Topology of Graph Configuration Spaces

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Prerequisites: None, but if you've taken Topology or Abstract Algebra, you might find this particularly attractive.

Description: Consider the situation in which n robots move throughout a factory along a connected system of one-dimensional tracks, avoiding collisions. Mathematically, we can view the tracks as a graph Γ and the robots as n distinct points on Γ. The space of all configurations of n robots on Γ is called a *Graph Configuration Space*. If the robots are labeled $1, 2, \ldots, n$, we say that the robots form an *ordered* configuration. For example, Figure 1 (left and middle) shows two distinct ordered configurations. Here, we have $n = 3$, and the robots are represented as small squares on the graph. There are also *unordered* configurations, in which the labels of the robots are ignored; see Figure 1 (right).

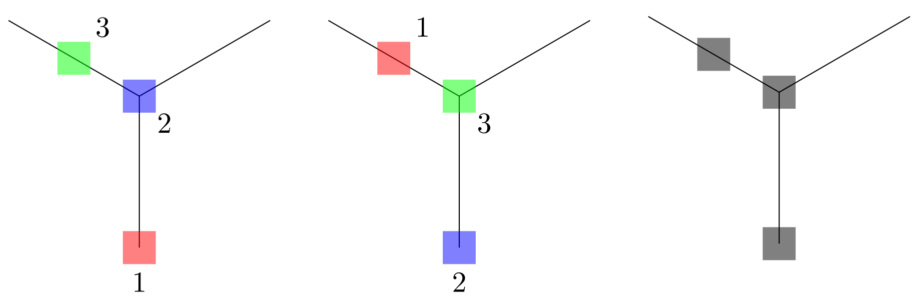

Figure 1: Two distinct ordered configurations (left and middle); an unordered configuration (right)

There are *discrete* versions of these configuration spaces, which can be modeled using a *cube complex*. This means we can get a picture of the entire configuration space by "gluing together" cubes of various dimensions. For example, Figure 2 shows the cube complex which models the unordered configuration space of three robots on the graph in Figure 1.

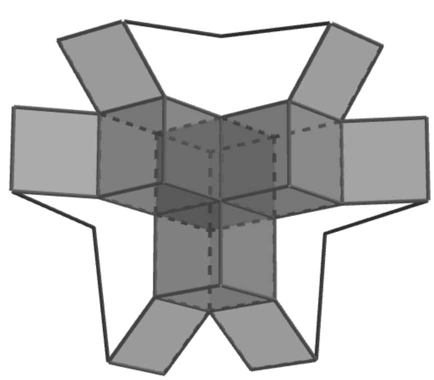

Figure 2: The unordered configuration space of three robots on the graph in Figure 1

In other words, every unordered discrete configuration of three robots on the graph in Figure 1 corresponds to exactly one point in the cube complex in Figure 2. The ordered configuration space would have more cubes, since there are more distinct configurations.

There are many interesting topological questions that can be asked about these configuration spaces, such as "is the configuration space homemorphic or homotopy equivalent to a familiar space?" If you're familiar with these concepts, you might recognize that the cube complex in Figure 2 is homotopy equivalent to a wedge of three circles. We can also ask questions about algebraic structures on the configuration spaces, such as the fundamental group and/or homology groups.

Furthermore, there are also versions of these configurations spaces in which the robots are labeled, but some of the robots have the same label, as in Figure 3. These configuration spaces would sit somewhere between the ordered and unordered configuration spaces. It doesn't appear that many people have looked into this type of configuration space, so there is potential to discover some new results.

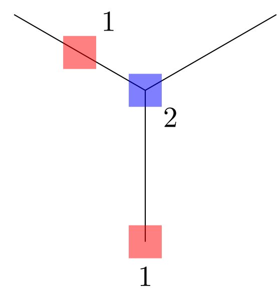

Figure 3: A configuration in which multiple robots share the same label

The project should be largely self-contained. We will start by building the necessary topological background (including the notions of homeomorphisms, homotopy equivalences, and cell complexes) before applying these ideas to graph configuration spaces. If we decide to discuss the fundamental group and/or homology groups, we can learn about abstract algebra as we go along.