Baire Category, Automatic Continuity, and Kuratowski-Type Problems

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Comps Term: Fall 2019/Winter 2020

Prerequisites: Real Analysis I or Topology. It is helpful if you have studied groups before, but not necessary as we can learn the basics from scratch.

Problem Description: *Part I: Baire Category and Automatic Continuity*

Question 1: Suppose a map $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Must f be a linear map, i.e. must $f(x) = ax$ for some a?

Question 2: Consider the additive group $\mathbb{R}$ and the additive group $\mathbb{R}^2$. Are they isomorphic?

It may not be immediately obvious, but these two questions are intimately related. If the answer to the first question is "Yes," then every group homomorphism $f : \mathbb{R} \to \mathbb{R}$ is continuous. If the answer to the second question is "Yes," then there exists a discontinuous group homomorphism $f : \mathbb{R} \to \mathbb{R}^2$, and hence a discontinuous group homomorphism $f : \mathbb{R} \to \mathbb{R}$. Both answers are surprising and subtle: it turns out that they depend on the underlying axioms of set theory that we assume in the background.

For the first part of this comps project, we will study some of the technology needed to understand the phenomenon of *automatic continuity*, i.e. minimal conditions which force homomorphisms (maps which satisfy a functional equation) to be continuous. We mainly want to learn the machinery of *Baire category*: meager, nonmeager, and comeager sets; sets with the Baire property; the Baire category theorem and its relationship with Axiom of Choice; sets without the Baire property; Vitali sets; Bernstein sets; and more as time permits. You are likely to enjoy this if you like "pathological objects" in mathematics, i.e. you like to try to visualize things that are difficult (perhaps impossible) to visualize.

If you are unfamiliar with it, you should think of Baire category as like a topological analogue of measure theory, where meager sets play the role of measure zero sets. But it is easier to define and understand than Lebesgue measure, and it is applicable in spaces where "good" measures like Lebesgue measure are not necessarily defined. Also, Baire category is "orthogonal" to Lebesgue measure in some surprising ways—for instance, we will see that there are many ways to decompose $\mathbb{R}$ into a disjoint union $A \cup B$ where A has measure 0 and B is meager.

For the first part of this project, we will meet twice a week, and it will take the form of a directed reading. I will ask students to take turns giving brief presentations on certain assigned readings. I expect we will have a good background after a few weeks, and then we can decide as a group whether to move on to Part II as described below, or go in another direction.

Part II: Kuratowski-Type Problems (if time and interest dictate)

In the 1920's Kuratowski proved the following well-known and amusing theorem: if you start with a subset $A \subseteq \mathbb{R}$ and repeatedly apply the closure operator, and the set complementation operator, in any order as often as you want, the largest number of distinct subsets it is possible to obtain is 14.

Other authors have expanded upon this result in clever ways. In the 1980's Soltan proved that using the closure and complementation operators, and in addition the boundary operator, it is possible to obtain 34 distinct sets. In their 2019 comps project, three Carleton students and I proved that using these same three operators, together with another closure operator and another boundary operator from a second topology on $\mathbb{R}$, it is possible to obtain 120 distinct sets!

During Part I of this project, we will encounter and learn about several natural and important set operators associated with Baire category. It would be interesting to try to pose and solve problems of Kuratowski type using these operators: how many distinct subsets of the reals may be obtained by applying certain collections of these operators, in any order as often as you want, to a single initial set? This can become a very fun combinatorial-type problem, with potential for genuinely new results.