**Bernoulli Numbers and Irregular Primes**

Fall/Winter Comps with Professor Caroline Turnage-Butterbaugh

**Background.** The **Bernoulli numbers** $B_n$ are a sequence of rational numbers that can be defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

For $n \geq 2$, the Bernoulli numbers can also be defined recursively

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k,$$

as well as by the complex contour integral

$$B_n = \frac{n!}{2\pi i} \oint \frac{z}{e^z - 1} \frac{dz}{z^{n+1}},$$

where the contour can be taken to be the unit circle with positive orientation. It is known that $B_0 = 1$ and $B_n = 0$ for every odd integer $n > 1$. The first few Bernoulli numbers $B_n$ are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730}.$$

The Bernoulli numbers occur often in analytic number theory. For example, one can compute special values of the Riemann zeta-function in terms of Bernoulli numbers. Indeed, if $n \geq 1$ then one can prove that

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

Note that this shows that the Riemann zeta-function has infinitely zeros! But don't get too excited – these are the so-called *trivial zeros* of $\zeta(s)$, as opposed to the *non-trivial zeros*, which are the subject of the Riemann Hypothesis.

An odd prime $p$ is said to be **B-irregular** if $p$ divides the numerator of at least one of the Bernoulli numbers $B_2, B_4, \ldots, B_{p-3}$, and **B-regular** otherwise. The first ten B-irregular primes are

$$37,\ 59,\ 67,\ 101,\ 103,\ 131,\ 149,\ 157,\ 233,\ 257,\ 263.$$

In 1850, Kummer proved that Fermat's Last Theorem is true for all prime exponents $p$ which are B-regular. Kummer conjectured that there are infinitely many B-regular primes, but this has yet to be proven. It *is* known, however, that there are infinitely many B-irregular primes.

**Possible Lines of Investigation.** In this project we will explore the distribution of the B-irregular primes. We will also explore some generalizations of the B-irregular primes, in particular those defined in terms of Euler numbers and Genocchi numbers. Depending on the backgrounds and interests of the group, we can study relationships to Artin's Primitive Root Conjecture, define and study our own notion of irregular primes, etc.

**Prerequisites:** We have a lot of flexibility in this project. The only required prerequisite is Math 312 (Elementary Number Theory) OR Math 342 (Abstract Algebra I). Certain background will lead the project in different directions, but those directions will be decided upon *after* the group is formed. (For example, if the group members have seen Math 261/Math 361 (Complex Analysis), then there are certain directions we can take.)