimum number of colors necessary to color all maps with these constraints is four, better known as the **four color theorem**.

Unfortunately, the four color theorem would be too advanced to prove here, but we will cite some resources below if you wanted to learn more about it. To speak more about the history, the problem was first thought of in the nineteenth century as the statement suggests by Francis Guthrie working with a map of England. Despite noticing that the map only required four colors to satisfy the problem, it took mathematicians and computer scientists a hundred years later in the 1960s to prove it using brute force.

Those that proved the four color theorem constructed a comprehensive list of maps and used computers to prove every single case. By coloring every single map in this comprehensive list, they showed that all maps were four color-able. It was also a moment in mathematics and computer science history because it was the first computer-assisted proof which brought significant criticism due to the proof not illuminating anything about the underlying mathematical structure.

In the meantime, there were numerous other proofs that claimed to have shown the four color theorem but they have all been disproven.

Like we said previously, the four color theorem, and even the five color theorem, which claims that all maps are five color-able, is beyond the scope of the lesson plan. What we wanted students to get out of this Circle are some of the logical techniques they can use to satisfy the properties of the problem. Below will be resources on more information and an undergraduate-level proof for the five color theorem which is less involved.

Our Experiences & Suggestions

Note: this problem was not conducted in person due to the COVID-19 pandemic. Instead, we sent students a video summarizing the problem and let them go about it on their own, unsupervised.

Here are some areas that might cause issues when running this Circle in person:

- Please be sure that the students understand the constraints thoroughly, specifically the idea that two neighboring states cannot have the same color. This idea of *any* two pairs of states holding this property may be confusing.
- Students may be frustrated by the fact that their early colorings of states will impact their ability to color later states. Encourage the students to either continue and just add another color or backtrack and try a different arrangement of colors to maintain their current number of colors.

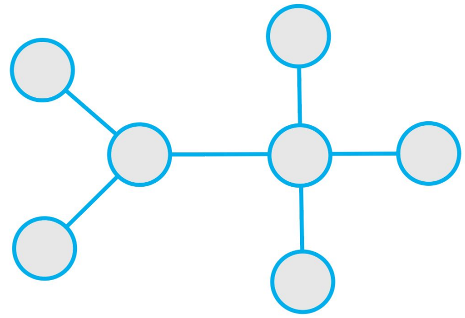
Taking It Further

- Have students draw different maps that require as many colors as possible to color them correctly, i.e. can you draw a map that requires a lot of colors?
- Does this problem change when you have to color islands that are connected by bridges? What if the bridges can cross between each other? This may lead students more naturally to graph notation, but introduce the idea of planar and non-planar graphs defined in additional resources.

Sources & Additional Resources

The Wikipedia article on the four color theorem is a good resource to learn more about the history of the problem and its respective proof. Additionally, if you want an undergraduate-level proof by contradiction of the five color theorem, a proof can be found here. We also mentioned graphs significantly in this lesson plan. Here is another resource that converts the problem from maps to using graphs. If you wanted students to color different maps online, then this tool and also be useful for online/distance learning. Finally, we briefly mention the concept of planar graphs in the last section. To learn more about planar graphs, refer to the Wikipedia page that covers it thoroughly.

Trees: Dots and Lines



Keywords
Graph Theory, Nodes, Edges, Degrees

Materials
Pencil and scratch papers for every student

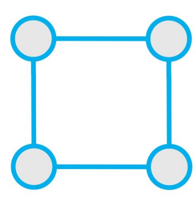
Problem Statement
The main goal for this Circle is to allow students to draw different types of tree graphs. As you continue the activity, you will be introducing more "terms" so that they could draw more complicated graphs with more restrictions.

This Circle was inspired by a 1997 movie called, *Good Will Hunting*. In particular, it is inspired from a scene in the movie where the protagonist draws a bunch of trees on a board outside a classroom.

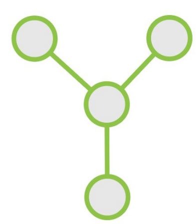
For about seven to ten minutes, students can play around and draw a bunch of trees with different numbers of dots. We started with drawing trees with two dots, then three, four, five, and six. For the next five minutes, they can choose a tree with a larger number of dots and count the number of lines that each dot is connected to. Let us call this number the "branches" of a dot. Then for about ten minutes, they can observe trees that form cycles and those that do not. For another ten minutes, they can check how two seemingly different trees might in fact be the same if one rotates the paper or tilts the edges of one of the trees. For the remainder of the time, students can try to draw trees with six dots of particular feature: 1) no two trees are the same and 2) no dots have two branches (that is, the number of branches of any dots in any trees can be any number except two).

For the remainder of the time, allow students to draw trees with six dots where two trees are not the same and no dots have two branches.

Math Background
There is no math background needed in the beginning when the students are asked to draw patterns using dots and lines. In graph theory, there are terms such as nodes, edges, degrees, and isomorphic. For the purpose of this lesson, we have replaced these terms with dots, lines, branches, and "same" respectively. A tree is any figure where dots are connected with lines such that there is no cycle. A tree has a cycle if you can trace a path from a dot across the lines and come back to the starting dot. Figure on the right shows what is a cycle and what is not a cycle.



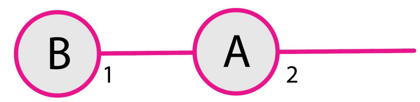
A Cycle



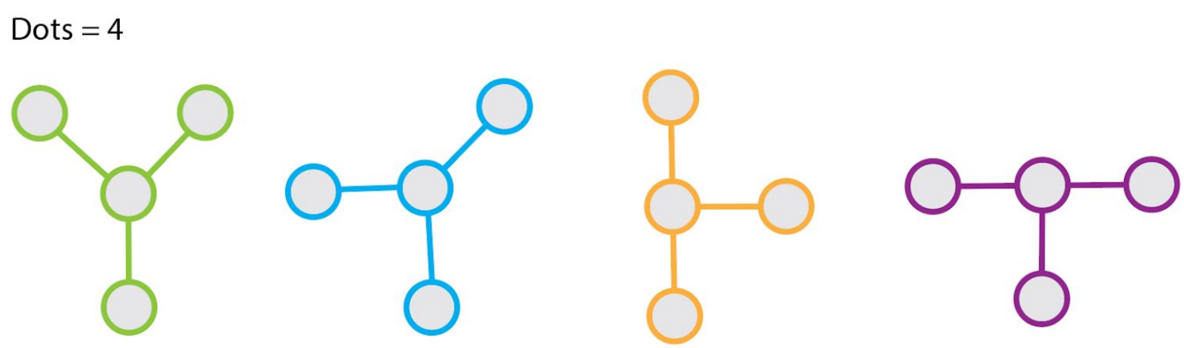
Not a Cycle

As mentioned in the problem statement, the number of lines connected to a dot is how many "branches" that dot has.

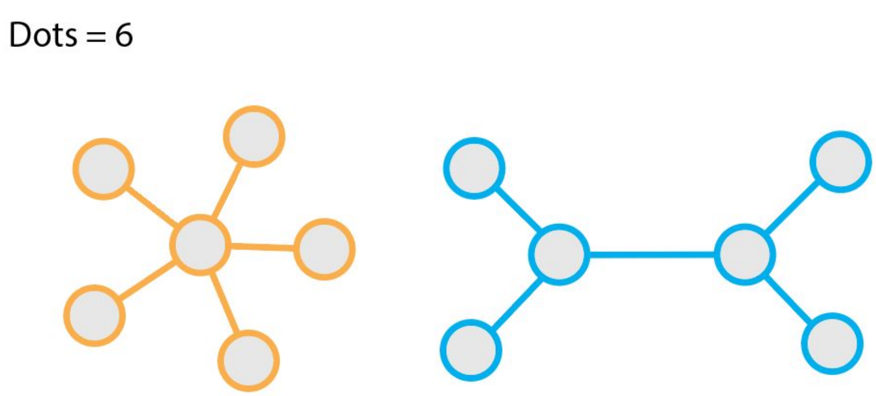
The figure on the right shows that dot A has one branch because there is one line connected to it and dot B has two branches because there are two lines connected to it.



The same trees can be drawn in many different ways. In the figure below, we have shown how trees with four dots can be drawn such that they seem different but are in fact just alterations of each other, and thus they are all the "same". Notice that the left-most dot on the first tree can be brought down a little and it will give us the second tree. Similarly, if the right-most dot on the second tree is brought down, you will get the third tree. If you rotate your paper anti clockwise by 90 degrees, you will get the forth tree. Therefore, these seemingly different trees are actually the same. It is important that the changes made to each tree are such that two adjacent dots remain adjacent even after the alterations are made.



With these extra labels or terms, students will be able to draw more advanced trees. Below is the solution for the final task with six dots.



Our Experiences & Suggestions

Note: this problem was not conducted in person due to the COVID-19 pandemic. Instead, we sent students a video summarizing the problem and let them go about it on their own, unsupervised.

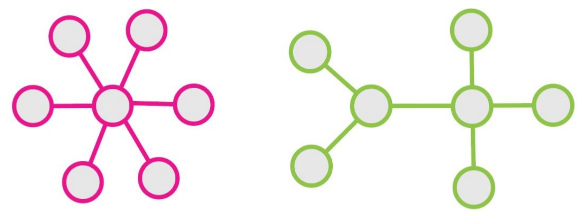
Even though we were not able to run this Circle in person, here are some suggestions:

- As always, we encourage the students to come up with their own terms, especially for this Circle where there are many terms. They can call a node “dot,” “circle,” or “round shape,” and an edge can be “line” or “stick,” etc.
- Let the students explore their creative sides. There’s no rush or need to get to the final task at all. Take your time with this Circle!

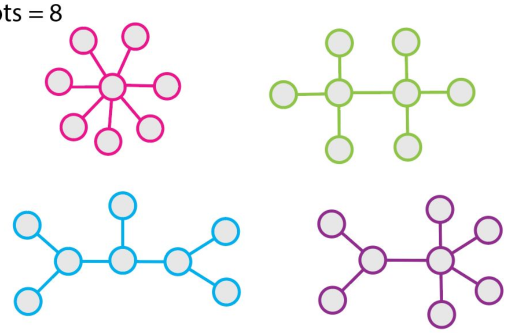
Taking It Further

If time permits, students can try the final task on seven, eight, nine, and ten dots. To reiterate, the final task was to draw trees with a certain number of dots such that no two trees are the “same” and no dots in any of the trees have two branches (i.e. the number of branches can be one, three, four, five, and so on, but not two). In the aforementioned scene from Good Will Hunting, the protagonist draws ten such trees when asked to use ten dots.

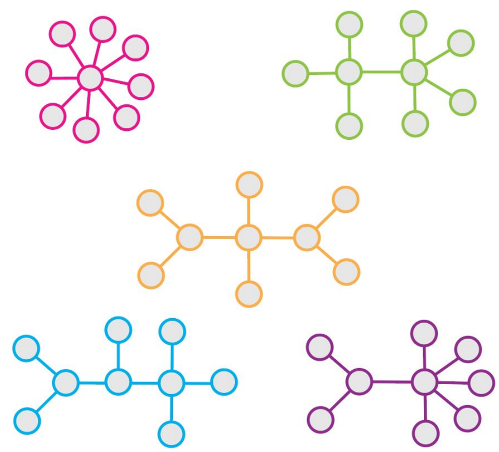
Dots = 7

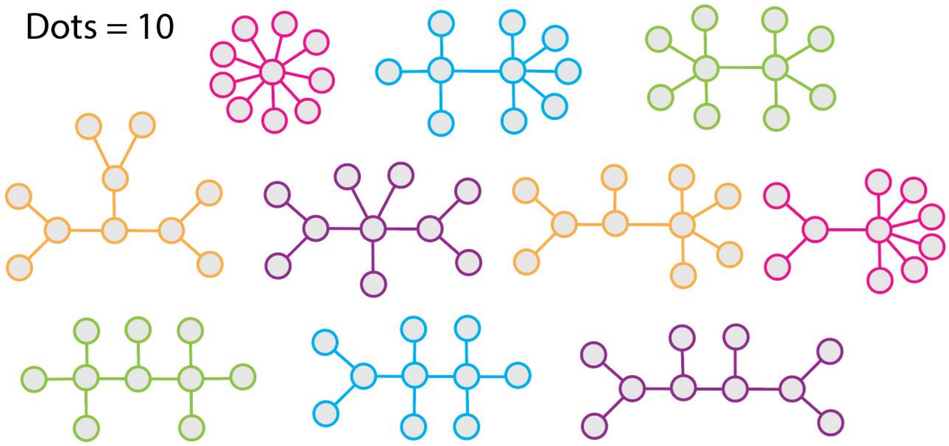


Dots = 8



Dots = 9

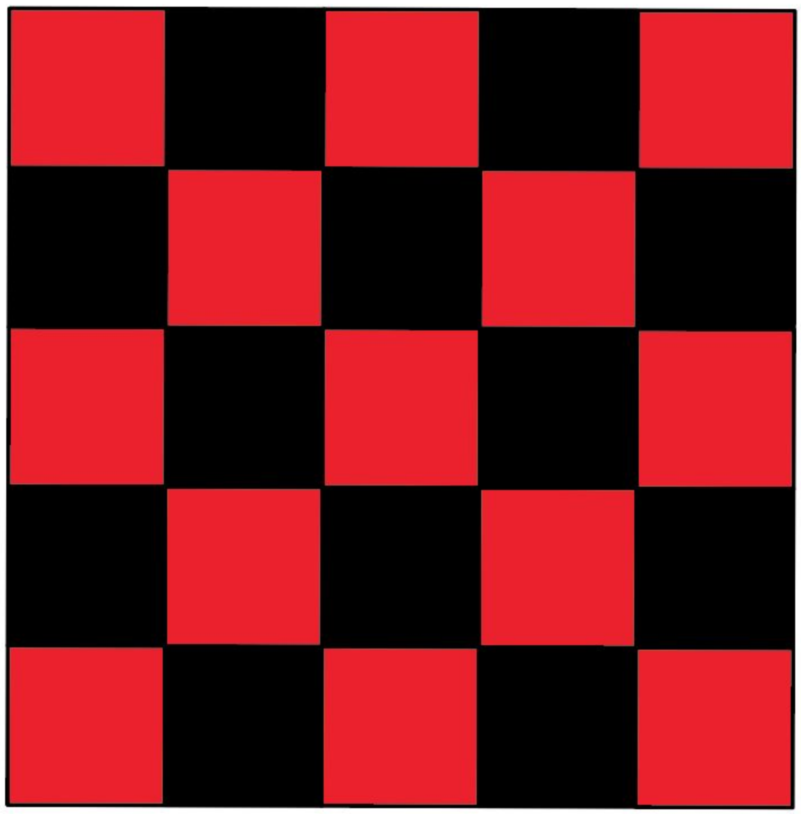




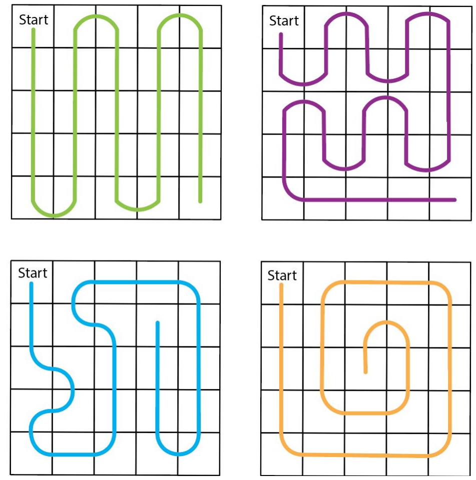
Sources & Additional Resources

Here is a paper about these graphs, with all the solutions on pages 161-162.

Boards and Spatial Reasoning



Filling Grids

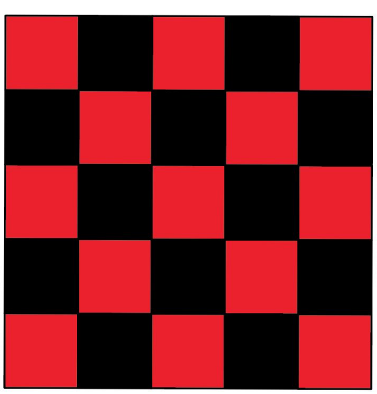


Keywords
Even and Odd, Proving Impossibility, Surfacing Underlying Structure

Materials
A whiteboard or chalkboard will suffice.

Problem Statement
Draw a five-by-five grid of squares on the board. The initial question is this: “Can you draw a single line on this board that goes through every square on the board, but doesn’t go through any square more than once? The one rule is that at each step the line can only go up, down, left, or right — no diagonals.”
This, as the students will quickly see, is pretty easy. However, there are other more difficult questions waiting behind this one, once the students have drawn a few such lines. One is: “How many such lines can you draw?” However, we recommend going down this path: Ask the students to draw one of these lines starting in a square immediately next to a corner square. Is it possible? If not, why not? If not, what other squares are impossible?

Math Background
Counting the number of paths on a five-by-five board is difficult, and we will not give a full explanation here of how to do so. The general strategy would be to exploit symmetry as much as possible. Finding one path can give you many other paths “for free,” either by swapping starting and ending points, flipping the path vertically or horizontally, or rotating the path. From there, the trick is to find methods to count that are systematic and error-proof; one must check at every step that one is not missing any possibilities, and also that one is not double-counting a path that was already counted. On some boards there are many squares from which there are no valid paths, as will be discussed below, which makes the task of counting all paths somewhat easier. If you do pursue the counting-paths route in your Circle, starting with smaller boards (such as three-by-three or even two-by-two) is recommended.



However, the question of starting from the square next to the corner is, we feel, more interesting and accessible. It is impossible to draw a legal path starting from that square. Why? First, let us note that it is not just that square; if you were to color the board like a checkers board, with alternating red and black squares and red squares in the corners, then no path starting on any black square would work. In order to see why, let us think in terms of the checkerboard. If you consider the sequence of squares that a given path visits, note that it must always alternate red, black, red, black, and so on, because the path must always go between adjacent squares, and adjacent squares are always different colors.

Now, notice that since the board is five-by-five, there are 25 squares on the board, which is an odd number; that means there must be either more red squares or more black squares on the board (because there cannot be the same number — that would be an even number of squares total). There are, in fact, more red squares, because red squares occupy the corners. Additionally, note that any path on this board must start and end on the same color. To see why, number each square you visit along the path, starting with one and ending with 25. Note that odd numbers are all the same color, and even numbers are all the same color. Since one and 25 are both odd, they are the same color. Not only this, but there is one more square of the starting color than there is of the other color, because there are thirteen odd numbers between one and 25 and only twelve even numbers.

This is the crux of the problem. If you start on a red square, there is no problem with having thirteen red squares and twelve black squares along your path; those are exactly the squares that are on the board. However, if you try to start on a black square, then your path will need to include thirteen black squares and twelve red squares. Unfortunately, that is impossible on this board! And that is why you cannot have a path that starts on one of the squares on the five-by-five grid that corresponds to a black square on our conceptual checkerboard.

Our Experiences & Suggestions

- When drawing the initial grid, drawing attention to the fact that six lines are needed to draw five rows of empty space (perhaps by playing dumb and thinking that only five are needed) is a subtle way to get students to start thinking about even and odd, which will be useful later in the Circle.
- Let the students come up and draw different paths on the board. It is a lot of fun, and gets them engaged. Ideally, this free-form path-drawing would segue smoothly into more difficult questions, but if it does not happen naturally, don't spend forever drawing different paths. At some point, there is no harm in asking: "What if you start in this square?"
- Similarly, do not let students struggle forever trying to draw paths that are impossible. We found that students kept trying over and over, and did not give up or think that it might be impossible without prompting.
- Trying smaller square boards of both even and odd side lengths is a great way to get a handle on the problem. So is trying different starting squares, aside from just the corners, center, and next-to-corners, and seeing which do and do not work.
- It can be good to "play dumb" early on in this problem. As the first few paths are found — three, say — try saying: "ah, so there are only three paths that work!" This will outrage the students, getting them involved, and putting them in charge.

Taking It Further

There are a few different ways this Circle could be expanded. One interesting route is discussions of symmetry (both reflections and rotations); this would come up especially in the context of trying to count the number of paths, but also might come up when figuring out which squares are impossible to start from (students may notice that there are several squares that are in some sense equivalent to a given square next to a corner square).

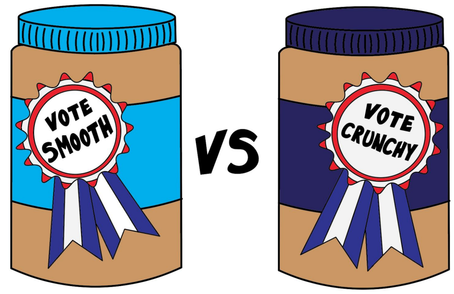
A lot of time can also be spent classifying different paths, and seeing how many different fundamental types of paths can be made (spirals, zig-zags, and so on). While doing something like this, ask questions that guide the students towards being careful about what they are doing: "How do you know you didn't draw a path like that before?" "Are you sure these two paths are fundamentally different?" "What do these two paths have in common?" "Are there other paths like these?"

Another great direction to explore is different sizes and shapes of board. What about a board with an even number of squares on each side? What about smaller boards, larger boards? What about rectangular boards? Or, even further: what if it were a hexagonal or triangular grid, instead of a square grid? What about three dimensions?

Sources & Additional Resources

This Circle was adapted from a demonstration video by Robert Kaplan, which was featured on the homepage of the Global Math Circle website as of April 8, 2020. As of that date, the video was also on YouTube, and was titled "Dr. Robert Kaplan: "Mathematics: Learning to Speak our Lost Native Language" | Talks at Google." Watching Kaplan work is also a great way to get general ideas about how to run Math Circles well.

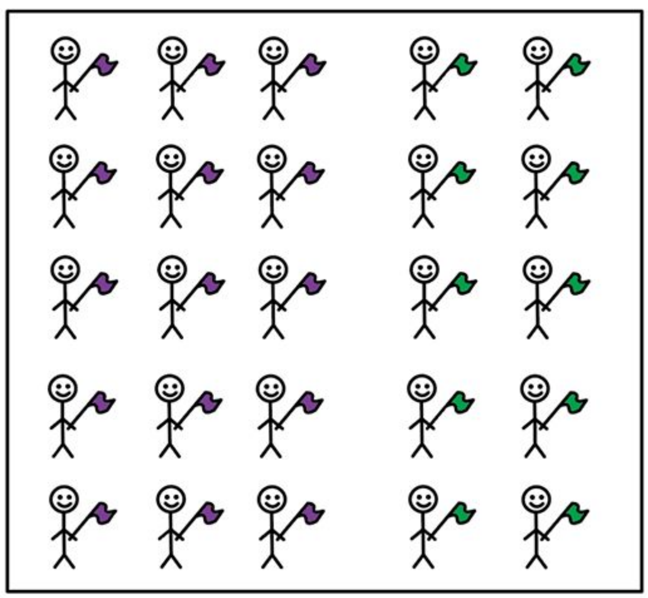
Gerrymandering Marcellusville



Keywords
Voting Theory, Social Choice Theory

Materials
A single worksheet of Marcellusville for each student.
Extra copies are encouraged, but not necessary.

Problem Statement
Students will be in control of voting districts in a fictional city called Marcellusville, named after the inventor of peanut butter, Marcellus Gilmore Edson. By drawing district lines, students will notice that the results of the election change based on the lines they have drawn. Their task is to determine how to draw "fair" voting districts to produce a result representative of the population in this fictional city (or determine that fair districts do not exist).



Pass out the worksheet (attached above) displaying Marcellusville and draw it on a whiteboard if available. Start the lesson with the statement below or a statement akin to it.

> "Welcome to Marcellusville! Named after Marcellus Gilmore Edson, who invented peanut butter, which all of our citizens are grateful for. In Marcellusville, we have a factory that staffs five workers. Every two years, the people of Marcellusville vote on who gets to work in the factory. Everyone in Marcellusville has a strong view on whether crunchy or smooth peanut butter is better. People who are crunchy voters are purple and smooth voters

> are green. Our job is to draw five voting districts that elect the factory members. Whoever has the most votes in a district gets to send a worker to the factory to make their type of peanut butter."

Consider drawing an example of voting lines on the board and determining the winner. That will allow them to see how the system works, if it is unfamiliar. As the statement above describes, there will be five voting districts, and whoever wins the majority gets to send their type of worker to the factory.

Math Background

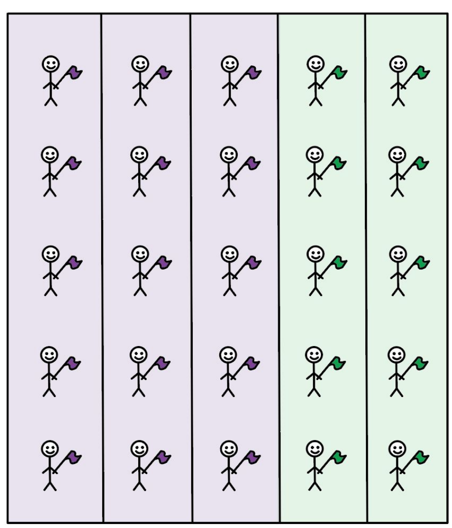
The types of questions the students should consider or discuss once released on this assignment of drawing voting districts are:

1) What types of voting districts can be drawn with these voters?
2) What types of results would be produced from those districts?
3) What should you do in case of a tie?
4) Are those voting districts fair?
5) What does it mean for a voting district to be fair?
6) What rules would you make to avoid unfair voting districts?

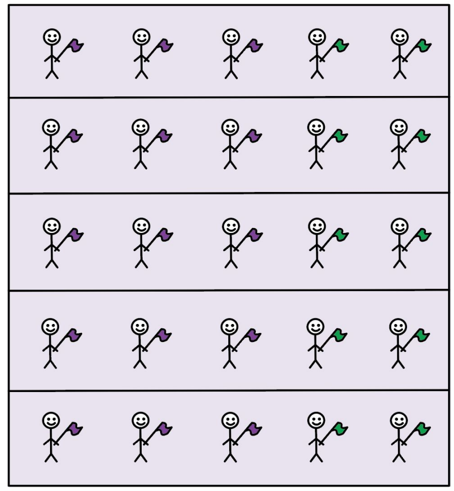
Before considering those questions, first we need to know the makeup of Marcellusville. We can see that Marcellusville has 25 citizens, fifteen of whom are crunchy voters and ten of whom are smooth voters. In percentages, 60% of them are crunchy voters and 40% are smooth voters.

Now, *what types of voting districts can be drawn? And what types of results are produced from those districts?*

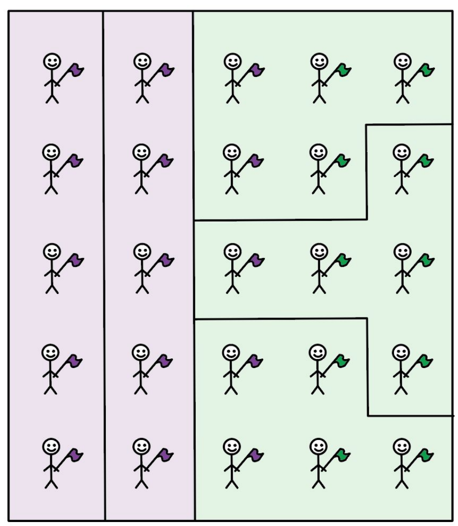
One potential drawing results in a **perfect representation**. For perfect representation, the voting districts would be columns. Crunchy voters would win three of them, and the rest of the districts would be won by smooth voters. The makeup of the factory would then match perfectly with the makeup of Marcellusville because there are three crunch factory workers and two smooth factory workers.



Another result is **majority rule**. For majority rule, the rows would be the voting districts and all districts would be won by crunchy voters. The factory will now consist of all crunchy factory workers.



One that is more tricky to come up with is **minority rule**. For minority rule, students may have to use the *packing and cracking strategy* which involves packing as many crunchy voters into a district and have the remaining districts be won with narrow majorities for the smooth voters. An example of this is below.



Regarding *ties*, we would encourage the students to figure out what would be fair. There is no right answer. Some answers may be that the majority party (crunchy voters in this case) get those districts, some may say that the minority party gets those, or perhaps even a coin flip to determine the result. Tie results would only occur when voting districts are not all the same size, so have the students consider whether different-sized voting districts are fair as well.

From the districts above, we should now consider which of them are “fair”. It is easy to disregard minority rule as unfair, but should we strive for perfect representation or majority rule? Do not use these terms, but refer to the drawn lines. Because fairness is subjective, have the students support their answers as to why a certain drawing may be fair.

Next, *what rules can we make to avoid unfair voting districts?*

There are three rules that are usually agreed upon to hinder attempts of organizing voting districts to the advantage of a specific party, i.e. gerrymandering. Those are:

1) All voting districts should have equal populations, or at least close to equal populations.
2) All voting districts should be contiguous, i.e. the voting district needs to be connected. That eliminates the potential of islands of voters being in the same district despite not being connected.
3) All voting districts must be compact to avoid distant citizens influencing the results of other citizens.

There are exceptions to these rules, but these are good guidelines, especially the equal size district rule.

Our Experiences & Suggestions

- Some of our students had difficulty developing rules to make fair voting districts. For example, the idea that equal size voting districts were fair compared to different sized ones. It could help to ask the student which district they would rather be in if they lived in Marcellusville and ask why that is the case. That should hopefully make them realize that there is an imbalance in voter significance.
- We were working with sixth graders, so it was not intuitive to them that drawing five voting districts in Marcellusville meant drawing four lines in total to separate them into five groups. It would help the students to do an example on the board of a trivial case or to draw five circles around the groups of voters

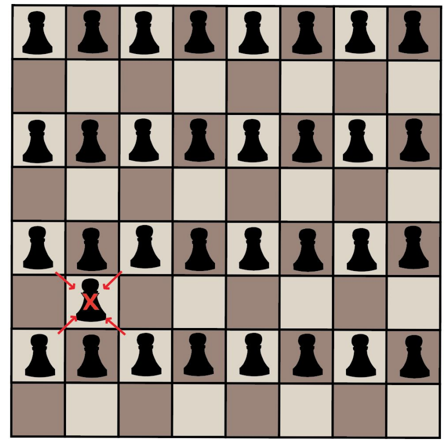
Taking It Further

- If you were a strong supporter of smooth peanut butter, how can you draw the lines to get as many smooth peanut butter factory members in the factory?
- Can you get more smooth peanut butter factory members than crunchy?
- What happens if we change the number of crunchy or smooth peanut butter voters there are?
- What happens if we change where they live in Marcellusville?
- Can you draw your own state and the voters that live in that state?
- What do the results look like in your personal state and how would you draw the voting district lines?
- Is it possible to always have fair voting districts?
- What happens if we have more voting districts and in turn more factory members?

Sources & Additional Resources

To compare this lesson plan to an undergraduate-level lesson plan, please refer to Duke University who has published a more indepth outline into the mathematics of gerrymandering. Also to satisfy your curiosity of how voting districts matter in real life, we have linked Minnesota's congressional districts that are redrawn after the census every ten years. To take a look at the rules that the federal government and different state governments implement to avoid gerrymandering, refer to the next citation.

Pieces on a Chessboard



Keywords
Logic, Game Theory

Materials
One chessboard and set of pieces for every three to four students.

Problem Statement
Before beginning the Circle, make sure that students understand the way pieces in chess capture others. How many pawns can you place on the board without capturing another no matter the color? What about rooks?

Depending on the group and the time constraint, you may choose to do the rest of the pieces, but we chose to let them try the first two before moving on to the next prompt. This was based on both the rowdiness of the group and the short time we have with the students. Once the students were familiar with the way each piece was captured, we asked them to start placing random pieces on the board to see how many they could fit. It may take them time to keep track of all the different pieces on the board and their capture rules; let them work out the kinks rather than correct their mistakes.

If time permits, move on to the "Knights on a Chessboard" puzzle. In the game, there are two players and one knight piece that is moved around the board. The knight can only move the way a normal knight moves and no spot that has been visited once can be visited again. The players alternate turns moving the piece around the board and the player who can no longer move to a previously unvisited square loses. To keep track of the places the knight had already visited, we used other chess pieces to mark the squares. Ripped pieces of paper may work just fine as well. Let the students play together against the Circle leader for a couple of rounds. If the students play first and the Circle leader plays optimally, then the Circle leader will always win. The video in the suggested resources does a good job explaining how to play optimally. After a couple of rounds, let the students play each other and have them share their strategies for winning.

Math Background
These are the ways different pieces in chess can capture other pieces:

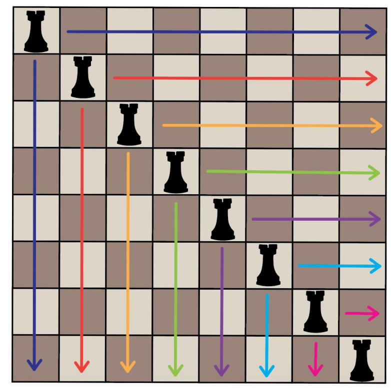
- Pawn: one square diagonally (in any four directions, for the purpose of this puzzle)
- Rook: any number of squares above, below, left, or right
- Knight: can capture like they move, in an "L" shape. That is to say, one square above, below, left, or right and then two squares above, below, left, or right such that an "L" is formed. The knight can also move three squares first and then the single square.
- Bishop: any number of squares diagonally
- King: one square above, below, left, right, or diagonally
- Queen: any number of squares above, below, left, right, or diagonally

Here are the optimal methods of placing them on a chessboard such that no two pieces or more capture each other:

Pawn
Since pawns cannot capture pieces on either side of them, it is safe to fill an entire row without any issues. Like in the example pictured above, if a piece is placed underneath a filled row, then it can be captured by two pieces above it. Thus we must fill every other row to maximize the number of pawns placed without capture. The example image for the lesson plan, with the exception of the pawn x-ed out, shows a solution to this question.

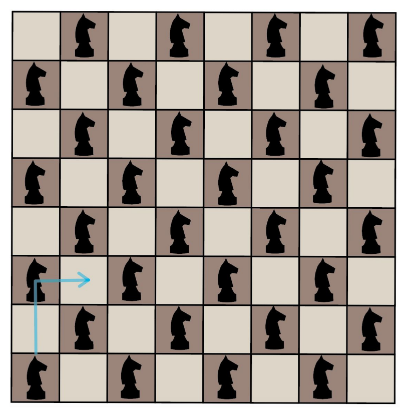
- *Answer:* 32 pieces

Rook
Since rooks can move any number of squares in their column or row, each rook must have their own column and row so that they capture no other piece. Thus, to maximize the number of pieces on the board, the number of column/row pairs must be maximized. The squares on the diagonal of the board correspond to the set of maximized column/row pairs. It is on these squares that the rooks can be placed, as shown here. You can similarly place the rooks on the other diagonal or in many other arrangements that have one rook per column/row.



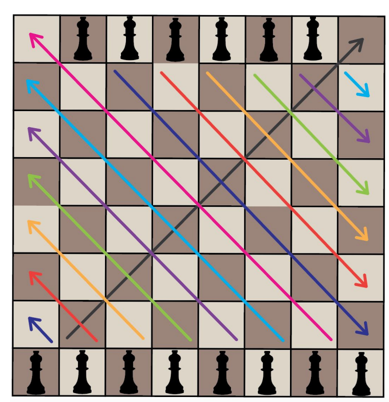
- *Answer:* eight pieces

Knight
To understand the way knights are maximized, it helps to consider how they move and the colors of the squares they move to and from. If the knight starts on a black square, then it can move two squares in one direction moving from black to white to black. It then moves one non-diagonal square, thus changing colors once more to white. No matter which "L" is formed, if the knight starts on a black square, it will always end on a white square. The opposite pattern holds true for a knight that starts on a white square: if it starts on a white square, then it will end on a black square. Therefore, if we place knights on squares of the same color, they could never capture each other. An example board is shown here. We recommend you try moving the knights to see how the pattern described works (the blue arrow is our first example).



- *Answer:* 32 pieces

Bishop
Bishops face a similar problem as rooks, but rather than capturing along columns and rows, they capture pieces that lie any number of squares on their diagonal. To maximize the number of bishops on the board, we must maximize the number of non-overlapping diagonals. At first glance, it may seem as though there are fifteen non-overlapping diagonals,

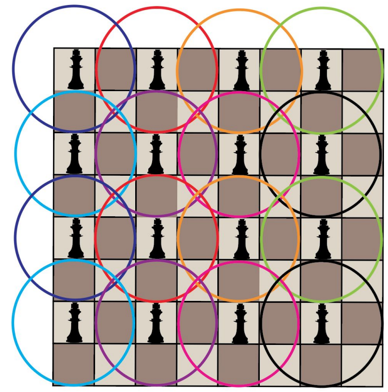


but the two consisting of single corner squares are connected by the center diagonal, and thus only one can count. The figure shows this diagonal with a black arrow that crosses all the others perpendicularly. The figure also shows the other bishops and their corresponding diagonals for the solution.

- *Answer:* fourteen pieces

<u>King</u>

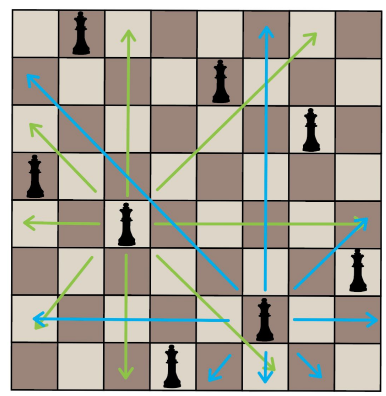
A king can capture any piece within one square of it in any direction. Imagine the king has a "personal space" with an area of nine squares. To maximize the number of kings, the total area of personal spaces around each included king must be minimized. The best way to do this is to overlap the "personal space" of each king just enough so that they cannot capture each other. In other words, the "personal spaces" must overlap by exactly one square. The figure illustrates one optimal way to minimize the area of all the "personal spaces." Each color represents the "personal space" for the king that is centered within. Personal spaces are represented as circles in the image.



- *Answer:* sixteen pieces

<u>Queen</u>

There are many different ways for queens to capture other pieces, which can make it difficult to find solutions. Since they can capture any number of squares along diagonals, columns, and rows, there must be only one queen on each. There are fourteen non-overlapping diagonals, which we solved for in the section regarding bishops. There are eight unique rows and columns. Thus we are limited by the number of columns and rows and so we can only place at most eight queens on the board. The next step is to find a way to place them all in a way that does not cross diagonals. One example is shown here. Only two queen capture paths were included to make the image more clear.



- *Answer:* eight pieces

<u>Knights on a Chessboard</u>

To understand the solution to the version of Knights on a Chessboard used in this Circle, we highly recommend the video linked in the section *Sources & Additional Resources*. They do a much better job at explaining with a video than we could do with only words.

Our Experiences & Suggestions

- Many students found the pieces distracting and would play with them rather than try the puzzle. We chose to hold on to the pieces so that they would ask us for only what they needed. This would also force students to think more precisely about what pieces they wanted to experiment with and focused their strategy rather than feeling overwhelmed with options.

- We had over ten students during this activity and only two chessboards. It was difficult for everyone to share their ideas and from the beginning one student on each board became the soul voice of their respective group. A couple of students, to avoid having to share the boards, drew their own. Another possible solution could be to hand the pieces to different students rather than always the same student.
- Many students did not know the rules of chess. At the start of the Circle we asked the students with prior knowledge to teach the other students the rules and help them practice so that they could all enter the problem at the same level of understanding.
- Our students enjoyed placing as many pieces as they could at first, but after two types of pieces they were ready to move on to something more challenging. Be prepared to have another puzzle ready for them to tackle whether that is our example of Knights on a Chessboard or a completely different challenge.
- Because of their range of capture, it can be difficult to see that eight queens can be placed on the board. We never tried this example, but we believe it may take them a long time to solve. Of course, we always encourage them to explore on their own first, but if they want help, letting them know only eight pieces can be placed at a time can serve as a hint so they can find where and how they can fit them.
- When explaining "Knights on a Chessboard," playing a few rounds with the students will help them better understand how the game works. Because it is a two player game, we had the students team up to decide where to place the knight against the leader. Our students were very active during this Circle, so this helped bring them together and keep their focus on the task at hand. We find that any competition where all the students are working together to defeat the Circle leader often results in better focus.
- We ran out of time during this Circle and so we were unable to finish the Knights on Chessboard game. Thus we never discussed the strategies that students devised to win the game and they never showed us how they tackled the problem. We recommend quickly judging how interested students are in maximizing individual piece placing on the chessboard. If they seem to enjoy solving these puzzles, then this should be the focus of the Circle. If they are less interested, then use these puzzles as a way to get the students familiar with the logic of placing pieces on a chessboard, but move on to playing Knights on a Chessboard early on.

Taking It Further

- There are many different ways to make a puzzle out of knights on a chessboard. Here are some that exist and others that are made up:
  - One problem is more commonly known as Knight's Tour, and the math behind it may be trickier than our Knights on a Chessboard; however, the puzzle is definitely an interesting one and is a fun challenge for your students to try. This video captures the problem pretty well.
  - Place pieces on the board in any random squares, such that they cannot capture each other. Now place a knight on the board. How many squares can the knight travel to before it lands somewhere it could get captured?
  - Place a knight on the chessboard and select a final destination for the knight. Can the knight make it to its destination? What is the minimum number of moves it can make?

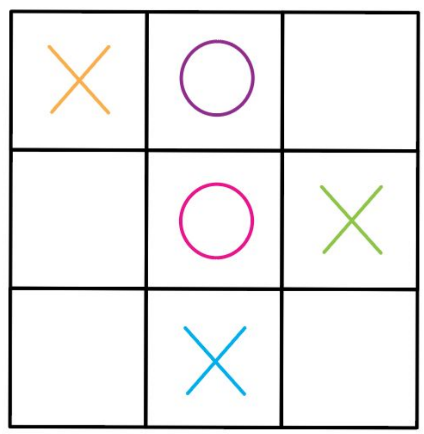
- Ask the students to place as many different kinds of pieces on the board as they can without any capturing the other. What pieces are the most limiting? What pieces are the best?
- What if you change the shape of the board? Make it a smaller square, make it a heart, make it a random collection of squares. How does that change the way you place pieces if they cannot capture across "holes"?

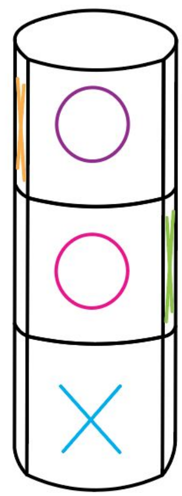
Sources & Additional Resources

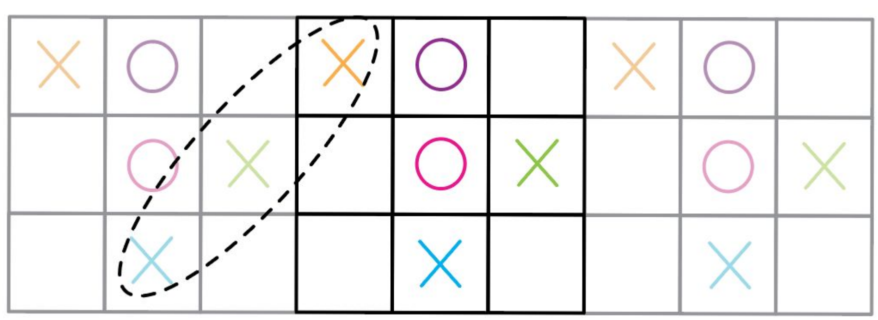
This link has all solutions to the question "How many queens can fit on a chessboard?" Additionally, there is a fun interactive application so you can try to solve it yourself!

We found this video very helpful for understanding the solution to our version of Knights on a Chessboard.

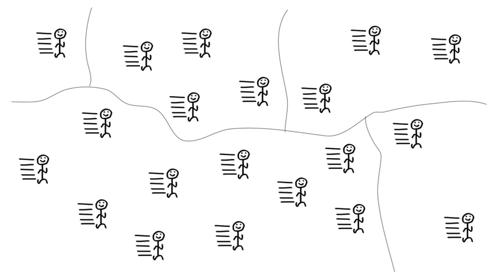
Games and Strategy







Game Strategies (Rock/Paper/Scissors)



Keywords
Case Analysis, Game Theory, Logic, Optimization, Probability

Materials
Pencil and Paper for each student

Problem Statement
There are four parts to this Circle, and each one of them is short enough to be done in under ten minutes, so you could cover one of these topics when you finish your planned Circle early for whatever reason, or you could cover them in one session. The games are the following:

Vote for the Land
This game is based off of the idea of gerrymandering. We start with a plot of land that is divided into five regions as shown above with twenty pre-drawn running figures. You give your students the option of drawing twenty different looking stick figures wherever they want on the paper in the five regions. However, to win the game, they have to win at least three regions out of the five, and to win one of the regions, they have to place more figures of their own than the preexisting stick figures in the region. For example, to win the region in the top right corner of the figure above, the students have to place at least four stick figures of their own in that region. Then you can ask the students to win the game with the least number of figures used. For example, in the figure above, they have to use at least nine figures to win the game.

Rock, Paper, Scissors
This game just starts by asking the students to play rock, paper, scissors against each other. After they have played a couple of rounds, ask the students if there was a strategy to win. They are all supposed to say it is a game of luck, since each tool wins a third of the time, loses a third of the time, and draws a third of the time. The point of doing this game is to have the students start thinking about this case analysis against different moves. One way the students can do that is by making the table pictured below, where their moves are represented by the columns and the opponent's move is in the rows.

	Rock	Paper	Scissors
Rock	Tie	Win	Loss
Paper	Loss	Tie	Win
Scissors	Win	Loss	Tie

<u>Monkey vs. Rabbit</u>
Monkeys have almost the same hands as humans, more specifically, monkeys have fingers. However, rabbits do not have fingers, they have paws! If they wanted to play rock, paper, scissors, they could not ever play scissors. They can only play rock and paper. Now you have your students play this game where they split up into pairs with one being the monkey and the other being the rabbit. Have the students think about optimizing strategies. Will the game be "fair"? Note that if the monkey avoided playing rock, then it would win half of the time, but the rabbit cannot adopt such a strategy since it only has two options but against three tools, each tool wins a third or the time.

	Rock	Paper	Scissors
Rock	Tie	Win	Loss
Paper	Loss	Tie	Win

<u>Rock, Paper, Scissors, Sword</u>
This game is the same as rock, paper, scissors, except we add a sword that beats scissors, beats paper, and loses to rock. Again, have the students play this game, and then decide what the best moves are. We get the following table for all the cases:

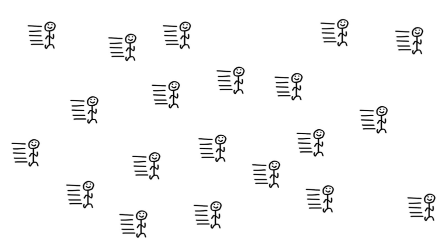
	Rock	Paper	Scissors	Sword
Rock	Tie	Win	Loss	Loss
Paper	Loss	Tie	Win	Win
Scissors	Win	Loss	Tie	Win
Sword	Win	Loss	Loss	Tie

Note that here we have both rock and sword winning half the time they are played.

Our Experiences & Suggestions

As mentioned above, these topics are pretty quick to go through, so we would only recommend doing them all together, or as time fillers.

Taking It Further



- For the first game, you can take it further by having the students figure out the least number of figures required to win the game if they could also draw the boundaries. For example, given the image to the right, how many figures do you need? The answer is three with each figure being alone in its region! So you're drawing 3 empty circles (regions) and putting one of your figures in each.

- In the other games, you can also have the students rank the moves. What would be the best strategy? What would be the worst strategy?
- Add another game! You could try to look up game theory topics online. One example would be Deal or No Deal.

Sources & Additional Resources

You can look up resources for game theory online. A useful keyword would be “Nash Equilibrium”

Game Strategies for Nim

Keywords

Game Theory, Number Theory, Logic, Case Analysis

Materials

Each pair of students should have twenty tokens. The tokens can be items like pennies, paperclips, straws, pieces of scratch paper, or pencils. For our case, it was pennies.

Problem Statement

Nim is a two-player game where players take turns removing one to three tokens from a thirteen-token pile. The player who takes the last token loses, so the objective is to force the other player to take the last token. With this arrangement, the player who goes second can always win when playing optimally (refer to math background section for explanation). Please note that other rulesets exist for Nim where the player who is left with no tokens to take loses.

After practicing the optimal strategy for Nim, introduce the rules of the game above to the students and demonstrate that you can always win despite not going first, even though that is essential to the strategy. If they insist that you go first, then you are still likely to win because they do not know the optimal strategy. An example statement is below.

> "Hey everyone, today we will be playing a simple game that I have been practicing lately. The game is called Nim, and to play, two players take turns taking [insert your token] from the pile and the player who takes the last [insert your token] loses. Each player is only allowed to take one to three tokens, but must take at least one token on your turn. I will play an example game with someone, and I will even let you go first because I have gotten so good at the game."

Play the game against a few students and then arrange them in pairs to play the game amongst themselves. Allow them to try and construct different strategies and play against them during this activity if they request.

The objective of the lesson is to allow students to develop this optimal strategy on their own through experimentation and reducing the game to simpler configurations, e.g. a game where the pile only has five tokens.

Math Background

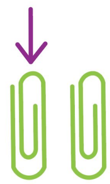
The questions students should consider are:

1) How can you simplify the game to make it easier to analyze different strategies?
2) Are all the tokens in the pile of equal importance? Or is one token essential to a win?

3) Does a particular player, the player who goes first or second, have an advantage in the current configuration?
4) Does there exist a strategy that allows a certain player to always win?

We will go through each of these questions for this section. First off, like in many other math problems, we should simplify the game to make it easier to analyze the mechanics of the game. Even trivial configurations will help us answer some of our questions.

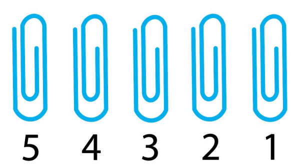
Consider the game where there are only two tokens. By considering the case with only two tokens, we can already answer two of our questions. *Are all the tokens in the pile equal in importance?* The answer is no, the tokens are not all equal. Whoever takes the second to last token wins the game because it will force the other player to take the last token making them lose. Now the perspective can change from avoiding the last token to developing a strategy to obtain the second to last token.



Does a particular player, the player who goes first or second, have an advantage in the current configuration? When the game only has two tokens, then yes. The first player will always win when playing optimally. Playing optimally here would mean the first player will take only one token.

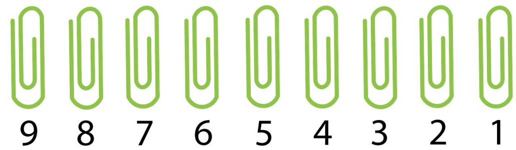
Because the first player is allowed to take at most three tokens, this optimal strategy works for Nim games that have two to four tokens. When there are four tokens, then the first player can still take the first three forcing the other player to take the last token.

Consider the case with five tokens now.



In this case, the first player cannot force the second player to take the last token because they can at most take three tokens leaving the second to last one for the opposite player to take. Now, the advantage has moved over to the player that goes second rather than the first. That proves that the number of tokens determines the winner when playing optimally.

If we extend this case from five tokens to nine tokens, then the player who takes the sixth token has the opportunity to win because it results in the same situation above.



If we keep appending four tokens onto this pile then we can see that the player who takes the $4n+2$th token will always win when playing optimally for some non-negative n. That means the player who takes the second, sixth, tenth, fourteenth token and so on has the potential to win when playing optimally.

To win the original configuration with thirteen tokens, you would need to take the tenth token, then sixth token, then second to last token. By having thirteen tokens, the first player cannot take the tenth token because they can only take the thirteenth, twelfth, and eleventh token. A good rule of thumb would be to take the number of tokens so that the sum of the tokens between the two players equals four in this configuration.

Our Experiences & Suggestions

- Students may suspect that allowing them to go first in the initial demonstration is a part of the optimal strategy. If they do, then allow them to go first and attempt to still win using the optimal strategy. In all likelihood, you will still win by knowing the game better than them.
- Keep in mind that the competitive nature of this game might dissuade certain students from participating. If you notice this happening, then try and spend some time with that pair of students to see if you can mitigate it. Although we did not face this problem, always keep this in mind.

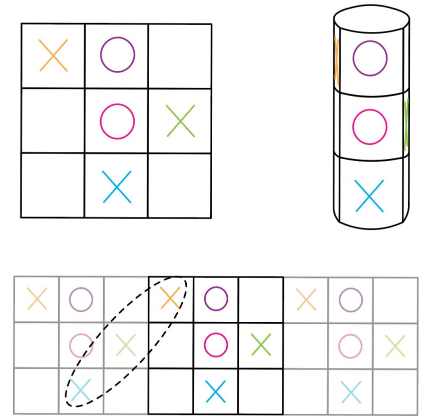
Taking It Further

1) What if we change the number of tokens a player is allowed to take, e.g. players are allowed to take one to five tokens now, does the strategy change?
2) What if we change the number of tokens in the pile? Does the strategy change?
3) Can we find a strategy that will allow someone to win no matter the number of tokens that the player is allowed to take or that are in the pile?
4) What would happen if we had more than one pile? How would the strategies change in this case?

Sources & Additional Resources

General information about the game Nim and other modified versions can be found on the Wikipedia page. Please note that some versions of Nim have it so that the player with no tokens left on the pile to take loses. The University of Maryland also has a worksheet covering one-pile and two-pile Nim games at an undergraduate level.

Tic-Tac-Torus



Keywords

Game Strategies, Visualizing Surfaces, Topology, Cylinders, Tori

Materials

Paper, markers, and either tape or a stapler (something that can hold paper together in three-dimensional shapes).

Problem Statement

In this Circle, students first think quantitatively about good strategies for tic-tac-toe, then use those analytical techniques to explore what the game would look like on surfaces other than a flat board. After making sure everybody knows how to play the basic game, pose this question: “What’s the best starting square in a game of tic-tac-toe?” After letting the students play against each other and take a stab at the question, eventually guide them to think about how many winning positions each given starting square opens up for you. With this line of reasoning, it will come clear that the center is the best starting square.

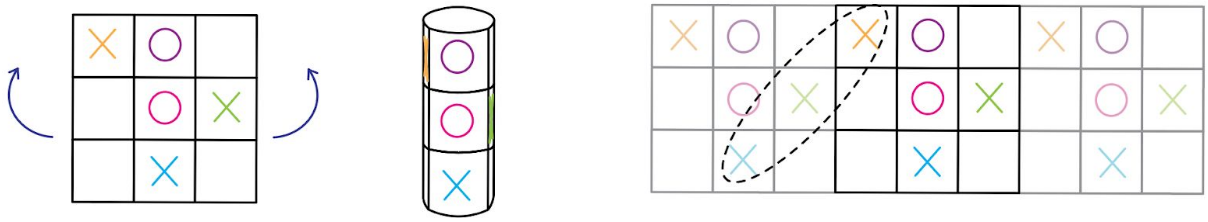
Then, the next layer: what if you play the game on a cylinder? Connect two opposite sides of the board and see what happens. (More on how this works is below, in the “Math Background” section). Once that game has been explored, other possibilities open up — what about a torus, where you connect two opposite sides like a cylinder, but also connect the other two opposite sides, creating a donut-like shape? How can you represent these game boards in a more manageable way? Are there other shapes you could try? (Perhaps spheres, or Möbius strips?)

Math Background

If X goes first, counting the number of winning positions that have X in a given square is a pretty good proxy for how “good” that square is as a starting square — it quantifies how much potential for victory a given square gives you. Counting these winning positions out, and marking each square with its score, gives you a table like the following for a flat board:

3	2	3
2	4	2
3	2	3

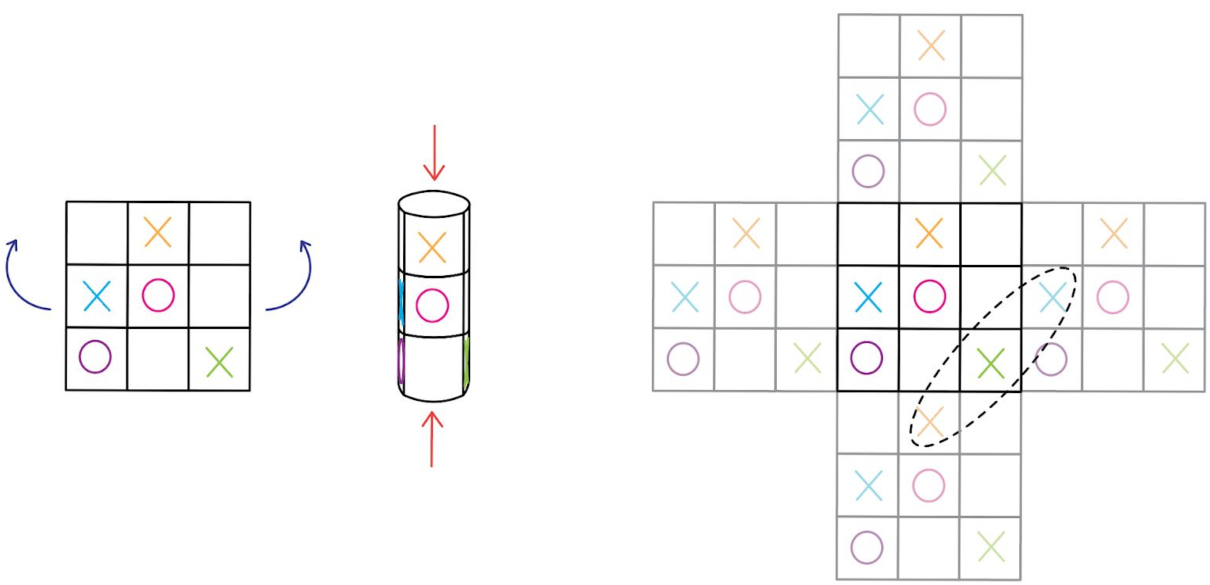
To create a cylinder, we then “identify” the left edge with the right edge. That is to say, we join the two edges together so that they are one and the same — so that when you go off the right edge, you come back on the left edge. This is demonstrated in the following image.



The rightmost image is a two-dimensional way of visualizing how the game works on a cylinder. The shaded copies of the board to the right and left of the main board represent how the board wraps around; when you go off to the left of the central board, you end up on the right side of that board again (represented by a copy of the board). Thus an arrangement of X's like the above, which is not usually a three-in-a-row, becomes a three-in-a-row on a cylinder. With this board topology (that is, shape), counting winning positions gives us the following:

4	4	4
4	4	4
4	4	4

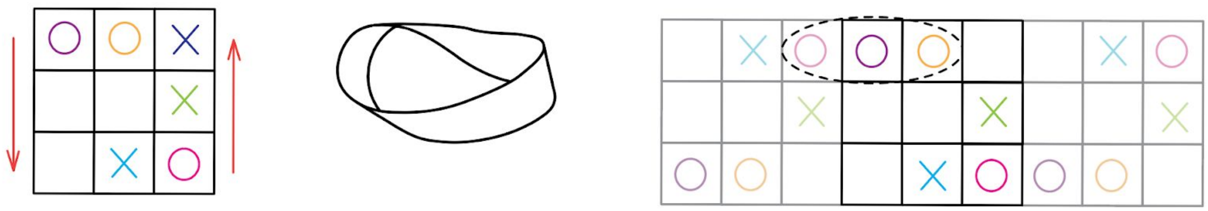
We can do something similar with a torus, where we not only identify the left and right edges, but also the top edge and the bottom edge (that is, the board loops in every direction). The image below demonstrates how this works, in a similar fashion to the image for the cylinder.



If we repeat the win-counting strategy from above for a torus board, we get the following:

4	4	4
4	4	4
4	4	4

This result is interesting; it may be worth asking the students, upon getting this result, whether this means the cylinder and the torus are effectively the same. (For the purposes of tic-tac-toe, they are, in fact.) You can also try playing on a Möbius strip, where the left edge is identified with the right edge, but flipped, so that going off the right edge from the top square puts you on the left edge at the bottom square. Thus, on a Möbius strip board, the following position is a win for O, because the top of the left edge loops to the bottom of the right:



This is interesting because there are now rows longer than three squares; the top edge and the bottom edge are the same, and together they are six squares long. The leader and the students will have to work out what this means for the game. Spheres, too, are tricky, and it is not entirely clear how tic-tac-toe should be played on them; this would be another interesting thing to explore with students.

Our Experiences & Suggestions

- Students have a lot of ideas about what good strategies for tic-tac-toe are, but we found that a lot of them tend to be more about psychologically tricking your opponent, and less about the number of winning positions you open up with your move.
- Avoid leading the students too much. In one of our Circles, we somewhat heavy-handedly tried to make a point about “duplicates” of the main board in the cylinder case, such as those in the image at the start of the lesson plan (these “duplicates” are explained in more detail in the Math Background section). This went over the heads of almost every student. Had the students come up with the idea themselves, they might have gotten it. The leader should avoid telling when possible; just keep asking questions.

Taking It Further

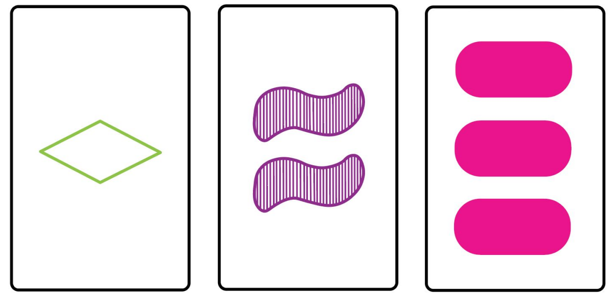
After trying cylinders and tori, see if the students can come up with other shapes to try. Try Möbius strips and spheres. Try larger boards, non-square boards, different game rules, and wackier shapes. Perhaps hexagonal or triangular boards, or something else entirely. What if you played the game in 3D? How could you represent all these things on paper, so that you can think about them clearly?

For each of these different cases, what does the game of tic-tac-toe look like, and what is your best starting move or general play strategy? Does any scenario have a starting move that guarantees your victory? Exploring the game theory of tic-tac-toe — that is, whether there are strategies that always win, and if so, what they are — is another, and very interesting, path that this Circle could go down.

Sources & Additional Resources

Many thanks to Stephen Kennedy for introducing us to this problem. For more, Chalkdust has an excellent article on exactly the topic of this Circle.

SET

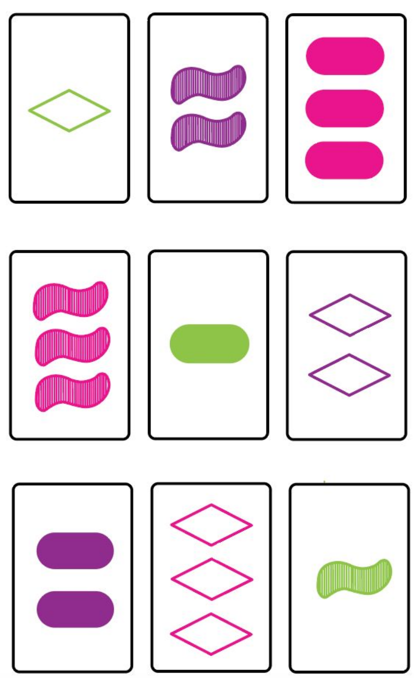


Keywords
Patterns, Magic Squares, Systematic Reasoning, Surfacing Underlying Structure

Materials
The card game SET — ideally one copy per three or four students.

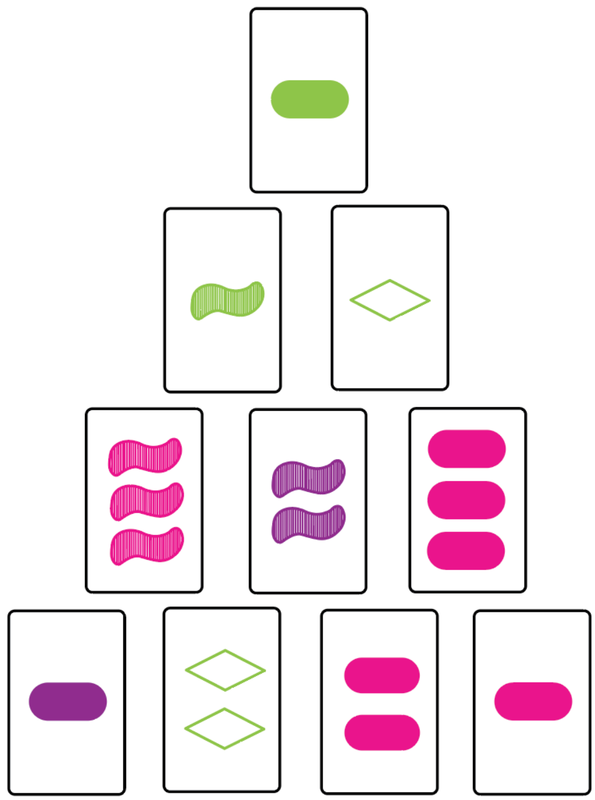
Problem Statement
The game SET is described in detail below. In order to introduce it to the students, ask them what features they see on the cards. Once they are familiar with all the features, explain the basic rule of forming sets. From there, there are many questions that can be asked. The most basic is to make SETs, and perhaps to make SETs such that there is no more than one trait in which all three cards are the same, or even a SET such that the three cards are different in every trait. Is there just one of these SETs?

Students can also be asked to make Magic Squares, which are three-by-three squares of SET cards such that every row, column, and diagonal is a SET. You can either start students out with three cards and ask them to fill in the rest, or just have them create one from scratch. One such square is pictured below.



Another form we explored was triangles of SET cards. There are multiple possibilities here. One is that you start with a row of SET cards — say, three of them — and then you create a row one longer right below it such that the edges between cards in one row line up with the centers of cards in the other, as in the picture below. This row should have the property that every pair of two adjacent cards forms a set with the card between them in the row above. This pattern can be extended indefinitely in a growing triangle.

This can also be done in the opposite direction, going "up" the triangle. Form a row of, say, four cards, then form progressively smaller rows such that each card in the smaller row forms a set with the two adjacent cards in the previous row. Is there a way to figure out what the card at the top of the pyramid will be without doing the whole thing out? Are there multiple starting rows that lead to the same final card?



Math Background

The SET deck consists of 81 distinct cards. Each card has the same four features—shape, shading, color, and number—but each feature has three different possibilities, and each card has a different combination of the possibilities of each feature. The standard game consists of a board of twelve cards, on which you aim to find a SET. A SET is a group of three cards such that, in each feature, the cards are either all the same, or all different. In other words, a SET is any group of three cards such that there is no feature where two cards are the same in that feature and the third card is different. In the normal game, you simply try to find as many of these SETs as you can. For this Circle, more creative endeavors are recommended. There is little else in terms of hard math background for this Circle; the focus is more on creative play.

Our Experiences & Suggestions

- For the second pyramid question, students tried to start with a row that was a set of three. They soon realized this would not work — you cannot form a SET with any pair of those cards because the third card that forms a SET is unique, and already in use.
- Students had fun coming up with their own names for the different card features.
- One student, when working on Magic Squares, was at first trying things more or less randomly — if an arrangement did not work, she would take out one card, and try another; if that did not work, she would take out a different card; and so on. Eventually, with a little bit of guidance, she began to think the problem through more carefully, and instead of grabbing for a random new card, started systematically looking for cards that would complete SETs with the cards she already had in place. This felt like an important learning moment.
- We had a lot of success with this Circle; we felt it was one of our best. The students were quite engaged.

Taking It Further

What other arrangements or patterns can you make with SET cards? Can you define sets such that they include four cards instead of three? Consider introducing the idea of "super sets;" these are sets of four cards, in two pairs, such that the two pairs would each form a set with the same fifth card (but that fifth card need not actually be present). In other words, if ABC and CDE are both sets, ABDE is a super set.

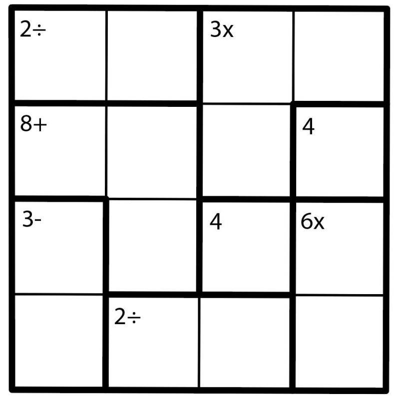
Another potential avenue of exploration is asking how many cards there are (without counting), and how many possible sets there are. Or — even trickier — what is the biggest number of SET cards you can get together without a SET between any of them?

Another line of inquiry: If you were designing the game SET again, how could you modify it? How would these questions be answered if there were more features, or more options for each feature? Would three-card SETs still make sense, or should the number of cards per SET change?

Sources & Additional Resources

Most of the inspiration for this Circle came from one of the Julia Robinson Mathematics Festival activities, which, at the time of this writing, can be found here. Credit is also due to SET, the board game, by PlayMonster LLC.

KenKen Puzzles



Keywords
Puzzles, Arithmetic, Factorization

Materials
At least one print out of a set of KenKen puzzles from the sources below.

Problem Statement
KenKen puzzles take the popular game of Sudoku and add a mathematical spin to it. KenKen puzzles allow students to practice their basic arithmetic skills, as the puzzles require students to be comfortable with case analysis and the fundamental operations of arithmetic: addition, subtraction, multiplication, and division.

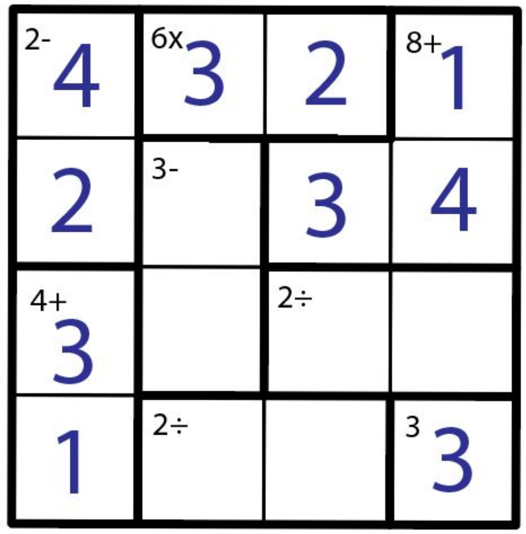
This Circle does not require an understanding of Sudoku puzzles, as all the rules for KenKen puzzles will be covered in this lesson plan.

The rules of an n-by-n KenKen puzzle can be summarized in four constraints.

1) Each square of the KenKen must be occupied by a number from one to n.
2) Each row must use the numbers from one to n only once. For example, in a four-by-four KenKen puzzle, numbers one through four appear in the top row exactly once.
3) Similarly, each column must use the numbers from one to n only once.
4) KenKen puzzles are divided up into "cages." Cages are groups of squares surrounded by a bold border, with a particular number and operation in the top-left most square. Unlike the rows and columns, cages can contain the same number more than twice as long as it follows Rule 1 and Rule 2. The numbers in each cage must fulfill the number and operation specifications; this will be explained more specifically below.

Rule 4 introduces cages, the mathematical spin that KenKen puzzles add to traditional Sudoku puzzles. Again, each cage is characterized by a particular number and operation. The operator next to the number states the operation to use on the group of numbers within the cage. The number represents the result of using the operation on all of the numbers in the cage.

For example, consider the 4+ cage on the partially solved four-by-four KenKen puzzle below. The cage contains the numbers three and one, and with the plus operation, we can observe that three and one do sum to the number next to the operator, four.



The operation can be applied in any order. Consider the 2- cage on the top-left. Since the minus operation can be applied in any order, the result of that particular cage can be four minus two or two minus four. Only one of the cases needs to satisfy the result.

A few more notes about the rules regarding cages: if a cage contains more than two squares, then the same operation applies to all numbers in the cage. Consider the 8+ cage; here we calculate the total summation of one, three, and four, which results in eight. Additionally, cages with only one square, known as "freebies," only require you to fill them in with the result listed. As an example, look at the cage marked with a 3 on the bottom-right of the puzzle. For cages like these, just fill in the number denoted on the top-left.

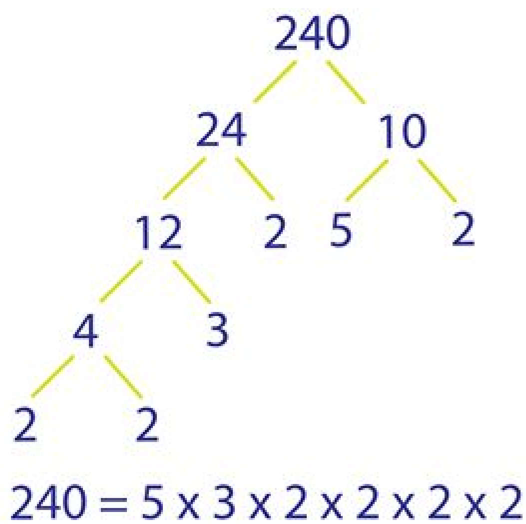
In summary, an abridged ruleset is below, which can be written on the board. The rest of these rules can be explained verbally.

1) Each square must be filled with a number between one and *n* (fill this in with the size of the particular board).
2) No row or column may contain a number more than once.
3) The numbers in each cage must result in the number marked in the cage using the operation listed in the cage.

Math Background

There are a variety of strategies to solve KenKen puzzles, but most of them can be distilled down to knowing the possible answers to cages and eliminating possibilities with incoming information as you solve the puzzle. Although this is a good strategy, it will still require students to backtrack from unsolvable positions, as well as require a decent familiarity with the possible summations, factors, and divisors particular numbers may have.

For example, for a size four 240x cage on a five-by-five KenKen puzzle, students would need to know that either the set of numbers two, four, five, and six or three, four, four, and five need to go into the cage, because those four numbers result in a product of 240. To assist students with finding four numbers for this complex cage, consider using a prime factorization tree (pictured below), where numbers, like 240, are continuously broken down into smaller factors until the factors are prime.



Knowing the possible options and trying them to deduce if a particular set of numbers for a cage is impossible is the trial and error students need to go through to solve KenKen puzzles.

With extra time, we also want students to consider whether they can determine if particular puzzles have multiple solutions or only one unique solution. The answer to this question is out of the scope of the lesson plan, as it requires an understanding of algorithms, but a citation containing further information on the topic is listed below. Regardless, the reasoning skills needed to determine if particular puzzles have more than one solution are still productive to think through.

Our Experiences & Suggestions

- Students are often excited to go straight into five-by-five or even six-by-six KenKen puzzles for the challenge. Although it may be exciting at first, students may become frustrated, as they do not have the practice from smaller KenKen puzzles that allows them to sift through the numerous possible arrangements in larger puzzles. Encourage inexperienced students to practice with the smaller-size puzzles with a limited number of operations.
- To help students understand the logic behind these puzzles, spend some time with each student and do a think-aloud with them. A think-aloud is where the students verbalize their thoughts when solving KenKen puzzles. This will allow them to self-analyze their strategies and make progress faster than they would silently brute-forcing the solution.

Taking It Further

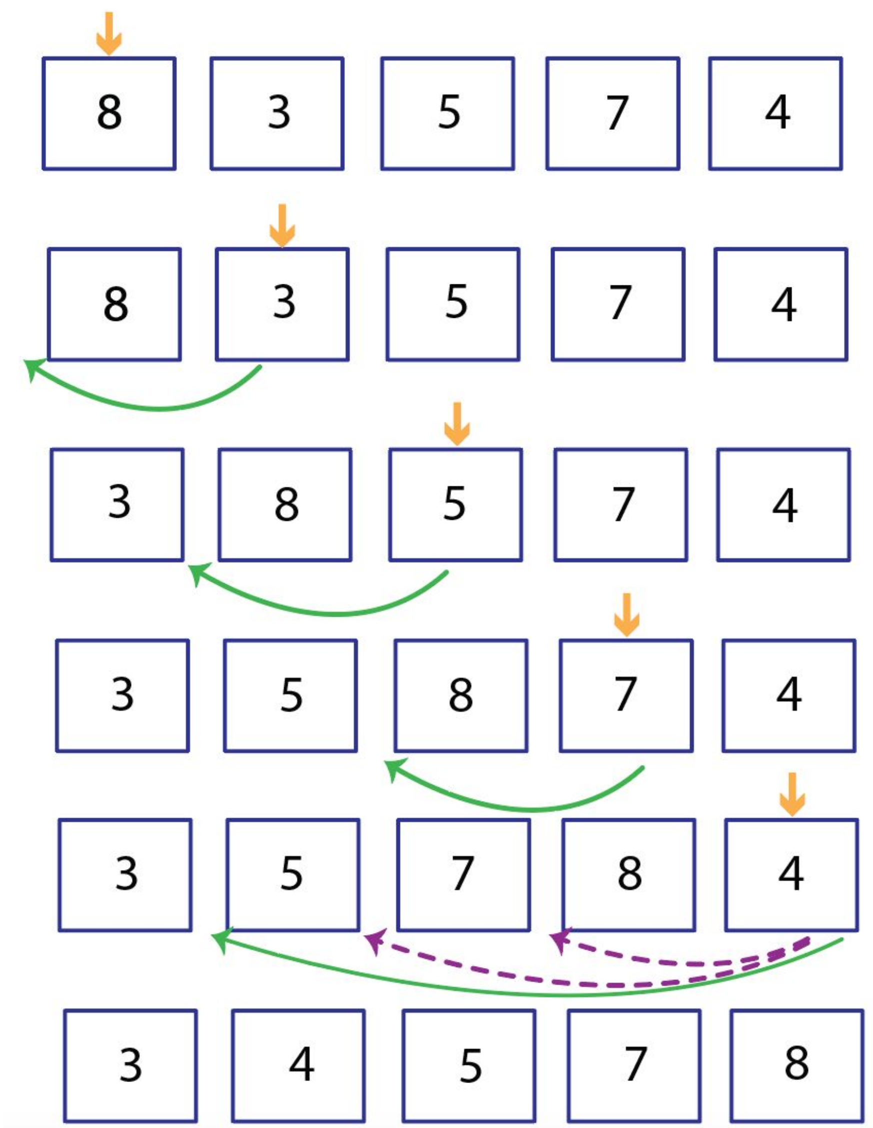
- Is there a way to determine whether or not there are multiple solutions to a KenKen puzzle? Can you make a KenKen puzzle where multiple solutions satisfy it? Can you make a puzzle that only takes one solution? How do you know that it only takes one solution?

- How would you go about creating your own KenKen puzzles? Do you want to write out the cages first? Or the numbers first and arrange the cages afterwards?
- Are there KenKen puzzles that have no solutions? How do you know that they have no solutions?

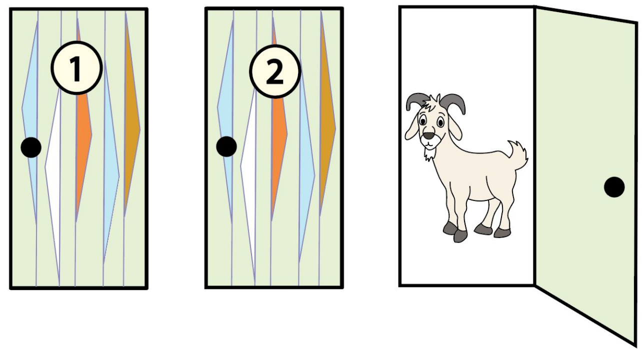
Sources & Additional Resources

As referred to in the Materials section, the website krazydad provides printouts of KenKen of various difficulties and sizes. Please select the difficulty you believe best fits your students and their comfort levels with addition, subtraction, multiplication, and division. To compensate for different ability levels, we recommend printing out a variety of difficulties. If unfamiliar with prime factorization trees, please refer to the Math is Fun citation, which introduces primes, factors, and factor trees. As a broad overview of KenKen puzzles, the Wikipedia page can serve as a good resource. For your own curiosity, here is a citation discussing the provable uniqueness of particular KenKen puzzles solved by using Mixed Integer Programming (MIP). This citation is a bit advanced and requires an understanding of computer science, specifically algorithms.

Logic and Probability



The Monty Hall Problem



Keywords
Probability, Conditional Probability, Case Analysis, Logic

Materials
At least one set of Monty Hall props which include: objects to represent three doors (e.g. construction paper) and three prizes (two duds and one desirable item, e.g. pictures of two goats and a picture of a car). A whiteboard to represent the problem will suffice too. If able, having more than one set will allow students to play the game amongst themselves. Otherwise, a demonstration set will suffice.

Problem Statement
In front of the class, place the three prizes behind the doors without the students seeing their arrangement. Memorize the positions of each prize and begin the game show, *Let's Make a Deal!*

To conduct the game show, present the three doors to the students, allow one student to select a door. After selecting the door, do not reveal what is behind it, but instead reveal the position of one of the goats. Once that goat is revealed, offer the student the option to win the prize behind the door they originally selected, or switch and win the prize to the door they did not select.

After setting up the doors and secretly placing prizes behind them (again, be sure to remember where they are), explain the Monty Hall problem to the students—perhaps using the dialogue below—and invite students up to play the game.

> "Welcome to *Let's Make a Deal!* I'm Monty Hall. Here are the rules of the game. We have three doors. Behind two of them are goats, and behind the third is a brand new car! Once you've picked a door, I will open another door, showing you the location of a goat. Once you see the goat, you may either switch to the other closed door, or stick with the one you chose originally. Who wants to come up and try?"

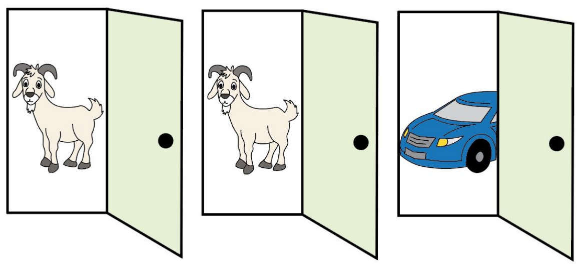
Once the students have tried the game a few times, pose the question: "Once the goat is revealed, is staying or switching the better strategy, and why?"

Math Background
The optimal strategy for the contestant is to always switch instead of staying. Always switching will result in a $\frac{2}{3}$ probability of winning the car, and staying will result in a probability of $\frac{1}{3}$. The probability of winning when you always stay is $\frac{1}{3}$ because, when

picking randomly, the game contestant will have a $\frac{1}{3}$ chance of picking the car with their initial guess because the car is behind one of the three doors.

To understand why $\frac{2}{3}$ is the probability of winning the car when always switching, we need to realize that the strategy of always switching will always result in the car when your initial chosen door has a goat behind it. We know this because if we initially picked a goat, then another goat will be revealed to us. Once that goat is revealed, then switching off of our original door with the goat behind it will result in the car. Thus, picking a goat initially and switching will always result in the car making it a $\frac{2}{3}$ probability under the always switching strategy.

Consider the door configuration below:



If the game contestant always stays, then the probability of their initial choice being door 3 will be $\frac{1}{3}$.

Another way of understanding why $\frac{2}{3}$ arises is to consider the instances when you get a goat when always switching. If the game contestant always switches, then the only time they lose is when their initial guess was door 3 and they switched off of it. Picking door 3 first gives us the probability of $\frac{1}{3}$ and because the probabilities must sum to 1, then the probability of always switching resulting in a win is $\frac{2}{3}$.

These solutions can also be derived through decision trees, in regards to a specific arrangement, if the students are familiar with them. They can also brute force the probability by playing the game multiple times and recording their win rates or notice the trend by trying all three distinct situations with respect to where the car is placed.

Our Experiences & Suggestions

- Many students struggled with how to begin the problem, and had trouble thinking it through by cases. For example, if the car were behind the first door, what would be the probability of winning when you pick a random door and always switch? What about the probability of winning when you always stay? Guide them to think about specific arrangements of prizes and calculate the probabilities for those arrangements.
- Students found it entertaining to believe that the objective is to win the goat. Be prepared to steer them away from that.
- Students may also be stubborn in their initial beliefs on the optimal strategy. We found that they, along with multiple mathematicians, were quick to guess that there was a 50/50 chance of winning the car, and were resistant to believe otherwise.

- Consider having the students run experiments and calculate the experimental probability to see whether it aligns with their viewpoint.
- In our Math Circle, we eventually took on the problem as a group and played out all the different door-picking scenarios for each possible car location on the board. They noticed a pattern once the different outcomes were written on the board, but there was less collaboration among the students and more leader input than we had wanted.

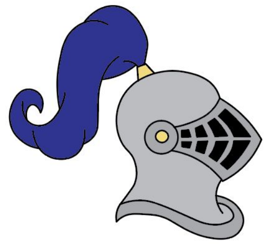
Taking It Further

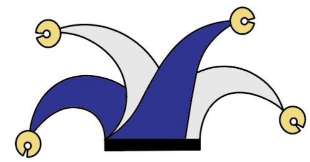
- What are the exact probabilities for each strategy?
- Does the strategy you come up with change when you change the number of doors, goats, cars? Does the strategy change when Monty Hall changes the doors he opens? For example, if he opens two doors in a four-door game, how do the probabilities change?

Sources & Additional Resources

Wikipedia is a good resource to learn more about the history of the problem and how it puzzled mathematicians. Wolfram also covers this puzzle more concisely along with an explanation for a four-door Monty Hall variant. The University of California San Diego has an online resource to play *Let's Make a Deal!* in your web browser. Finally, the techniques we used to explain this problem are well demonstrated in this video.

Knights and Knaves



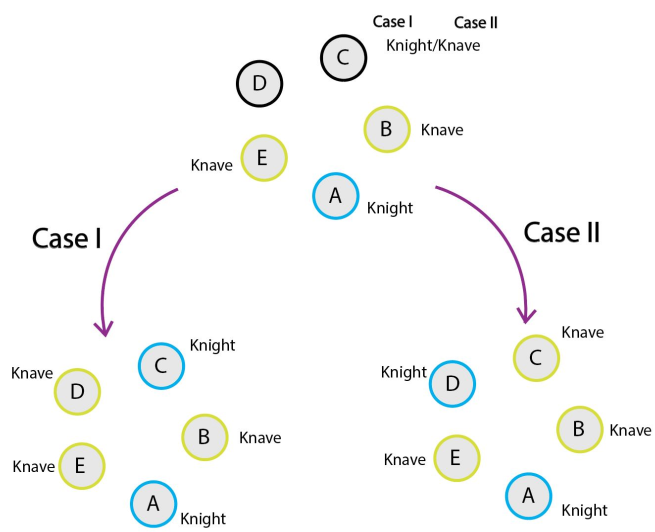


Keywords
Logic Puzzle, Case Analysis,
Reasoning Skills

Materials
Twelve chess/Go/Othello pieces for each student (at least six white pieces and six black pieces to represent knights and knaves).

Problem Statement
The goal is to use logic to find the number of knights and knaves on an island. The knights and knaves puzzle is made popular by a Mathematician, Logician, and Musician Raymond Smullyan in his 1978 book *What is the Name of this Book?* The knights always tell the truth and the knaves always lie. In our setup, there are five people on an island sitting in a circle. Each person says, "people on both of my sides are knaves." The objective of the Circle is to figure out who are the knights and who are the knaves, and in turn the total number of each. The number of people for this scenario can be any number greater than or equal to three. We started off at five because it gives an exact number of knights and knaves, as opposed to, say, six, where the solution can vary. The Circle leader(s) can demonstrate this puzzle by getting in a circle with other people and each person in a circle claiming people on both their sides are knaves.

Math Background



Logic puzzles are great ways to understand the mathematical concept of deduction. We always make deductions in daily lives. However, when the problems become bigger and more complex, they seem more challenging to solve. This is a puzzle that is relatively more challenging than daily situational puzzles. For the knights and knaves puzzle, we have established that there is an island where five people are sitting in a circle. Everyone claims that people on either side of them are knaves. We cannot have everyone be knights because then when the first person who says that people on both sides are knaves, he is not telling the truth, so he cannot be a knight. We also cannot have everyone be knaves because if the first person says that people on either side of him are knaves, he is actually telling the truth so he cannot be a knave. Therefore, we proceed with the newfound assumption that there is at least one knight and one knave in the circle. In an island of five people namely A, B, C, D, and E sitting in a circle, let's say person A is a knight. As shown in the figure, when A claims, "People on both of my sides are knaves," he is telling the truth. Therefore, B and E are both knaves. Now, C can be either knight or knave because either way, B would remain a knave. Similarly, D can be either knight or knave because either way, E would remain a knave. Assuming that C was a knight, D must be a knave in order for C to be a truth teller. However if C was a knave, D must be a knight because otherwise C would be telling the truth. Therefore the total number of knights and knaves are two and three, respectively, where in the first case, A and C are knights and B, D, and E are knaves. In the second case, A and D are knights and B, C, and E are knaves.

If there were eight people on the island instead, the problem becomes much bigger but the process of finding the number of knights and knaves is similar. Unlike the first puzzle, however, we will find that the number of knights and knaves can be four and four, respectively, but it can also be three and five, respectively.

Our Experiences & Suggestions

- This is the sort of problem that quickly becomes intuitive to some students and confusing for others. We recommend that the leader(s) match the leading pace to the general students' pace of understanding the puzzle.
- It is also a fun project for students who understand much faster to learn to explain puzzles like this to other students. This will make the experience more challenging and therefore, they will eventually learn more. This also enables good team work and other collaborative practices.
- In an effort for the students to understand the puzzle well, it would be best not to put them in groups. We recommend making this an individual-based puzzle. Each student can try to figure out the puzzle solutions on their own. The leader(s) can then reconvene and allow students to discuss their solutions and thought processes. They can return to work individually for a different number of people on the island. This process can be continued for a varying number of island populations.
- In cases when there is more than one answer to the number of knights and knaves on the island, students can get competitive. This is also another opportunity for them to collaborate and explain to each other.

Taking It Further

There are many variations to this logic puzzle. Here is one: what happens if there are only two people on the island, say Amy and Bob. Amy says, "we are both knights" and Bob says, "Amy is a knave." Who is the knight and who is the knave?

Sources & Additional Resources

This puzzle was made popular by mathematician Raymond Smullyan in his 1978 book *What is the Name of this Book?* If you want to learn more about the knights and knaves puzzle, here is its Wikipedia page. There are many variations to the puzzle. For example, here is a link that has 382 such Knights and Knaves puzzles generated by a program written by Zachary Ernst.

Sorting Algorithms

Keywords
Algorithms, Programming, Computer Science

Materials
At least six paper cups for every pair of students with (any) numbers written inside them

Problem Statement
We say that a machine is "working" if it is able to follow instructions that we feed into them. Algorithms, in essence, are a set of rules or procedures that a machine must follow in order to get the desired result. The goal of this Circle is for students to learn how to write those instructions to introduce them to the world of algorithms. For this activity, given a set of cups with numbers written inside them, students are asked to come up with a method(s) to sort numbers in ascending order. We give them at least 25 minutes to ask them to write instructions to implement the method(s) that they discovered. We then allow at least twenty to 30 minutes for checking if these methods work and correcting them if need be. If another student is able to follow these instructions and get a sorted number as a result, the method works for this task of sorting numbers!

This Circle will not only make them think of every process in sorting numbers but also motivate them to think about different ways of sorting these numbers. The best methods of sorting numbers are generally the ones that take the least amount of time and computer memory. This can be analogized to a person solving a math problem using a piece of paper. The best methods are the ones that do not take a lot of time to solve and that do not require much space for the human to solve.

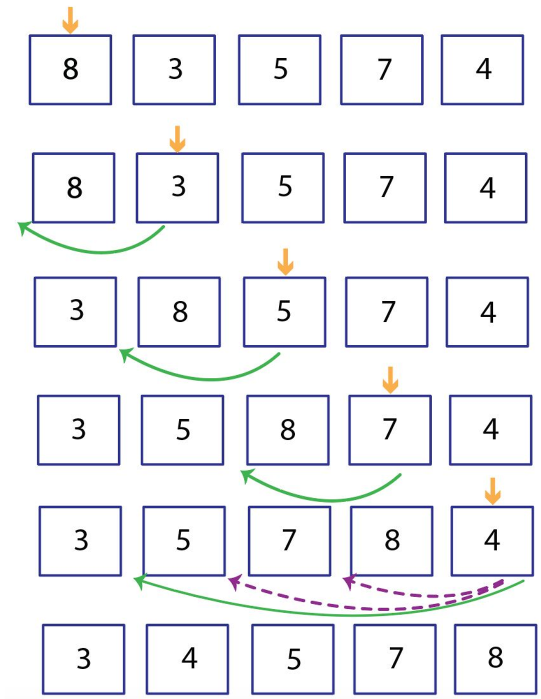
Math Background
According to Wikipedia, an algorithm is "a finite sequence of well-defined, computer-implementable instructions, typically to solve a class of problems or to perform a computation." We were inspired to do this Circle because it is a fundamental skill in writing proofs and instructions in both mathematics and computer science.

The instructions can be written in plain language. For example,

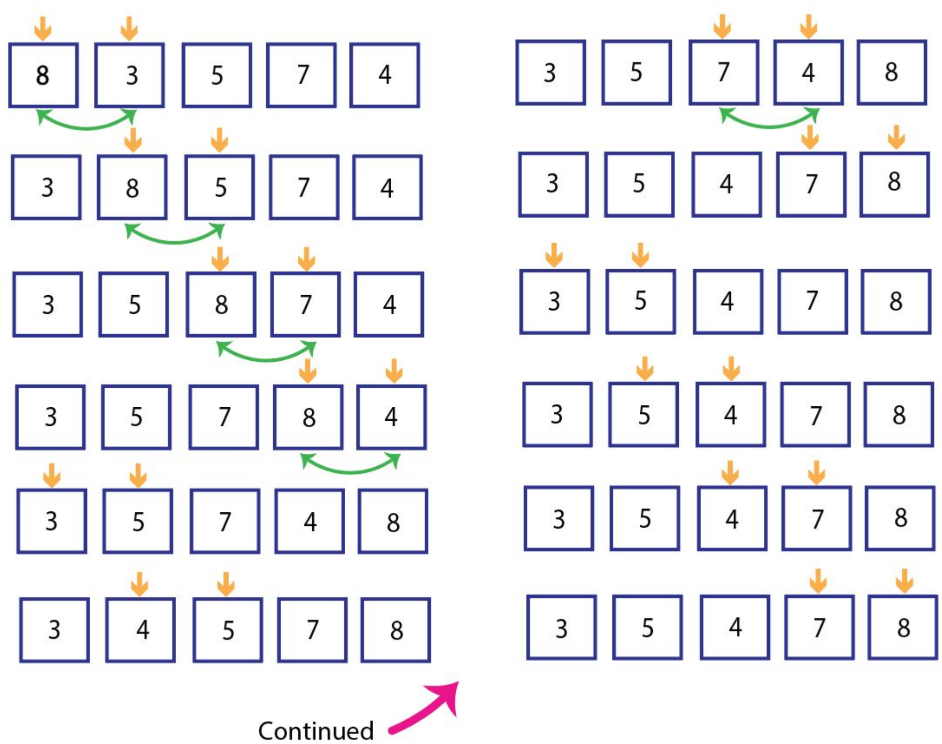
"Check the number in the first cup. Then, check the second number and if it is bigger than the first, move it before the first.
Then, …
If the next number is bigger, …
If it's smaller, … "

There are numerous ways to sort numbers. We let the students play around and discover methods on their own. Some students came up with a sorting method that is formally known as Insertion Sort.

This is a sorting algorithm where we look at one item at a time and is not very efficient for big lists. The figure on the right is an example of insertion sort for a set of cups in a random order, in this case 8, 3, 5, 7, 4. To get them sorted in ascending order (3, 4, 5, 7, 8), insertion sort looks at one number at a time, starting from the left-most, and compares it to the number(s) to its left. Then, if the number to its left is bigger, it moves before it and continues until no number to its left is bigger than itself. In the figure, we start at 8. Since there is no number before 8, we keep the order as it is and look at 3. Since 8 is greater than 3, we move 3 before 8. Next we look at 5 and find that it is smaller than 8. So we move it before 8. We continue this process until we reach the right-most number 4, which is smaller than 8, 7, and 5. Our random set of cups are now in the desired order.



Some students also discovered another method known in the academic world as Bubble Sort. Here, the adjacent numbers are compared with each other and swapped if they are in the wrong order. This repeats until we compare every pair of adjacent numbers and no swaps is required. Figure on the right shows an example of how bubble sort is used to sort the same random set of cups. First, 8 is compared with 3. Since 8 is greater than 3, the numbers are swapped. Then we move on to compare the next two numbers: 8 and 5. We repeat this procedure until the biggest number is at the end of the list. Then we compare the first two numbers and repeat until the second biggest number (which is 7) is placed just before 8, and so on and so forth. The last four comparisons require no swapping and therefore, we



know that the numbers are fully sorted.

In doing the Circle however, there is no need to know the names of the algorithms, such as "insertion sort" or "bubble sort." Let the students come up with names for methods that they discover!

Our Experiences & Suggestions

- We strongly recommend that the numbers are written inside the cup because that way, students will not simply look at the numbers and put them in the right order. This Circle is to get them thinking about how a seemingly easy task can be broken down so that computers and other machines can understand.
- We suggest that this Circle is done in pairs. This will help in the later process of checking if their instructions work. After writing the instructions, they can place the cups upside down and shuffle them around. In a pair, one student can read the instructions (without allowing them to look at the numbers inside the cups) while the other student can look at the numbers as they follow the instruction. That way, they could ensure that their instruction works for any unsorted list of numbers.
- Another part of the lesson is to see if they could find different ways of sorting numbers. Some students discovered sorting algorithms such as Insertion and Bubble Sort. If time permits, we recommend that they try to find even better sorting algorithms.

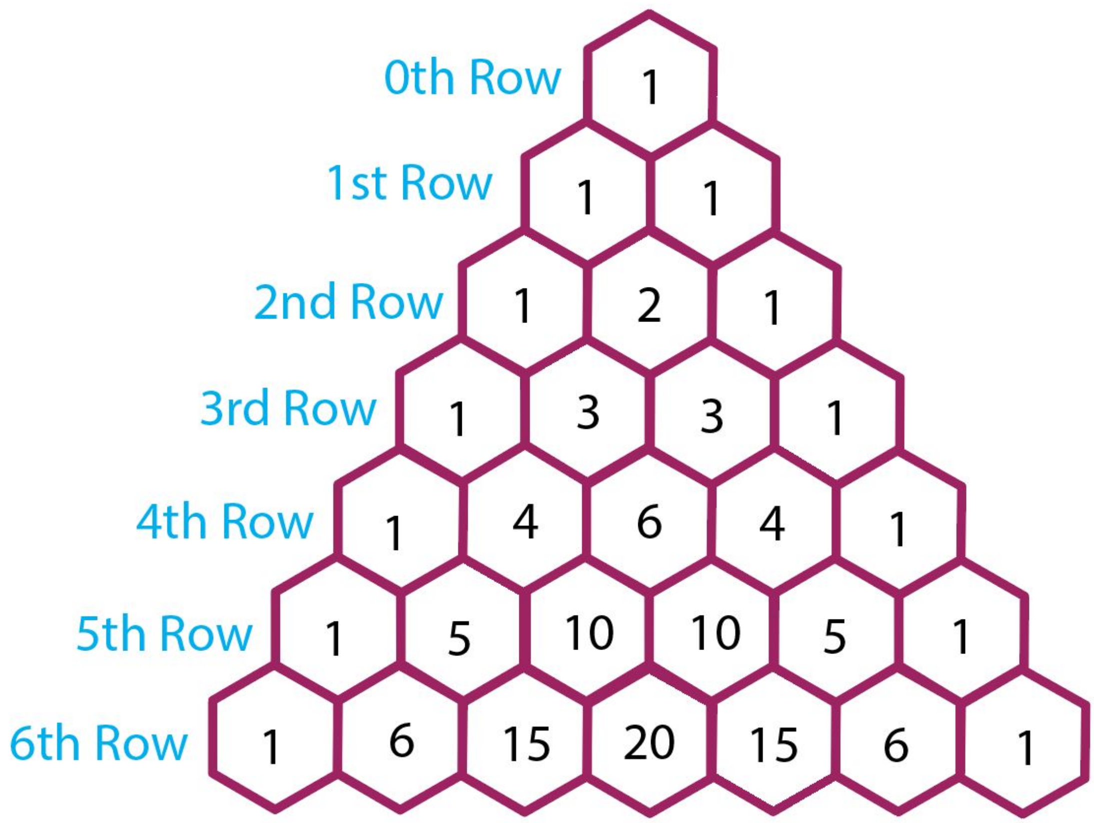
Taking It Further

There is another sorting algorithm that could get students thinking more seriously about time and computer memory: Merge Sort. This sorting algorithm is known for its divide and conquer way of ordering numbers. The list of random unsorted numbers are divided into two groups, which in turn is divided into two groups, and so on until each "group" consists of only one number. Each "group" is then sorted before bringing it back with other numbers. This takes a lot less time than Insertion and Bubble sorts.

Sources & Additional Resources

If you want to learn more about sorting algorithms, here is a Wikipedia page about them.

Counting and Combinations



The Handshake/High Five Problem

Keywords

Combinatorics, Counting, Graph Theory

Materials

None, aside from scratch paper and pencils.

Problem Statement

We will present this problem with a story about a mathematician going to a fancy party. And with some mathematicians, they are not the most social bunch so instead of socializing, they will try and calculate the number of handshakes that happened at the party.

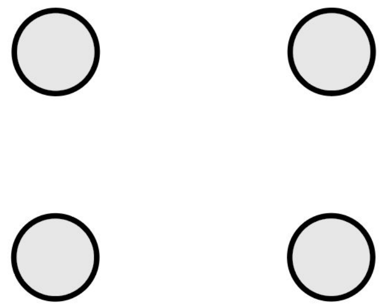
> "Today, I will tell you all a story about a mathematician who goes to a fancy dinner party. During this dinner party, it is polite and necessary for everyone to shake hands with everyone else. While at the party, the mathematician steps aside and thinks to themself, how many handshakes happened during the party? To put ourselves in their situation, we will do the same with (high fives/handshakes) and see if we can figure it out."

The leader should then supervise the students during this activity and ask the students for their initial thoughts on the number of high fives/handshakes. When their initial thoughts are off, or they do not know where to go, then guide them to trying smaller examples, e.g. parties with only four people, five people. Have a permanent table of the number of people, and number of handshakes on the board.

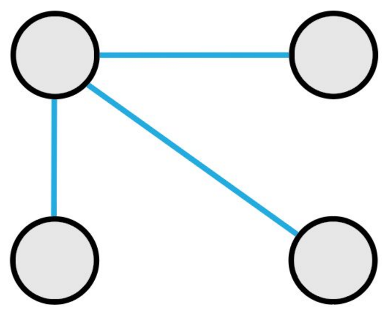
Number of People at Party	**Number of Handshakes**
2	
3	
...	

Math Background

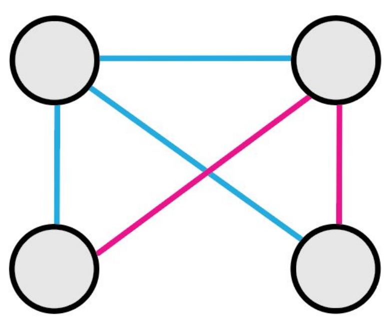
An intuitive path to take to understand the number of handshakes that occur at a party with n people is by drawing a graph and drawing the number of connections/edges each person needs to shake hands with one at a time. Below, we will demonstrate that with four people, represented by the circles/nodes, but the method can be used for any number of people.



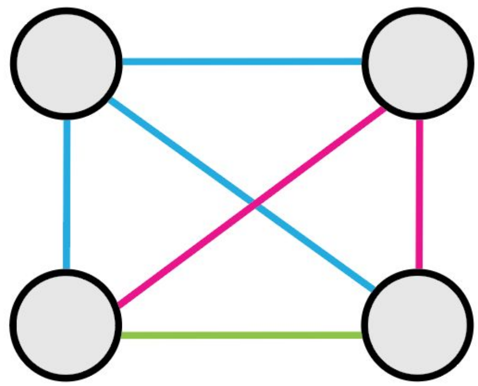
The first person at the party, denoted by the top-left circle, needs to shake hands with the three other people at the party.



Thus the total number of handshakes is now three, but we still have more handshakes to count. The second person, say the top-right person, needs to shake hands with only the bottom two people because the top-right person has already shaken hands with the first person denoted by the edge/connection between them.



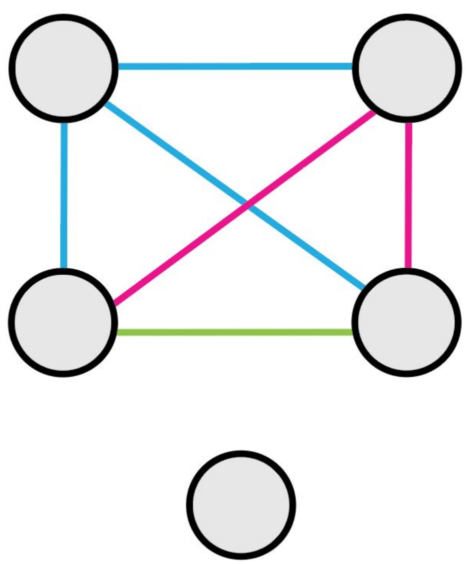
Now the total number of handshakes is three plus two which is five, i.e. $3+2=5$. Next we need the bottom-left person to shake hands with everybody. Notice that they have already shaken hands with the top two people, so the bottom-left person only needs to shake hands with the bottom-right person.



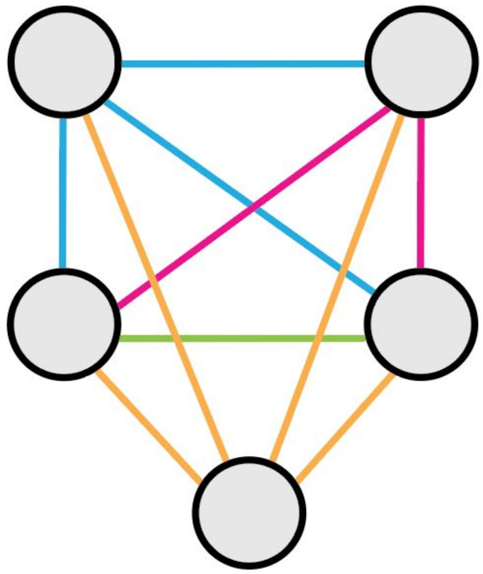
Now the total number of handshakes is now three plus two plus one, which is six. Notice not the total sum, but the summation which is $3+2+1$. We can see that the sum is three at the start and falls by one each time the next person needs to shake hands with everybody.

This trend holds for any n number of people which can be shown through a proof by induction formally, but it can also be shown by just adding one more person to the party each time.

Consider the above party of four, what happens when we invite another person to the party? How many people does this individual need to shake hands with?



The answer is the rest of the four people at the party making the summation to $4+3+2+1=10$ following the pattern above.



Informally, the solution would be that for an *n* person party, the number of handshakes that would occur is $(n-1)+(n-2)+(n-3)+...+2+1$, or more formally, $\sum_{i=1}^{n} i$.

This story is a disguise for the graph theory problem of calculating the number of edges in a *n*-node complete graph.

Our Experiences & Suggestions

- Some students may be hesitant towards touching one another. If you believe this will be the case for your cohort, or could be a potential issue, then moving towards high fives, or even elbow bumps should suffice. Another idea would be to bring cups and have them cheer (tap cups) one another with their cups to avoid touching completely.
- Remember, the first activity is to introduce the problem and the setting. Once that is accomplished, move onto smaller parties so that students can find the summations more easily and slowly increase the size of the party.

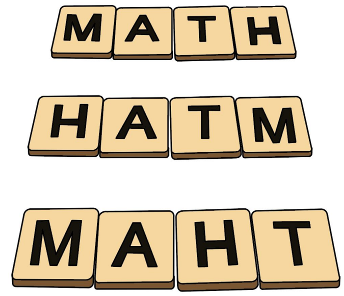
Taking It Further

- Instead of calculating the summation, is there a closed formula that would calculate the summation? In other words, is there a faster way to do $10+9+8+7+6+5+4+3+2+1$. Or even $99+...+3+2+1$ for parties with a hundred people.
- If two different families were having a party together, and each family member needed to shake hands with all other members of the opposite family, how many handshakes would there be? Let the family sizes be equal at first, and then have different-sized families. The disguised question is, "how many edges are in a complete *n, m*-bipartite graph, citation below.

Sources & Additional Resources

To calculate the sum of all whole numbers from *x* to *y* efficiently and other variants, refer to this citation that discusses it with examples. The Wikipedia page is also a good resource to learn more about complete graphs and complete bipartite graphs mentioned briefly in the lesson plan. There are also multiple solutions to the handshaking problem. A solution that requires the Fundamental/Basic Counting Principle and eliminating duplicates can be found here. A video that demonstrates another solution using permutations and combinations can be found here. The solution to the two families problem in the Taking It Further section can be found on this website discussing it as computing the number of edges in a complete *n, m*-bipartite graph.

Anagrams



Keywords
Combinatorics, Probability

Materials
Enough Scrabble tiles such that each student gets five distinct tiles, with one extra repeated tile. In our case, we used Bananagrams tiles, but you can use any tiles that have letters or numbers that you can rearrange.

Problem Statement
Start by asking the students to pick three distinct tiles from their piles to play with, and then ask them to find all three-letter words (whether they are sensical or nonsensical) that are made up of these three letters. They must use each letter of the three exactly once, so if you ask the students how many words there are, they should get six. For example, if a student has the letters *a, b,* and *c,* then you want them to reorder their letters to get: *abc, acb, bca, bac, cab, cba*. After they are done with making three-letter words, you can ask them to make four-letter words, and then five-letter words. Note that you should start by using unique letters, and give students time to figure out how many anagrams an n-letter word has. Then you can start using letters more than once. This is when the students can start using their extra repeated letter. Maybe have them check if *abcc* still has the same number of anagrams as *abcd*? They should just try to provide a formula or explain their thought process of how to figure out anagrams of a certain word with repeated letters.

Math Background
<u>Unique letters</u>
Given an n-letter word, there are $n \times (n-1) \times (n-2) \times ... \times 2 \times 1 = n!$ different ways of rearranging the letters to get a unique word. Students can think of starting with n letters, and having the choice of using any letter for the first one in the word, and then having $n-1$ choices for the second, $n-2$ for the third... multiplying the different options gives us $n!$ (called "n factorial") different ways.

<u>Repeated letters</u>
Students can start by making n-letter words assuming that the letters are unique. This gives them n! words that are not different. To account for the overcounting of words, they can divide by certain factors. This could be shown in an example. The most popular one, is the different permutations of the word MISSISSIPPI. For every permutation of Mississippi, you can change the location of each *S* in four different ways to still get the same word. You can also change the location of each *P* in two different ways, and *I* in four different ways to still get the same word. This means that you divide the $n!$ by $4!$ to account for the *S*'s, $4!$ for the *I*'s, and $2!$ for the *P*'s. so given an n-letter word with a letter that is repeated k times, the answer would be $\frac{n!}{k!}$. Note that if there is also another letter that is repeated m times, then the answer would be $\frac{n!}{k!m!}$ and so on. In particular, if we look at the MISSISSIPPI example, we note that we start with $11 \times 10 \times 9 \times ... \times 2 \times 1$ and then divide by $4 \times 3 \times 2 \times 1$

(because of the four occurrences of *S*), then divide again by $4 \times 3 \times 2 \times 1$ (This time because of the *I*'s), and by 2×1 (to account for the *P*'s).

Our Experiences & Suggestions

- It was a good idea for us to build off of the same letters when increasing the length of the word in this activity. This is because some students noticed that if you already had all anagrams of *m-a-t*, then you just add *h* to the end you find six of the anagrams you are looking for. Then some students thought that they could repeat this process by finding the number of anagrams of *t-a-h* and adding *m* to the end etc...
- The students could be challenged to try figuring out passwords for electronics. For example, we asked them to try to unlock a phone if they knew that the password consisted of four unique letters. This got the students excited to apply what they just learned.

Taking It Further

You can ask students about probabilities! It is of course easier to guess a six-letter code if we allow repeated letters, but is it easier to guess a six-letter code with unique letters than guess ten-letter codes with two repeated letters? Explore!

Sources & Additional Resources

Any beginning probability or combinatorics textbook covers this kind of material. We first recommend looking for resources online on permutations and combinations, but if you would like a more rigorous and detailed approach, we wholeheartedly recommend the book *Probability: With Applications and R* by Robert P. Dobrow.

Ciphers

```
def encode(text , key):
    take all the letters and transform them using the key
    loop through the text that you want to encode
    and encode using the key
    return the new enconded message
```

Keywords
Cryptography, Computer Science, Logic

Materials
None, aside from scratch paper and pencils.

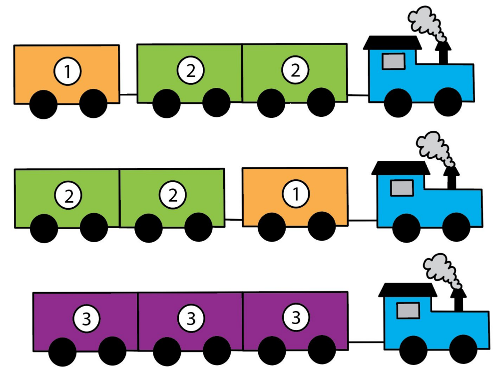
Problem Statement
This is more of an explorative Circle for students to be creative. Start by telling the students that they need to send each other secret messages that only their friends can read, and if a stranger finds that message, they should not be able to understand it. In the beginning, students will need to send each other long messages just to get comfortable with this activity. Ask the students to try different encryption strategies, and then you ask them about ways to try to understand these encoded messages if they do not know what the code is, so ask them to try to crack each other's code! As the students try to decipher each other's code, the Circle leader should ask some guiding questions that would make it easier to figure out some letters. For example, one could use the frequency of some letters such as vowels, or how I's and a's could be found separate from other letters.

Math Background
There is no math background required for this Circle, but we recommend that you read about different examples of ciphers if you want to guide your students towards a specific one. See additional resources below

Our Experiences & Suggestions
Since this activity is more explorative and is dependent on what the students do, it is important to keep them on track. Some students enjoyed the messaging part more than the cracking part, but it was still important that they worked on the cracking a little bit.

Additional Resources
You can learn about substitution cipher here, transposition cipher here, and pigpen cipher here.

Counting Trains



Keywords

Counting, Combinatorics, Pascal's Triangle

Materials

Several "train cars" of various whole-number lengths (in arbitrary units); we used rectangles made out of construction paper, with a unit length of around two inches. Cuisenaire rods, if available, are excellent for this activity. For a medium-sized classroom, we recommend around 40 length-one cars, twenty length-two cars, ten length-three, and so on, although this depends heavily on the number of students and on how many groups you plan to break them up into. Nothing longer than, at most, a couple of cars of length six should be needed. Cars are best when color-coded by length.

Problem Statement

First, introduce students to the train cars available (unlimited cars of any integer length — but without the big words). Asking them what they would like to call the unit of length for the cars is fun. Then, the question: "You work at a train yard. One day there's an earthquake and all the tracks are out, so you have nothing to do. To keep yourself entertained, you decide to try and figure out how many different trains of length four [units] you can make with your train cars." Make sure it is clear that the order of cars matters; a car of length one followed by a car of length two is different from a car of length two followed by a car of length one.

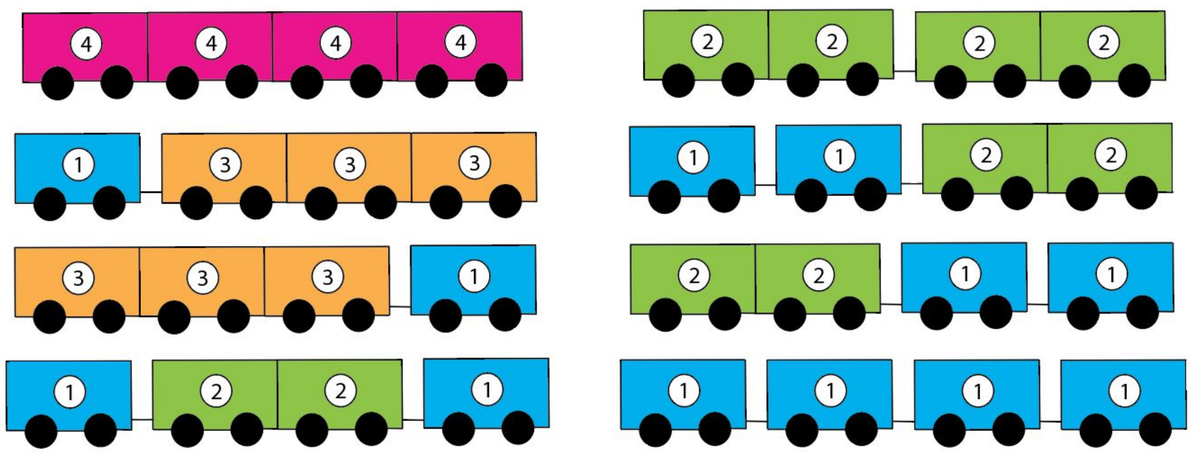
After this first question is resolved, likely with significant work in small groups, ask about other lengths of trains. See if the students can find a rule for how many different trains there are of any given length, and see if they can figure out how many there are of length ten without counting.

Math Background

This question is asking, in essence, how many different ways there are to split a row of n units into an arbitrary number of contiguous groups. Each group is a train car, and n is the length of the train. We will notate a train as a sequence of numbers, where each number indicates the length of a train car, and the numbers are written in the order of the cars. So, for example, if you are making length-four trains, a length-one car followed by a length-two car followed by another length-one car is a valid train, and we will write this as 1-2-1.

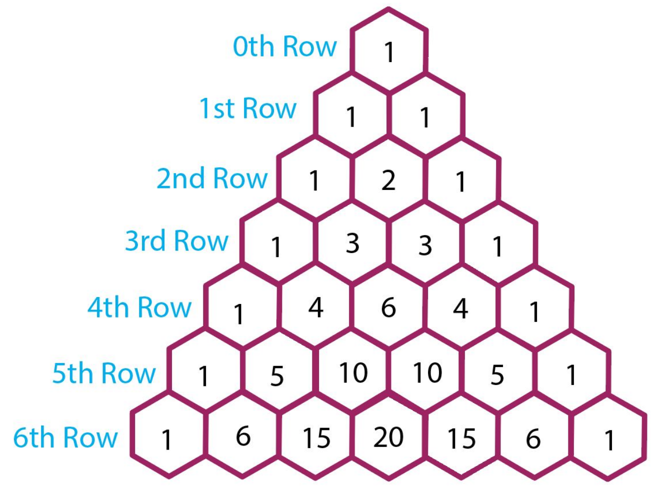
There are several ways to count how many different trains of a given length there are. For now, let us think about where to place boundaries between train cars. Any boundary between two units can either become a break between cars or not, and you will get a different train depending on whether you choose to break the units apart at that point, or not. For example, the top two cars in the figure below differ only in that they are broken in the middle — that is, at the second boundary between units.

Now, for length-four trains, let us count all the possibilities. Each boundary between units has two options, "break between cars" or "no break", and every different combination of these options results in a different train. The number of boundaries in total is three, one less than the length of the train. Since each boundary can independently be set to "break" or "no break," they each multiply the total number of possible trains by two. Thus we get $2 \times 2 \times 2 = 8$ possible trains, pictured below.



It follows that, in general, for trains of length n, there are 2^{n-1} possible trains. Note that there are many valid ways to come to this answer; this is just one argument.

Pascal's Triangle is another concept that may come up in this Circle, as described below in the "Taking It Further" section. Pascal's Triangle, the first few rows of which are pictured below, is a figure created with a simple rule: each number in the triangle is the sum of the two numbers above it, starting with the number one by itself at the top of the triangle. In the second row, each number only has one above it, so they are both one. In the next row, however, we get something more interesting: one, two, and one. The rule of adding the two numbers above continues forever, generating patterns with interesting connections to other concepts, which some of the resources in the Sources & Additional Resources section point towards.



Our Experiences & Suggestions

- There was some confusion in our Circle over whether the order of train cars mattered; that is, whether the train 1-2 was different from the train 2-1. We intended that order does matter, and that should be made clear from the start.
- Students referred to train cars by their color rather than their length.
- Starting out by splitting students up into small groups to find all the trains of length four worked well. However, it meant that we ran short on pieces of paper to use as train cars.
- We had trouble abstracting the problem onto paper; students were attached to the physical representations of the train cars.
- Introducing the question with a little bit of a motivating, fun story — as suggested in the Problem Statement — went well for us.
- In our 40-minute Circle, we got to the point where students recognized that the number of trains you can make doubles with each unit you add to the length.

Taking It Further

A further step can be to ask students how many trains of a given length there are that only use a certain number of cars. For example, how many trains of length five are there that use only two cars? These numbers, if put in a table of train length versus number of cars, will give Pascal's Triangle (see Math Background, as well as Sources & Additional Resources, for more on Pascal's Triangle). The first few rows will look like this:

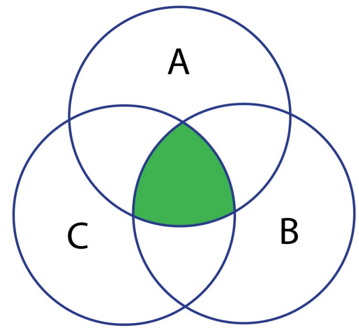
Train Length	1 car	2 cars	3 cars	4 cars
1 unit	1 way	0 ways	0 ways	0 ways
2 units	1 way	1 way	0 ways	0 ways
3 units	1 way	2 ways	1 way	0 ways
4 units	1 way	3 ways	3 ways	1 way

Once you have Pascal's triangle, the possibilities are endless: Is there a rule that gives you a row from the previous row? What do the rows sum to? How do the numbers on the diagonals change as you move along them? What if you colored in all the odd numbers?

Sources & Additional Resources

Adapted from the *Trains of Thought* task found in *Ways to Think About Mathematics* by Benson, et al., published jointly by Corwin Press and Education Development Center, via the Teaching Experience for Undergraduates at Brown University. The technical term for the concept that this Circle centers around is "compositions," about which there is a Wikipedia article. For more on Pascal's Triangle, see Math is Fun, Wikipedia, and this video by Numberphile.

But How Many Are There?



Keywords
Set Theory, Combinatorics, Logic, Inclusion-Exclusion

Materials
Six straws and four markers, or six of any object or four of another

Problem Statement
We envision the Circle as starting with the dialogue below, but you can phrase the question however you want.

> "I had some people fill out surveys for me, and I recorded some of the responses, but some of the information got lost. In particular, the survey had two yes-no questions, the first asking if the respondent has a pet cat, and the second asking if the respondent has a pet dog. I can't figure out how many people actually took my survey! I only know that six people said they had cats, four people said they had dogs, and two people indicated that they had both. I was wondering if you can help me find out how many people took my survey?"

Have the students work on solving this problem in groups for a bit, and then come back together as one group to discuss their solutions! After that, try to give a little demonstration to the students. Have some students volunteer to represent the respondents. If a respondent has a cat, then they will get a straw, if the respondent has a dog, then they will get a marker. If the respondent has both, then they will get both. Have two students volunteer to be the ones with both cats and dogs, and give both a straw and a marker to each one of them. These students represent the respondents who answered "yes" to both questions on the survey. Then have two other students volunteer to represent the rest of the people who answered "yes" to the dog question, because you have two who already said they had a dog, and give the two new students a marker each. Finally, give four straws to four new students. Ask the students the following questions:

- How many students were given a straw?
- How many students were given a marker?
- How many students were given both a straw and a marker?
- How many students volunteered in total?

Now you could ask them if they could have solved the initial question you asked them using just the information you gave them at the beginning. That is, could they have figured out the total number of respondents using just the numbers of people who have cats, people who have dogs, and people who have both? The answer is yes, and we will discuss how below!

Math Background

This problem is best explained in terms of sets, and we will provide two different approaches that give the same answer. A set is a collection of any kind of unique objects or elements. In our Circle example, we have a set of people who have cats (C), a set of people who have dogs (D), and a set of people who have both (B). The elements of each of those sets are the individual people. Note that the number of people in C is six, and the number of people in D is four, and the sum of the sizes of both sets is ten. However, this is not the number of total respondents. This is because we double-counted some people. In particular, there are two respondents who have both a dog and a cat (and they are in set B).

We are interested in finding out how many people have a dog or a cat, which we will call set O. Note that we must be careful which conjunction we use when phrasing these questions. The solution becomes more clear when we write out the sets. We will write these sets and give the respondents unique names so we can easily keep track of information:

- C = CatSet = {Kate, Laura, Rob, Luisa, Hugo, Rita}
- D = DogSet = {Kate, Maggie, Sam, Laura}
- B = CatAndDogSet = {Kate, Laura}
- O = CatOrDogSet = {Kate, Laura, Rob, Luisa, Hugo, Rita, Maggie, Sam}

Looking at these sets, we see that the number of people who have a dog or a cat (those in set O) is eight. The two people who were double counted are Kate and Laura.

We can also solve this in a different way, without specifically naming every person in each set! That is done by summing the sizes of the sets C and D, and subtracting the number of people in the set B because those were double counted. Whenever we want to find the total number of people in either one of two different sets, we can add the sizes of those two sets, and subtract the size of their intersection, the set that includes all the elements that are shared by both sets.

Our Experiences & Suggestions

Note: this problem was not conducted in person due to the COVID-19 pandemic. Instead, we sent students a video summarizing the problem and let them go about it on their own, unsupervised.

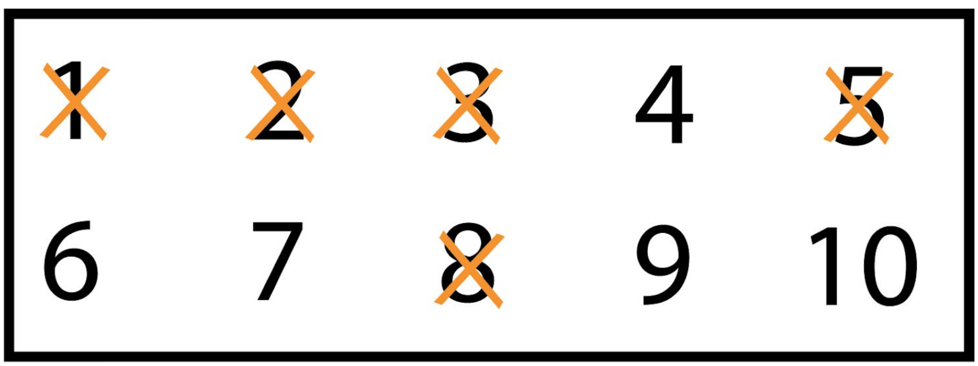
Taking It Further

You can take this further by having the students find a similar formula for the case where there were three questions — one asking about having a cat, another about a dog, and another about a guinea pig!

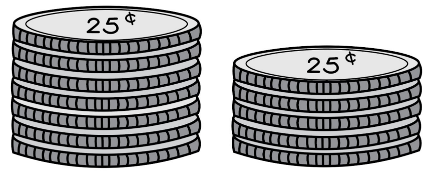
Sources & Additional Resources

This Circle implicitly covers the Inclusion-Exclusion Principle, which you can read about here. If you would like a more detailed approach, we recommend any introductory combinatorics textbook.

Numbers and Factors



Piles of Triangular Numbers



Keywords
Combinatorics, Factoring, Triangular Numbers

Materials
You can use any set of objects that are easy to pile up such as pennies. Each student should get eight or nine of the object of choice.

Problem Statement
Start by splitting up the students into pairs. Each pair gets two piles of n pennies, one pile per student. Try to keep the number less than eleven so that the students do not spend too much time on arithmetic. The way the game works is that each student starts with zero points, and then the two students take turns in splitting their piles into two until they get n piles of individual pennies. In each round, a student scores by multiplying the number of pennies in the two piles that they had just produced and adding that to their cumulative score. For example, if the student starts with five pennies, and in the first round splits it into a pile of three and a pile of two, then the student's score in that round is six. In the next round, the student can either score one point by splitting the two-pile into two one-piles, or score two by splitting the one three-pile into a two-pile and a one-pile, and so on.
Have the students play against each other and see who wins. They will eventually notice that the game will always end with a draw. Have them explore if the score will always be the same. Meaning that if the students start with an n-pile of pennies, will they always score a specific score for each number of pennies they start with? The answer is yes! It will always be $\frac{n(n-1)}{2}$ (a nice triangular number)!

You can let the students figure out this pattern by working together and finding the final score for the first couple of numbers. A good strategy would be to work with small numbers and then go up until they see the logic of how it works.

Math Background
The triangular numbers are a sequence of numbers that describes how many units you need to draw a filled in equilateral triangle of side length n as in the Penny Shapes problem. An explicit way to find the nth triangular number is to sum $1, 2, 3, 4, ..., n$. This equals $1 + 2 + 3 + ... + n = \frac{n(n-1)}{2}$. Being familiar with the triangular numbers would help understand this activity, but that is not necessary. The students can start by looking at the case when they start with 1 and notice that they cannot split this pile into more one-piles, and so the score for this game will end up being zero. Starting with a two-pile would give you a cumulative score of one. Starting with a three-pile would give a cumulative score of three. We start having a choice of how to split things when we start working with a four-pile. We can either split it up into a three-pile and a one-pile, and then add $3 \times 1 = 3$ to the cumulative score of starting with a three-pile, or we can split it into two two-piles, and

add $2 \times 2 = 4$ to the cumulative score of starting with two two-piles. So here we see the connection between the n-pile and the $(n-1)$-pile.

Our Experiences & Suggestions

- We introduced this activity after we had done the Handshake problem and the Penny Shapes problem, which also covered triangular numbers, so some students were able to figure things out more quickly when we reminded them of the previous problems. We recommend covering those two Circles first, since the connection to the triangular numbers is more apparent there.

Taking It Further

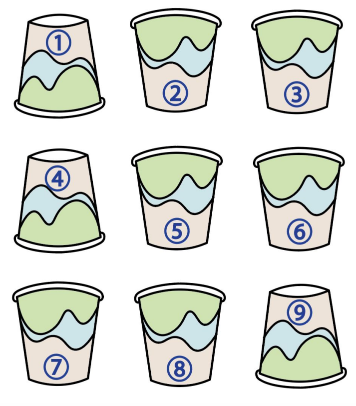
- You can take this further by working on proving the formula using induction, and introduce the students to the logic of induction by telling them to assume that the formula is true for the first ten numbers, and they have to show that it is true for the eleventh number. You can read more about mathematical induction in the resources below.
- You can check out Tom Davis' resource below to see how you can have the students explore the handshake problem and connect it with this one. It is always a good idea to look at a concept from different angles.

Sources & Additional Resources

This problem was adapted from Tom Davis. For a list of Math Circle topics by him, you can check out the following page here. You can learn about mathematical induction here. If you are interested in a more detailed explanation or reasoning, you can check out any introduction to proofs textbook.

Cup Flips

Keywords
Square Numbers, Factorization



Materials
Paper cups or some other object that can easily be numbered with two clearly different sides. You will need one set of 45 labeled cups for each group of three or four students

Problem Statement
Present the students with a small set of cups turned upside down and in a row. We suggest ten, since the number is unassuming and familiar.

Counting down the row, starting at the same end each time, flip every other cup, then flip every third cup, and so on until you flip every tenth cup. In this example, the cups that are flipped in the first round are the cups numbered 2, 4, 6, 8, and 10. For the second round, the cups that are flipped are numbered 3, 6, 9. Continue this pattern until the ninth and final round, when every tenth cup is flipped; in this case, the only cup flipped is the one marked 10. Which cups are still facing down? Can they predict which will stay down? Ask the students if they want to try again with the same number just to make sure they understand the rules.

Increase the number, preferably increased to another number that includes an additional square number before it. Try twenty, for example. Before beginning, what cups do they think will remain down this time? After they get the hang of it, have them work in pairs to solve some on their own. They can compare their guesses and answers. We had all students work together on 45 cups, an arbitrary number. Our group of six was small enough and divided itself into two naturally, so the table was not too crowded. If the group is larger, we recommend having more than one set of 45 cups. A set for each group of five or six students should suffice. What kinds of cups are staying down? What if we played with 100 cups?

Math Background
Starting with the first round, every second cup is flipped, in other words, only cups divisible by two are flipped, then only cups divisible by three are flipped, and so on. Notice that we never flip the first cup, so it will remain upside down. As for the rest of the cups, to explore the pattern we will explore some examples. A reminder that we do not flip every first cup and thus we will not consider the fact that all numbers are divisible by one. 3 is flipped once: when we flip every third cup. Three is divisible by three. 4 is flipped twice: when we flip every second cup and every fourth cup. Four is divisible by four and two. Fast forward to 12, which is flipped five times: it is flipped on every second cup, third cup, fourth cup, sixth cup, and every twelfth cup. Cup 16 is flipped four times: when we flip every second cup, fourth cup, eighth cup, and sixteenth cup. Sixteen is divisible by two,

four, eight, and sixteen. Notice a pattern? Square numbers are flipped an even number of times while all other numbers are flipped an odd number of times. It all comes down to factors and factor pairs. For example, two and six are factor pairs for twelve since 2x6=12. For a square number, however, the square is multiplied by itself and thus only counts once towards the total number of factors for a given number. Thus for every non-square number, we count itself and any additional pairs of factors which results in a total of an odd number of factors. For square numbers, we count itself, additional pairs of factors, and the single square root of the number, which results in a total of an even number of factors. If we flip an upside down cup an even number of times, it will remain upside down and if we flip it an odd number of times, it will no longer be upside down. Therefore only square numbers remain upside down.

Our Experiences & Suggestions

- We wanted to make sure that the students understood the task, but we also wanted to make sure they were focused and were not fighting each other for the right to flip the cup. So when we showed them the rules, we had the Circle leader set out all ten cups in a line, hover their finger over them, and move along the line from 1 to 10. As the finger moved, we asked the students to shout "flip" when the finger was hovering over a cup they wanted flipped. Sometimes we messed up on purpose so they could tell us why we were wrong. This helped build a better understanding of the rules and perhaps start to create some intuition. We also hoped that giving the control to the leader would let each student learn without having another tell them they were wrong. Their focus was on our mistakes rather than each other's.
- Once we moved on to the next set of numbers, we asked some of the students that were not participating as much if they wanted to record the number of times each cup was flipped as we flipped them. This way they were able to participate without us forcing them to be as forward as other students. To do this, they created a 2xC table, where C represents the number of cups included. Each cell in the first row was labeled for each cup in the set of C. Instead physically flipping cups, they added a tally. For example, on their first run through the table, they added a tally for every other cell. Below is an example of this for ten cups (C=10).

1	2	3	4	5	6	7	8	9	10
	|	|	||	|	|||	|	|||	||	|||

- When we moved on to paired work, we split off into two groups. Half of them wanted to try actually flipping all 45 cups, while the other half wanted to write the flips down as tallies. We thought this was a good way to compare results in the end and have them work together to resolve any differences they had.
- The students who chose to flip the cups were some of the more energetic ones and we found that it was a good way to let their energy out. One student would flip every second cup, then run all the way back to the front while the other followed flipping every third cup. Then they continued flipping every fourth, fifth, and so on. Together they tackled the challenge and corrected each other as they completed their own sub-tasks.

- The students who chose to tally started to work separately from each other but on the same board and so they lost track of what had already been tallied; we recommend making sure that they work together or completely apart otherwise the numbers will be too difficult to parse.
- Many of the students noticed that cups only flipped if we were counting off by their factors. We had them consider and count all the factors of the cups they had left upside down. Then we had them flip the cups according to the number of factors they found to check for mistakes.

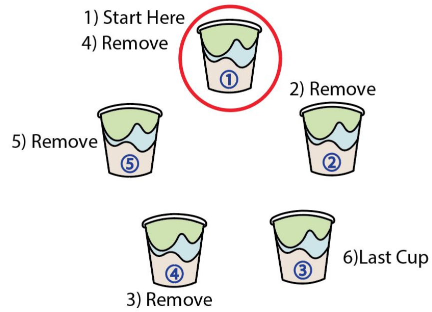
Taking It Further

- What happens if you flip only odd numbers? Only even numbers?
- What if we remove the cups instead and each round starts as though the other cups never existed?
- What if there was a specific cup, say number five, that we want to flip more than once? Since five is a prime, what other ways can we flip it? We could flip numbers that are in the Fibonacci sequence, for example!

Sources & Additional Resources

This Circle was taken from James Tanton's *Solve This*. Both Tanton's book and ours use cups to pose the problem, however, it appears in several other forms on the internet and as a result, we were unable to find any official name for it. Some internet threads and chat rooms discuss its logistics, but we found that this video does the best job. Bonus: he is very enthusiastic.

Counting with Cups — The Josephus Problem



Keywords
Counting Game, Powers of Two, Number Patterns

Materials
Paper cups or some other object that can easily be numbered and removed. Have sets of ten prepared for each pair of students at least, if not for each individual student. We also recommend having extra blank cups in case students want to try more.

Problem Statement
Form a standing circle of students, preferably around a table with chairs so that students can sit when asked to. Select one student to start as well as a direction to start removing students (counterclockwise or clockwise). Starting from the designated start student, tell every other student to sit down. All sitting students should not be reconsidered as we continue counting off in the circle. Keep removing students until only one student is left standing. The image above is broken out into steps below to show how this can work with five cups/people.

Now do it again! Start with the same student and move in the same direction. Who will be the last person standing this time? The answer should not change, but repeating the exercise will help them understand the rules, the goal of the exercise, and the different possible set up variations. If they have not already suggested it, have the students change some of the aspects of the exercises; for example, have them change the starting student or the direction. Before starting, have them once again guess who they think will win. Try as many times as necessary before changing the number of people in the starting circle. Who will win for each variation?

After the students start to get the hang of it, move on to paper cups. Have the cups numbered and have them try to recreate the game they just played. If the students are split up, have each team try different numbers of people in the starting circle and keep track of the number on the cup that is left at the end for each variation on the whiteboard. Collect enough class data before asking: What kind of pattern exists if any? If given a starting number, is there a way to know who will win without playing the game? We recommend collecting data for one to twenty cups in the circle.

Math Background

If the starting number is a power of two, for example $1 = 2^0,\ 2 = 2^1,\ 4 = 2^2, \ldots$, then the first cup will win. The pattern becomes more interesting when we consider numbers that are not powers of two. If the number of cups in the starting circle is one that immediately follows a power of two, for example seventeen immediately follows sixteen, then the final cup is the first odd number after one: three. For each number after that, increase the index of the final cup to the next odd number. Once the next power of two is reached, start all over again. For example, four is a power of two. The last cup for a circle of four is indexed as the first cup. The last cup for a circle of five is indexed as the third cup. The last cup for a circle of six is the fifth cup. The last cup for a circle of seven is the seventh cup. The last cup for a circle of eight is the first cup, since it is a power of two. Now the cycle begins again. The start of the pattern for this problem is summarized below.

Number of Cups In the Starting Circle	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Number of Last Cup (# spaces from the first cup)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Our Experiences & Suggestions

- It was easier for some of our students to understand the game when using themselves rather than the cups. The preference may depend on your group.
- Having them stand around the table and even take victory laps when they correctly guessed the last person that remained standing gave them a good way to channel their end-of-the-day energy. By now we have found that if students can move around rather than sit, they pay more attention to leaders and are more likely to interact with the tasks at hand.
- When we started to keep track of the numbers on the “last cup”, many students got different answers. We had them all try the examples with inconsistencies so that they could focus on working together to get the right answer rather than focusing on the student who had given the wrong answer.
- When we could, we tried to keep track of numbers without letting them all know who produced what so that the process could be anonymous. We did this for similar reasons mentioned above.
- We know that students have some sort of basic understanding of which numbers are powers of two because of previous Circles, so it is possible that this did not impede their ability to see a pattern.

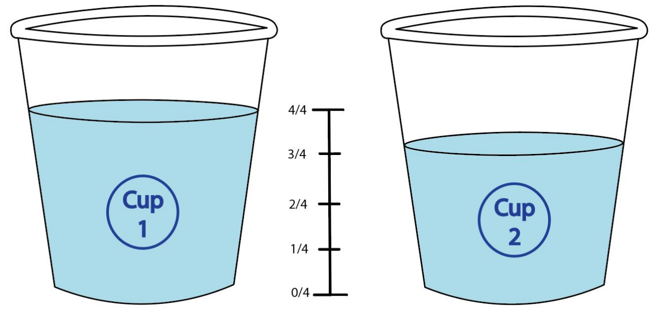
Taking It Further

- How does the problem change if we remove only every third cup? Every fourth cup? Fifth cup?
- How does the problem change if we remove the second cup, then the third cup, then the fourth and so on?
    - What if we start from the first cup every time we increase the cup number removed?

Sources & Additional Resources

The Josephus Problem was created by Flavius Josephus after he found himself in a similar situation during the siege of Yodfat. The history is a little gruesome and involves capture, drawing lots, and execution. For obvious reasons, we adapted the problem. More information on the history can be found on this Wikipedia page. On this page, a detailed mathematical solution can be found as well, but the formatting of the equations is hard to follow.

Water Cups



Keywords

Logic, Estimating, Volume, Fractions

Materials

A set of two cups for each student. One of the cups should be marked for one measuring cup worth of water while the other should be marked for three fourths of a measuring cup. Label these Cup 1 and Cup 2, respectively. Alternatively, each student can bring their own measuring cups for one full cup and one three quarters cup. You will also need a supply of water and bucket(s) to pour out water. We recommend that students have water bottles ready so that they do not have to wait for their share of water.

Problem Statement

You have two cups: Cup 1 can hold a cup of water and Cup 2 can hold three fourths of a cup of water. You can fill each cup up until the mark but no more. You can also dump water out and refill the cups as many times as needed. Can you get exactly half a cup of water in Cup 1? How can you estimate half a cup? Why is this hard? What if the cup is not a perfect cylinder? How can we be sure there is exactly half a cup in Cup 1? If the students struggle with this problem and want a hint, ask them to try to get only one quarter cup of water in Cup 1. How can they double it?

Math Background

There are a couple of ways to solve this problem. The first is to estimate. A student can simply eyeball half a cup in Cup 1. Another way to estimate is using water displacement. They first fill Cup 1 to the mark, then stack an empty Cup 2 into Cup 1 until the marks meet. Since Cup 1 can hold four fourths of a cup and Cup 2 holds three fourths, stacking will displace three fourths of a cup of water out of Cup 1, leaving one fourth in Cup 1. Pour the one fourth cup into Cup 2, refill Cup 1 to the mark, and displace the water again. At this point there should be a fourth of a cup of water in Cup 1 and in Cup 2. Pour the water from Cup 2 into Cup 1. There should now be about half a cup of water in Cup 1. Again, this is just an estimation since we are not taking into consideration the volume that is displaced because of the width of the Cup 2.

To solve this problem without estimating, consider the following steps:

1) Fill Cup 2
2) Dump all the water in Cup 2 into Cup 1
3) Cup 2 should be empty and Cup 1 should have three fourths of a cup of water in it
4) Fill Cup 2
5) Fill Cup 1 using the water from Cup 2, but stop pouring water once Cup 1 is full
6) At this point Cup 1 should be full and Cup 2 should have half a cup of water in it
7) Dump out Cup 1
8) Pour the half cup from Cup 2 into Cup 1
9) Cup 1 has half a cup of water!

Here is another way to solve the problem:

1) Fill Cup 1
2) Using water from Cup 1, fill Cup 2, but stop pouring water once Cup 2 is full
3) Cup 2 should now have three fourths of a cup of water and Cup 1 should have one fourth of a cup of water in it
4) Dump out Cup 2
5) Pour Cup 1 into Cup 2
6) Cup 2 two should now have one fourth of a cup of water in it and Cup 1 should be empty
7) Fill Cup 1
8) Pour Cup 1 into Cup 2 until Cup 2 is full. Once again, stop pouring out water as soon as Cup 2 is full.
9) Cup 1 has half a cup of water!

Our Experiences & Suggestions

Note: this problem was not conducted in person due to the COVID-19 pandemic. Instead, we sent students a video summarizing the problem and let them go about it on their own, unsupervised.

- We believe that it would not take the full 40 minutes. Some ways to add to the Circle are mentioned in the *Taking it Further* section.
- We suggest using measuring cups if possible, this way students are sure not to go past the appropriate mark.
- Having a half cup measuring cup is a good way for them to check their work. Because of human error, however, it is possible that even if the student solves the problem they still might not perfectly fill up the half cup. If you choose to use the half cup as a check, make sure you communicate the flaw to the students so they are not discouraged if it is not completely filled.

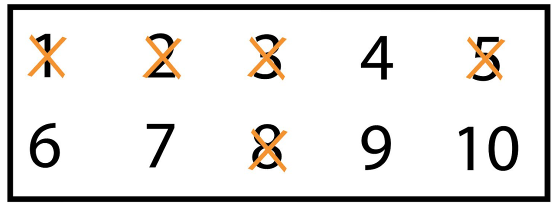
Taking It Further

- How can you try to solve this problem without using physical cups? What sort of actions are done on the cups? What kinds of equations can we write using these actions? How can we represent the cups in these equations?
- What if our cups were bigger? What if we had a 5/4 cup and a 7/4 cup?
- For which cup combinations is it possible to end with half a cup left over? Which are impossible?

Sources & Additional Resources

Eric Egge, an unofficial member of our group, gave us this idea. This problem is well covered on the internet and can be found by searching "Water Jug Problem." One interesting solution translates it into a computer science problem by creating a graph (a collection of nodes connected by lines) and running a search algorithm.

Guess What?



Keywords
Search Algorithms, Factors, Primes, Numerical Relationships

Materials
Two sheets with numbers one through ten. In addition, about four sheets of paper per student. Each paper should have the numbers one through a hundred. Students should also have scrap paper so they write down their secret number and keep it hidden from the student they are playing against.

Problem Statement
"Ever heard of *Guess Who?,* but with numbers? I bet I can guess your secret number first!" In the original *Guess Who?,* two players have the same set of 24 characters in front of them. Each one picks a secret character from a stack from the same set of 24. To win, a player must be the first to guess which character their opponent has selected. The players go back and forth asking questions such as "Does your character have brown hair?" or "Does your character have blue eyes?" Depending on the answer, certain characters are eliminated from consideration.

For this version, start by asking for a volunteer and have the rest of the students team up with them. The leader should keep one of the sheets numbered one through ten and hand the volunteer the other. Together the students should decide on their secret. Make sure they write it down on a piece of scrap paper so that they do not forget it or change it. The volunteer and the Circle leader should take turns asking each other questions to help eliminate the wrong numbers. The students can help the volunteer come up with questions. Some examples:
"Is your number even?"
"Is your number prime?"
"Is your number a Fibonacci number?" (see first note in *Our Experiences & Suggestions*)

Eventually, one of the two players will figure out what the other's secret number is and will win the game. After this example, each student should receive a sheet with the numbers one through a hundred so they can break off into pairs and play with each other. The first round will give them the opportunity to ask several types of questions about numbers. Encourage them to ask weird questions such as the Fibonacci example above or "Is your number a triangular number?" After their first round, come back together to discuss how their games went. Ask them what kinds of questions they asked and why? How many questions did they need to win? Or what is the fastest way to guess the right number? How many kinds of questions are there? Have them share their thoughts, then play a few more rounds to explore these questions, and regroup again to see if their answers have changed.

Math Background
The math background for this lesson depends on the sorts of questions that are asked during the game. Questions can range from searching algorithms, to factorization, to

sequences and series. One of the fastest ways to guess the correct number is using binary search. Here is an example set of questions for the numbers one through ten:

- "Is your number greater than five?" *Answer: no*
- "Is your number greater than seven?" *Answer: no*
- "Is your number greater than six" *Answer: no*
- "Is your number smaller than six" *Answer: yes*
- The secret number is five

Notice that in this example five is not greater than five. In our binary search, we are considering relationships strictly greater than or less than, not equally to. In traditional binary search, the algorithm asks if it is either greater, smaller, or equal to, and thus would stop at our first example question. In the game *Guess Who?,* however, only one question can be asked at a time and therefore we can only ask strictly greater than/less than questions. Back to binary search, the idea is that with each question you are eliminating roughly half the set of possible answers. An additional video on this searching algorithm, along with other searching algorithms, is provided in the *Sources & Additional Resources* section. Other numerical relationships to inspire questions are provided in this section as well.

Our Experiences & Suggestions

Note: this problem was not conducted in person due to the COVID-19 pandemic.

- When asking about Fibonacci numbers, for example, always ask if other students know what this means before explaining it yourself. It will give them the opportunity to learn to communicate mathematics to others. It also serves as a fun tangent to the Circle.
    - Similarly, there are many other relationships that exist and can be used in the game. You can also encourage students to invent their own.
- Sharing ideas is essential to this Circle. It can be easy for students to play the game without thinking about why they are choosing some questions over other questions. Let them take their time to share their sample questions, but also give them time to explain why they chose them and how it might help them win the game.

Taking It Further

- What secret numbers are the best to pick?
- Remove random numbers in the set of one through a hundred. How might this change the kinds of questions you ask your opponent?
- What if you had to pick two secret numbers? The first is the number that the opponent must guess, same as before. The second, however, is one that you cannot eliminate on your own board. How should you pick that number? What sort of restrictions do different numbers pose? What if the opponent could also win by guessing the second number? How might they figure it out?

Sources & Additional Resources

Guess Who? is a game owned by Hasbro and the rules can be found here.
We also recommend checking out fun number patterns, sequences/series, and other relationships online to show the students and to use as questions for the game. We also recommend looking into different searching algorithms, especially binary search.

Additional Circle Ideas

We only had about twenty weeks with our students, and because of this, we were left with a significant number of Circles that we did not have a chance to lead and write about. These potential Circle ideas are listed in this section, and we encourage you to take a look at the resources we cite and develop your own plan for a Circle.

Clock Hopping

This topic comes from the Global Math Circle. The question is this: if you are taking steps around the numbers of a clock, how big do your steps need to be to make sure you land on every number at least once? If you step one number at a time, you will land on every number, but step two, three, or four numbers at a time and you will not have as much luck. Why do some numbers work and others do not? What if the clock has more or fewer numbers on it? This question centers around modular arithmetic, about which there is an excellent Wikipedia article.

Color Triangles

This is an activity from the Julia Robinson Mathematics Festival, which at the time of this writing can be found here. The Circle would center around special triangles made up of blue, red, and yellow colored dots. These triangles are generated from a single row of colored dots according to a fairly simple rule: if a pair of dots are the same color, the dot below them is also that color; if they are different colors, the dot below them is the third color. Although the rule seems simple, the question is not: given a row of colored dots, can you figure out what color the tip of the triangle will be, without generating the whole triangle?

Euler Squares

This is a puzzle almost like Sudoku. Start by taking out all the aces, jacks, queens, and kings from a deck of cards. Your goal is to place the cards in a four-by-four grid such that every row, column, and diagonal has one of each card (ace, jack, queen, and king), and every row, column, and diagonal has one of each suit (clubs (♣), diamonds (♦), hearts (♥), and spades (♠)). There are many solutions to this puzzle. We recommend that you try it out on your own before watching this Numberphile video that shows how you can solve this puzzle.

Gale-Shapley

The Gale-Shapley algorithm is famous for its application to a fictional marriage problem, where the goal is to arrange the most suitable and “stable” marriages based on people’s preferences, and it is therefore also known as the solution to the Stable Marriage Problem. This algorithm was first introduced as a solution to a college admissions problem, as seen here, before transitioning to the marriage problem. For this Circle, students would be given a list of n people and their preferred college or spouse. The students would also be given the list of colleges or spouses and their preferred applicants or spouses, respectively. Then the Circle would revolve around students trying to find what pairs of spouses or college/applicant would result in the most stable combination. Here is a Numberphile video that shows how this could be done.

Logic Gates

Logic gates are a fundamental concept in electrical and computer engineering. Their versatility allows for students to explore creations like binary adders and subtractors, and also trace through executions of already-made arrangements of logic gates. That could lead to students also discovering truth tables, an essential tool in boolean algebra. As the name suggests, students will familiarize themselves with logic, and hopefully be surprised that everything in computing is built upon these fundamental units. To learn more about logic gates, refer to this resource.

M&M Distributions

Throw ten M&Ms on a table and count up how many have the "M" visible. Do this over and over again, all the while plotting the number of "M"s visible. What will this graph look like? What number of "M"s do the students expect to see each time? This exercise is a classic way to introduce the concepts of statistics and distributions. Students could also plot the colors that come in an M&M package.

Paper Dolls

If you have some Disney princesses, but their dresses are all mixed up, how do you fix them so that each princess is wearing her own dress? For example, if Cinderella is wearing Snow White's dress, how do you fix that? This is a good way to introduce the students to permutations! You can read about permutations here. Loosely speaking, a permutation is a function that maps a set to itself while making sure that each element in the domain gets mapped to a unique element in the codomain, and each element in the codomain has an element in the domain that is mapped to it. What we wanted to do here was to have the students look at a permutation that maps each dress to the right princess, while writing the steps of constructing the map in detail. Note that since a permutation maps a set to itself, we are assuming that the set of dresses is equivalent to the set of princesses (each princess is the same as her dress). You can find another resource here.

Sizes of Infinity

To the surprise of many, usually undergraduates in their first math structures course, not all infinities are equal. Specifically, the size of the set of all real numbers is not equal to the size of the set of all integers. Intuitively, we get a sense that there are more real numbers than integers, but how do we know that? And how do we know that the rationals are the same size as the integers when it feels like they are different? These questions concern the definition of equality and how we know if two sets are "equal in size" to each other. To learn more, refer to the basics of set theory and Cantor's diagonal argument for the proof that the size of the set of all real numbers is not equal to the size of the set of all integers.

Tiling a Chessboard

Present the students with a chess board and ask them if they can cover the whole board with dominoes. Each domino covers two adjacent squares on the chessboard. This is a classic algorithmic problem and there are many examples online of possible problem statements. Students can try to figure out which size boards are tileable and which are not. The boards can be complete, or some squares can be labeled such that no dominoes can cover them. Depending on which tiles are labeled, some boards are not tileable.

Further Reading

This section is dedicated to the authors and creators who have been especially helpful and inspirational to us in the creation of our Circles. If you want to learn more about Math Circles or explore other potential Circle topics, we encourage you to take a look at these resources.

We took much of our inspiration from the book *Out of the Labyrinth: Setting Mathematics Free* by Robert and Ellen Kaplan. In many ways, we see this as the foundational text for our Math Circle.

We took several ideas from the book *Solve This* by James Tanton, which includes many math activities for groups similar to ours that are accessible to a wide range of students. We highly recommend it for additional Circle inspiration.

The YouTube Channel Numberphile was a valuable resource for our group. Their videos illustrate many different mathematical puzzles, largely revolving around numbers and logic. Several of their videos offer detailed explanations of puzzles presented in this book, and we are confident that other videos of theirs could also be adapted into Circles.

Keyword Index