Algebraic Curves and Dedekind Domains

Prerequisites: Math 342
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Terms: Fall/Winter
Description: By a unique factorization domain (UFD) we mean an integral domain in which every element factors uniquely into a product of irreducibles. The integers $\mathbb{Z}$ is an example of a UFD since its irreducible elements are precisely the prime numbers (up to sign). Consequently, the property of $\mathbb{Z}$ being a UFD is equivalent to the famed Fundamental Theorem of Arithmetic which states that every integer has a unique prime factorization (up to order and sign). In the 1800s, mathematicians began to study rings that closely resembled the integers and for a time believed that these rings were UFD's. An example of such a ring which is not a UFD and closely resembles $\mathbb{Z}$ is the ring $\mathbb{Z}\left[\sqrt{-5}\right] = \left\{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\right\}$ where we have the following two factorizations of 6 into irreducibles of $\mathbb{Z}\left[\sqrt{-5}\right]$:

$$\left(1+\sqrt{-5}\right)\left(1-\sqrt{-5}\right) = 6 = 2 \cdot 3.$$

In 1871, Richard Dedekind showed that an analog of the Fundamental Theorem of Arithmetic does hold in rings such as $\mathbb{Z}\left[\sqrt{-5}\right]$ if one focuses on ideals rather than elements of the ring. In his honor, rings closely resembling $\mathbb{Z}$ are known today as Dedekind Domains and his Theorem states that if $\mathfrak{a}$ is a nonzero ideal of a Dedekind Domain D, then the ideal factors uniquely into a product of prime ideals of D (up to order).

In this comps, we will develop the necessary commutative algebra to define Dedekind Domains and prove Dedekind's Theorem on the uniqueness of factorization of nonzero ideals. This content will allow us to consider the theory of algebraic curves and prove one of the main results for this comps: The coordinate ring of an algebraic curve (under suitable assumptions) is a Dedekind Domain. To motivate algebraic curves, consider your favorite polynomial $f(x)$. Next, let I be the principal ideal generated by $y - f(x)$ in the ring $\mathbb{C}[x, y]$. The coordinate ring of $f(x)$ is then defined to be the quotient ring $\mathbb{C}[x, y]/I$. As we will see, the (complex) points satisfying $y = f(x)$ are in one-to-one correspondence with the prime ideals of $\mathbb{C}[x, y]/I$. Therefore studying the points on the curve $y = f(x)$ is equivalent to studying the prime ideals of the quotient ring $\mathbb{C}[x, y]/I$.

Sources: This comps will have selected readings from the following three books:

1. Introduction to Commutative Algebra by M.F. Atiyah and I.G. MacDonald
2. Algebraic Number Fields by Gerald Janusz
3. An Invitation to Arithmetic Geometry by Dino Lorenzini