Heights and Dynamics

Supervisor: Rafe Jones

Terms: Fall 2020 and Winter 2021

Prerequisites: Math 236 or the equivalent. Some experience with number theory is helpful but not required.

Description:

Heights:

How “complicated” is a rational number? One notion of complexity is absolute value, but there are pairs of rational numbers, such as 1 and 1/1537, where the smaller number seems harder to describe, and hence more complicated. We can define the height $H(q)$ of a rational number $q = a/b$ (written in lowest terms) as the maximum of $|a|$ and $|b|$. Thus $H(1) = 1$ and $H(1/1537) = 1537$. This captures the idea of how complicated the number is to describe. One of the most important properties of the height is that for any constant $B > 0$, the set

$$\{q \in \mathbb{Q} : H(q) \leq B\}$$

is a finite set. (Note that this is not true if we replace $H(q)$ by $|q|$). The height turns out to be a remarkably useful tool in a variety of contexts, and we will explore its applications to dynamics in this project. As a side note, which the project could also potentially explore, it's interesting to think about how the definition of height could be extended to algebraic numbers, like $\sqrt{2}$.

Dynamics:

Suppose we have a rational function $F(x) = p(x)/q(x)$, where p and q are polynomials with integer coefficients. The field of dynamics studies *orbits* of F, that is, sets of the form

$$\{q, F(q), F(F(q)), F(F(F(q))), \ldots\} \tag{1}$$

where q is a rational number. We often denote the composition of F with itself n times by F^n, and we take $F^0(x) = x$.

In general, the set (1) will be infinite. But of particular interest are those q for which the orbit is finite. This can only happen when there are $m \neq n$ with $F^m(q) = F^n(q)$; in this case we call q a *pre-periodic* point for F. For instance, the point $q = 1$ is pre-periodic for $F(x) = \frac{x^3-25x+12}{12}$ because $F^5(q) = q$ (here $m = 5$ and $n = 0$). Its orbit is

$$\{1, -1, 3, -3, 5\}$$

A fundamental question is:

Question 1. *Can F have infinitely many pre-periodic $q \in \mathbb{Q}$? If so, characterize the F that do.*

Using heights, we will completely answer this question. It leads to another one:

Question 2. *If F has finitely many pre-periodic $q \in \mathbb{Q}$, how can we find them?*

We'll discover a (fast!) algorithm for doing this. And then we may ponder the following:

Question 3. *How does the height of a number change as we repeatedly apply F?*

This will lead us to the notion of the canonical height associated to F. We might have time to think about two famous unsolved problems:

Question 4. *Let $F_c(x) = x^2 + c$, where $c \in \mathbb{Q}$. Is there a bound k (independent of c) such that F_c has at most k pre-periodic points? Is there a bound k (independent of c) such that the canonical height associated to F_c at a point q must be at least k?*