Explorations in complex dynamics – Mandelbrot mysteries and moduli spaces

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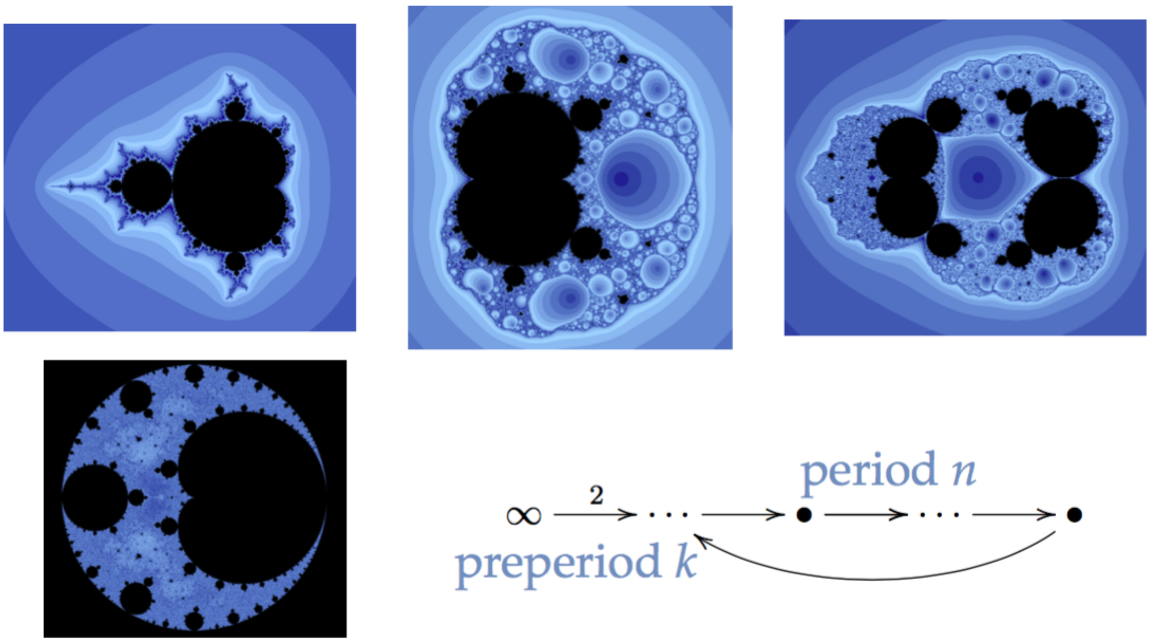
Terms: Winter and Spring 2020

Prerequisites: Math 321 or equivalent. You need some experience working with ideas of open and closed sets, and uniform and pointwise convergence. You might have gotten this experience by taking certain courses in Budapest, in an REU, or having done some reading on your own. The project will involve complex analysis, but it's fine if you haven't had experience with that.

Description:



This project is an introductory tour of complex dynamics – the study of iterating a rational function with complex coefficients. The central object of study is the orbit of a point under iteration: the sequence $z_0, R(z_0), R(R(z_0)), \ldots$, where R is a rational function and z_0 is a complex number. Complex dynamics seeks to answer questions about orbits. Which orbits behave in a regular, structured way? Which orbits behave unpredictably?

For instance, let $R(z) = z^2 - 1$ (recall that polynomials are special cases of rational functions). The orbit of 0 is

$$0 \to -1 \to 0 \to -1 \to \cdots$$

This orbit is a two-cycle, which is quite regular. Interestingly, if you start with $0.1 + 0.1i$, here are the first few terms of the orbit:

$$0.1 + 0.1i \to -1 + 0.02i \to -0.0004 - 0.04i \to -1.00159984 + 0.000032i \to \cdots$$

So the orbit of $0.1 + 0.1i$ appears to be converging to the orbit of 0. This is another example of regular behavior of an orbit.

Now let $R(z) = z^2 + i$. Then 0 has the orbit

$$0 \to i \to (-1+i) \to -i \to (-1+i) \to -i \to \cdots$$

Thus after two iterations, 0 maps into a two-cycle. However, if you start with $0.1 + 0.1i$, now you get

$$0.1 + 0.1i \to 1.02i \to -1.0404 + i \to 0.08243216 - 1.0808i \to -1.16133 + 0.82181i \to \cdots$$

This orbit appears to be getting *farther* from the orbit of 0, and it's not at all clear what its long-term behavior will be.

The project will begin by getting everyone up to speed on the necessary complex analysis, and then we'll begin studying the dynamics of the polynomials $z^2 + c$, where c is a complex number. This family turns out to contain the dynamical information for *every* degree-two complex polynomial, since every degree-two polynomial is linearly conjugate to one of this form, and linear conjugation preserves all dynamical information. Thus the set $\{z^2 + c : c \in \mathbb{C}\}$ is the *dynamical moduli space* of degree-two polynomials. It turns out to have a very special subset called the Mandelbrot set, which we'll spend significant time getting to know. It's the black set in the upper-left picture at the beginning of this description.

Even more interesting than the family z^2+c is the dynamical moduli space of degree-two rational functions. The set $\{z^2 + c : c \in \mathbb{C}\}$ is just one "slice" of this moduli space, and we may explore other interesting slices. A few of those can be found in the other pictures at the beginning of this description. You'll have the opportunity to learn how to make pictures like this with the program fractalstream.

If you'd like to get a more precise idea of how the project will begin, we'll start by working through at least the first two sections of this set of notes:

`https://math.iupui.edu/~roederr/complex_dynamics.pdf`