ORBITS OF NONCROSSING TREE PARTITIONS (Winter/Spring)

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A *set partition* of $[n] = \{1, \ldots, n\}$ is a collection $\mathbf{B} = \{B_1, \ldots, B_k\}$ of disjoint subsets of $[n]$ called *blocks* where $\bigcup_{i=1}^k B_i = [n]$. Such a set partition $\mathbf{B}$ is called a *noncrossing partition* if there does not exist $1 \leqslant i < j < k < \ell \leqslant n$ where i, k belong to one block of $\mathbf{B}$ and j, ℓ belong to some other block of $\mathbf{B}$. This condition can best be understood pictorially as in Figure 1. A set partition can be drawn as a collection of arcs connecting consecutive elements of its blocks where the arcs must lie above the elements of $[n]$; a set partition is noncrossing if the arcs may be drawn in a way where no pair of arcs intersect in their interiors.

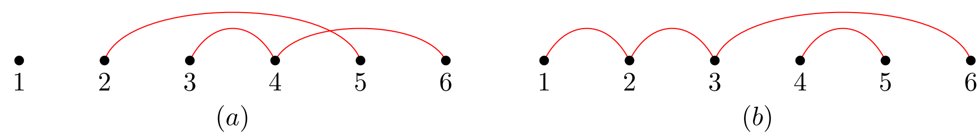

FIGURE 1. In (a), we show the set partition $\{\{1\}, \{2, 5\}, \{3, 4, 6\}\}$ of $[6]$. In this partition, 2 and 5 are in one block, 4 and 6 are in another block, and their arcs cross exactly because $2 < 4 < 5 < 6$. In (b), we show the set partition $\{\{1, 2, 3, 6\}, \{4, 5\}\}$ of $[6]$. Of the two, only the latter is a noncrossing partition.

Noncrossing partitions appear in many areas of mathematics, including combinatorics, representation theory, and probability. They also encode combinatorial and algebraic information about the symmetric group. Additionally, the number of noncrossing partitions of $[n]$ is the often-occurring *Catalan number*:

$$\frac{1}{n+1}\binom{2n}{n}.$$

The broad goal of this project is to understand enumerative and algebraic aspects of a generalization of noncrossing partitions called *noncrossing tree partitions* [2]. These noncrossing tree partitions are defined from the initial data of a tree T embedded in a disk (see Figure 2(a)). The vertices of T that are not incident to the boundary of the disk are called *interior vertices*. A *noncrossing tree partition* $\mathbf{B}$ is a set partition of the interior vertices of T where there exists a collection of pairwise noncrossing *red admissible curves* such that any two interior vertices in the same block of $\mathbf{B}$ are connected to each other by a sequence of these red admissible curves and two interior vertices in distinct blocks of $\mathbf{B}$ are not connected by any sequence of red admissible curves (see Figure 2(b)).

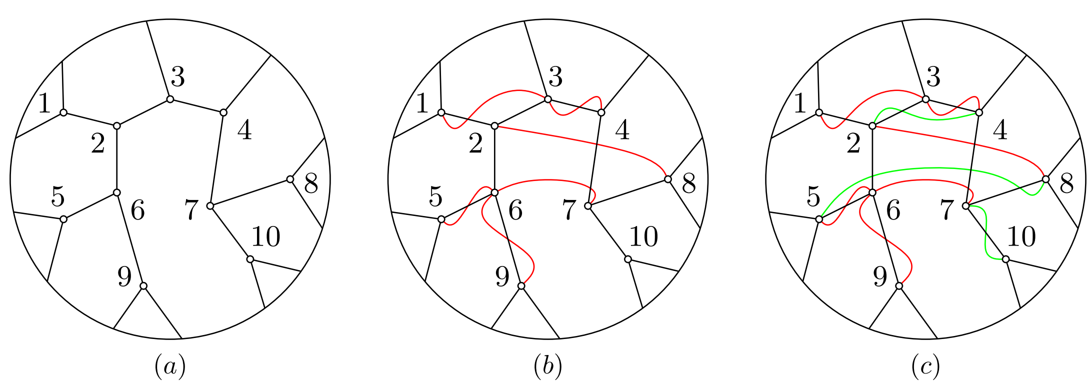

FIGURE 2. In (a), we show a tree T whose set of interior vertices is $\{1, \ldots, 10\}$. In (b), we show the noncrossing tree partition $\mathbf{B} = \{\{1, 3, 4\}, \{2, 8\}, \{5, 6, 7, 9\}, \{10\}\}$. In (c), we show the Kreweras complement $\mathrm{Kr}(\mathbf{B}) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 8\}, \{6\}, \{7, 10\}, \{9\}\}$.

There are technical details about the definition of a noncrossing tree partition and about subsequent definitions that we will not explain in their entirety here. One the first tasks of this project will be to fully understand these definitions. For now, we will focus on some examples.

One of the central objects in this project is the *Kreweras complement* of a noncrossing tree partition. Given a noncrossing tree partition **B**, there is a unique collection of *green admissible curves* such that when these are superimposed with the red admissible curves for **B**, one obtains a noncrossing tree on the interior vertices (after removing the black edges of T). The set partition of the interior vertices determined by the green admissible curves is the *Kreweras complement* of **B**, denoted $\mathrm{Kr}(\mathbf{B})$ (see Figure 2(c)).

We say that two noncrossing tree partitions **B** and $\mathbf{B}'$ are in the same *orbit* if $\mathbf{B}' = \mathrm{Kr}^m(\mathbf{B})$ for some $m \geqslant 0$. The main question of the project is the following: when are **B** and $\mathbf{B}'$ in the same orbit? See Figure 3 for an example of one orbit of noncrossing tree partitions; the tree shown there also has two other orbits. In addition to the goal of answering this question, we will want to work on some of the following related problems (depending students' interests).

(1) What are the sizes of the different orbits? Use this to determine the order of the Kreweras complement operator.
(2) Find formulas or recurrences for the number of noncrossing tree partitions for certain families of trees (see Figure 4).
(3) Investigate connections between Kreweras complementation and the related representation theory.
(4) Develop software to be used for generating examples of noncrossing tree partitions and orbits of noncrossing tree partitions.

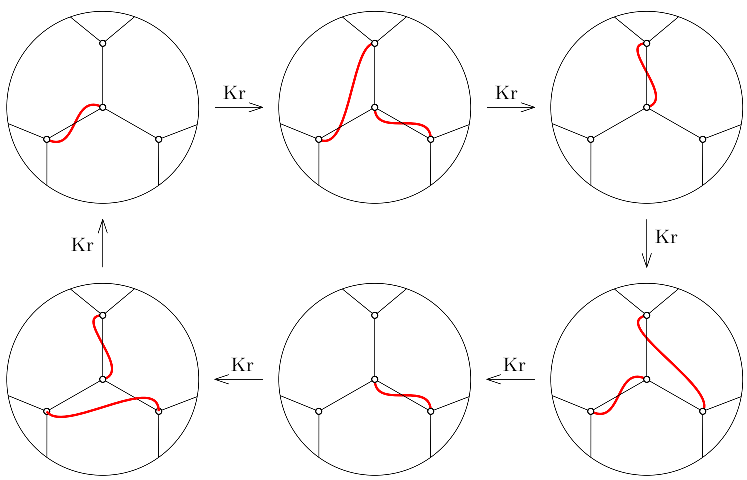

Figure 3. Here, we show one orbit of noncrossing tree partitions for the tree appearing above. There are two other orbits for this tree, another of size 6 and a third a size 2.

Prerequisites

It would be useful to have taken Math 342 (Abstract Algebra I) or Math 333 (Combinatorial Theory), but these are not required. Topics studied in Math 342 will be especially useful if students are interested in learning about the related representation theory.

References

[1] A. Clifton and P. Dillery. On the lattice structure of shard intersection orders. *http://www-users.math.umn.edu/ reiner/REU/CliftonDillery2016.pdf*, 2016.

[2] A. Garver and T. McConville. Oriented flip graphs of polygonal subdivisions and noncrossing tree partitions. *J. Combin. Theory Ser. A*, 158:126–175, 2018.

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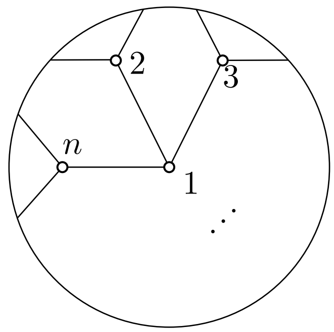

FIGURE 4. Consider the tree T_n with $n \geqslant 4$. The tree T_n has n interior vertices and $n-1$ edges between those vertices arranged in a cycle around vertex 1. The interior vertices $2, \ldots, n$ each belong to two edges connected to the boundary. It was show that the number $|NCP(T_n)|$ of noncrossing tree partitions of T_n satisfies the recurrence relation $|NCP(T_n)| = 2|NCP(T_{n-1})| + |NCP(T_{n-2})|$ [1, Proposition 7.4]. This sequence of numbers is known as the Pell–Lucas numbers.