On Torus Knots

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Contents

1 An Introduction to Knot Theory

1.1 The Basics

Knots arise intuitively when we take a string and creates a tangle in the middle by looping one end through and around the other, and then ties the two ends together. Knot theory is then exactly what it sounds like: the mathematical theory of such knots. What is surprising at first glance is that such an easy idea (take a string, tie a knot in it...) can lead to deep and challenging mathematics that relies on important results from a number of fields. In this paper we explore one such subfield of knot theory, torus knots, and present results, some that are new and some that are well known. Before we begin, however, we must cover some important knot theory basics. What follows in this section and the next is adapted from [1] and [4].

Definition 1. *A* **knot** *is a piecewise linear subset of* $\mathbb{R}^3$ *that is homeomorphic to* $\mathbb{S}^1$.

This definition is the mathematical definition of what was defined above. We can see that this definition requires our knot to be closed and a 1-manifold (since $\mathbb{S}^1$ is), but aside from that, not much else is required. We can generalize this definition to make our work of a broader scope:

Definition 2. *A* **link** *of* m *components is a subset of* $\mathbb{R}^3$ *that consists of* m *knots.*

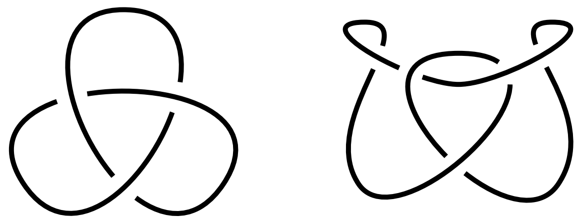
Intuitively, a link is just several knots considered together. They may be separate, (i.e. each knot of the link may be able to be placed in a sphere such that the knot contained in the interior of the sphere, the rest of the link is in the exterior of the sphere, and the link does not intersect the boundary of the sphere.), or they may not. It hopefully is clear that these definitions are very general, and allow for immensely complicated knots and links. Throughout the rest of this paper, we will refer to links primarily to maintain full generality of our results.

When one examines a link, one of its most immediate characteristics is its deformability: different parts of the link can be pulled, twisted and looped etc., and as long as no cuts or gluings are made on the link, its basic structure will be preserved. Mathematically, this idea is expressed via link equivalence and projection.

Definition 3. *Given two links* L_1 *and* L_2, L_1 *is* **equivalent** *to* L_2 *if there is an orientation preserving piecewise linear homeomorphism* $h : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $h(L_1) = h(L_2)$.

Definition 4. *The* **projection** *of a link* L_1 *is the projection of* L_1 *or any equivalent* L_k *into the plane* $\mathbb{R}^2$, *with the information about crossing order preserved.*

From the definition of projection it is clear that one link can have an infinite number of different projections. Two different projections of the trefoil knot are shown in Figure 1.

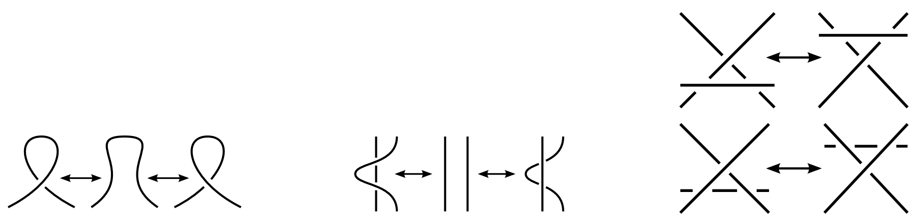


(a) A projection of the trefoil knot.

(b) A different projection of the trefoil knot.

Figure 1: Two projections of the trefoil knot.

To move between different projections of a knot we employ the three Reidemeister moves, shown below in Figure 2 . In each, we assume the projection of the knot remains unchanged except for the change depicted. The first Reidemeister move (Figure 2a) allows a twist to be added to or taken away from the projection of a knot; the second Reidemeister move allows two crossings to be added or taken away (as in Figure 2b); and the last allows one strand of a knot to be slid across a crossing (Figure 2c). It is a fact not proved here that two knots are equivalent if and only if there is a sequence of Reidemeister moves (along with simple non-crossing isotopy) with which one can turn one projection of a knot into another. Intuitively, these moves together with simple isotopy are a more rigorous way of expressing all possible isotopies of a knot; as such we will often use the isotopy as a general term referring to a collection of simple isotopies and Reidemeister moves.



(a) The first Reidemeister move.

(b) The second Reidemeister move.

(c) The third Reidemeister move.

Figure 2: The three Reidemeister moves

1.2 Knot Invariants in $\mathbb{R}^3$

Perhaps the most fundamental question in knot theory is "given two knots, show they are the same or different". With the above definitions, this question can be rephrased as "Given two link projections, show that there exists a series of Reidemeister moves that turns one link into the other, or show that no such series of moves exists". This question may appear simple but in reality is immensely difficult to answer - a complete, satisfactory answer has still not been given at this time. Although the question is not completely answered, there are many partial answers that exist, in the form of link invariants. In this section we give a brief overview of link invariants, specifically focusing on the use of polynomials to distinguish between different types of links.

Definition 5. *A link* **invariant** *is a characteristic of the link that is unchanged by any of the three Reidemeister moves.*

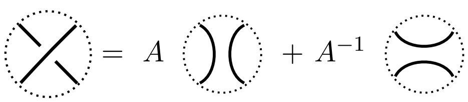
It should be noted that while an invariant is constant with respect to the projections of a link, it does not necessarily need to be unique to that given link. As an example of an invariant, we consider the crossing number of a link.

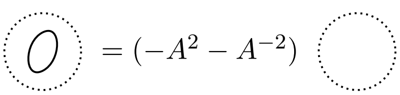
Definition 6. *A* **crossing** *in the projection of a link is when one strand of the link goes over or under another.*

Definition 7. *The* **crossing number** *of a link L is the fewest number of crossings that occur in any projection of the link.*

It is clear that the crossing number is a link invariant, since it is taken over all possible projections of the given link. However, it is usually exceedingly difficult to calculate such a crossing number rigorously. Nonetheless, were one to know the crossing number of two links, one would be able to discern that the two links were different if the crossing numbers were.

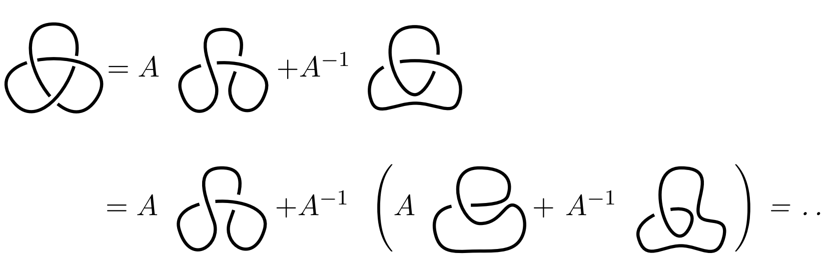
Although many link invariants exist, some of the most successful in discriminating between different links are those that take the form of a polynomial. Many such polynomials exist; here we consider one of the simpler ones, the bracket polynomial. To calculate such a polynomial, we will develop a scheme in which we trade crossings from our link for linear combinations of simpler links and polynomials in a single variable, A. This scheme will take the form of a skein relation with two rules.

Rule 1. 

Rule 2. 

It should be noted that this polynomial is actually only invariant for the second and third Reidemeister moves. Due to this, in the future we will consider only *framed* links - links which are considered different under the Reidemiester one move. Mathematically, this is equivalent to considering our links to have a thickness in one dimension - to be "strips" instead of lines.
In general, calculating the bracket polynomial for a given knot (even one a simple as the trefoil) is an involved process that grows exponentially with the number of crossings in the link. As such in Example 1 we show the first several steps of finding the bracket polynomial, and leave the remainder of the process to the reader. One should note that the polynomials we obtain from this skein relation are Laurent polynomials in A, and thus form a ring.

Example 1. *The Trefoil*



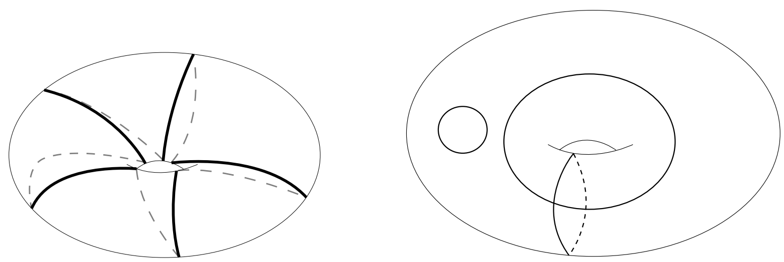
$= A^{-7} - A^{-3} - A^5$

1.3 The Torus and Torus Knots

So far we have only been discussing links that exist as subsets of $\mathbb{R}^3$. In this section we expand our scope and consider links that are subsets of the thickened torus. We focus on a specific type of link on the torus, the torus link, and introduce the relevant background definitions and theorems relating to them, in preparation for the next section.

Definition 8. *The* **thickened torus** *is $\mathbb{T} \times I$, where $\mathbb{T}$ is the usual torus in $\mathbb{R}^3$ and $I = [0, 1]$.*

Links can exist on the thickened torus just as readily as they can exist in $\mathbb{R}^3$. An example of this is in Figure 3a, where we have a knot that crosses the longitude of the torus five times and the meridian one time and has no crossings. Many of the definitions from above, especially those involving isotopy and deformation, hold in the case of the thickened torus as well. Thus, one can discuss different isotopies of the same link on the torus, as well as finding invariants to tell them apart. It should be noted, however, that links that are the same in $\mathbb{R}^3$ can be embedded in the thickened torus in a different way so as to make them distinct. As an example of this, one can embed $\mathbb{S}^1$, the unknot, in the torus in at least three different ways such that there is no isotopy between any two embeddings; see Figure 3b.



(a) A knot on the thickened torus which crosses the longitude five times, the meridian once.

(b) Three circles embedded on the thickened torus; each pair is non-isotopic to each other.

Figure 3: Knot representations on the thickened torus.

Within the domain of links on the torus, there is a specific type of link we will focus on: the torus link.

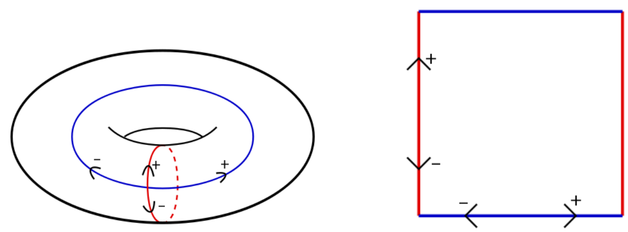
Definition 9. *A* **torus link** *is any link that when placed on the thickened torus has a projection with no crossings.*

Theorem 1. *Every link with 0 crossings has a unique minimal number of meridian and longitude crossings, and thus every 0 crossing link is a (p, q) torus link, for some p and q.*

The proof of this theorem is contained in [4] and is quite long. We do not include it here.

Notationally, (p, q) is the torus link that crosses the meridian p times and the longitude q times. It should be noted that for a given (p, q) there are two non-isotopic knots associated with it, one that goes around the meridian and the longitude in the positive direction, another that goes around the meridian in a negative direction and the longitude in a positive direction, where the orientation of the meridian and the longitude is as in Figure 4a; we denote these respectively to be (p, q) and $(-p, q)$. Note that this implies that $(p, q) = (-p, -q)$.

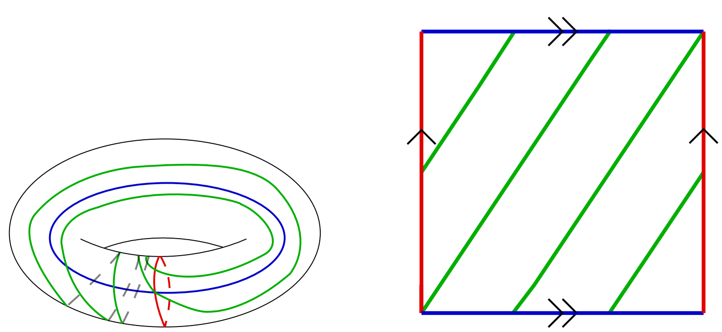
It is a standard topological fact that $\mathbb{T} \cong \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$, i.e. the torus is homeomorphic to the unit square $[0, 1] \times [0, 1]$ with opposite sides identified (see Figure 4). This homeomorphism is induced by the equivalence relation $\sim$ defined by $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow (x_1 - x_2, y_1 - y_2) \in \mathbb{Z} \times \mathbb{Z}$. (For future use, we the we will call the quotient map of this equivalence relation g; $g : \mathbb{R} \times \mathbb{R} \to [0, 1] \times [0, 1]$). Thus, we are able to picture links on the thickened torus as link projection in $[0, 1) \times [0, 1)$ much the same way that we pictured projections of links in $\mathbb{R}^3$ in $\mathbb{R}^2$. (Note that this implies that $\mathbb{R}^2/\sim = \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$.) The knot $(2, 3)$ is pictured in Figure 5 both as a knot on $\mathbb{T}$ and on $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$.



(a) The torus with oriented longitude (in blue) and meridian (in red).

(b) The oriented toroidal square with opposite sides identified; longitude is in blue, meridian in red.

Figure 4: $\mathbb{T}$ and $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$ are homeomorphic



(a) The knot $(2, 3)$ on $\mathbb{T}$

(b) The knot $(2, 3)$ on the toroidal square, $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$

Figure 5: The knot $(2, 3)$ in two different visualizations

With this picturing of the torus, we can make several definitions that will be of great use in the future.

Definition 10. *A* **strand** *is a section of a knot in* $[0, 1) \times [0, 1)$ *that goes from one side of the square to another.*

It is a fact that every torus link has a projection on the toroidal square composed entirely of evenly spaced strands of slope $\frac{q}{p}$. This projection is called the **standard projection**, and is defined rigorously below.

Definition 11. *Given a torus knot* (p, q) *such that* $\gcd(p, q) = 1$*, define the* **standard torus projection** *of* (p, q) *on the toroidal square to be* $g(\{(x, y)|y = \frac{q}{p}x\})$*. When* p *and* q *are not relatively prime, let* $k = \gcd(p, q)$ *and define the* **standard torus projection** *of* (p, q) *on the toroidal square to be* k *evenly spaced translations of the standard torus projection of* $(\frac{p}{k}, \frac{q}{k})$ *(with one of the copies being the standard torus projection of* $(\frac{p}{k}, \frac{q}{k})$ *itself).*

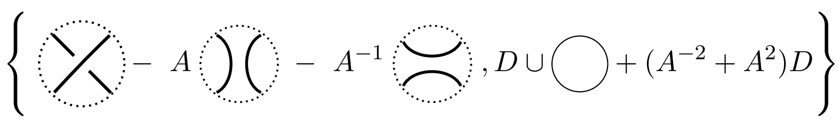
In what follows, we will assume that q is non-negative unless otherwise stated, and if q is zero, that p is positive.

Figure 5b is the standard projection of $(2, 3)$ - one can see that this projection has four evenly spaced strands and each is of slope $\frac{3}{2}$. From this definition it is clear that is p and q have the same sign, (whether positive or negative) we will get the same standard projection of the knot, while if p and q have opposite sign then the standard projection of the link will have an overall negative slope, rather than a positive one. It is easily shown that the standard projection of $(-p, q)$ is equal to the reflection of (p, q) across the line $y = \frac{1}{2}$.

1.4 The Skein Space S is an Algebra

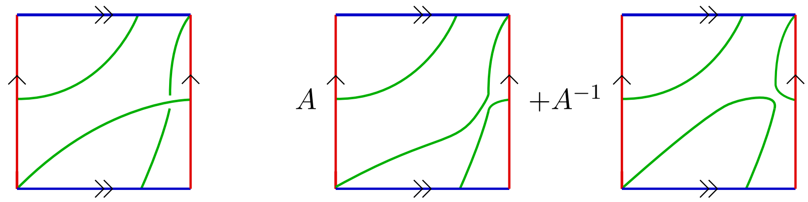
In this section we define an algebra over the space of torus knots subject to the skein relation. In what follows, definitions and theorems are adapted from [5].

To do so, first define V over $\mathbb{C}$ as the set of all finite linear combinations of pairwise non-isotpic links. Note that V is a vector space. Fix A in $\mathbb{C}$, and define V_0 as the subspace generated by all vectors of the form



where D is any element of V. Now, we define the skein space $S = V/V_0$. Intuitively, S can be understood as the space of all links in V subject to the skein relation. Algebraically, if $p : V \to V/V_0 = S$ is the quotient map, then for $L \in V$, $p(L)$ is going to be a linear combination of 0 crossing links (since these are the only links which when subject to the skein relation are unchanged) with coefficients polynomials the polynomials in A generated when we subject L to the skein relation defined above. Thus, when we consider A to be an element of $\mathbb{C}$, we see that S is a vector space over $\mathbb{C}$ with basis the torus links. (Alternatively, we can choose not to define A, and instead leave it as a variable. If we do this, then S is instead a module over the Laurent polynomials in A with basis the torus links. We will generally understand S to be a vector space.) Example 1 will (hopefully) make this idea clearer.

Example 2. Consider the element K of V in Figure 6a. When considered as an element of S, all crossings must be resolved in accordance with the skein relation.



(a) An element K of V

(b) The element K resolves to under the skein relation

Figure 6: An element K of V and its representation under p in S.

Now from Figure 6b we can see that $p(K) = A(2,2) + A^{-1}(-A^2 - A^{-2})$; this is then the representation of K in S.

We wish to show that S can be made into an algebra - to do so, we first give the definition of algebra:

Definition 12. *A vector space A over a field K is an algebra if along with addition and scalar multiplication A is endowed with a multiplication operation, $\cdot : A \times A \to A$ that is associative, distributive, and commutes with scalars.*

Algebras can be considered to have two different bases. One is the additive basis - the smallest subset of vectors in the algebra such that every element can be obtained through addition and scalar multiplication. The other is the multiplicative basis, (often called the set of generators), which is the smallest subset of vectors in the algebra such that every element can be obtained through multiplication, addition, and scalar multiplication. It is clear that the multiplicative basis is no larger than the additive basis, but in most cases it is smaller. A good example of this is $\mathbb{R}^{3\times3}$, which has a nine dimensional additive basis and a four dimensional multiplicative basis.

For elements of $F = \sum_{i=1}^{n} f_i(A)(p_i, q_i)$ and $G = \sum_{j=1}^{m} g_j(A)(r_i, s_i)$, where $f_i(A), g_j(A)$ are polynomials in A and (p_i, q_i), (r_i, s_i) are torus links, define $(\cdot)$ over S to be

$$F \cdot G = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(A) g_j(A)(p_i, q_i)(r_j, s_j).$$

This operation will serve as our multiplication for S (once we show it satisfies the properties mentioned above), but we first need to define the multiplication of torus links. (Note that in order to ensure the associativity of our operation, we will also need to check that whatever multiplication operation we define for torus links is associative.)

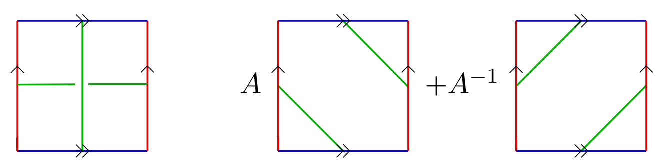
1.4.1 The Multiplication of Torus Links

We want to define an associative multiplication of two torus links K_1 and K_2, each of which is embedded in $\mathbb{T} \times I$. We will show that the operation $\beta : (\mathbb{T} \times I) \times (\mathbb{T} \times I) \to \mathbb{T} \times I$ defined as

$$\beta((a,b),(c,d)) = (a, \frac{b}{2}) \cup (c, \frac{1+d}{2})$$

will suffice. β essentially "wraps" the second thickened torus around the first, and normalizes the thickness of this new torus to be 1. Notice that if K_1 and K_2 are non-trivial links (i.e. not the empty link), then the β operation will "stack" these two on top of each other in the same torus, possibly creating more crossings that will be resolved under the skein relation. This is especially true for torus links, which have no crossings in S, but when multiplied might. Two torus links, both with no inherent crossings, are layered on a torus, and a crossing is produced; this crossing is then resolved under the skein relation to create a linear combination of torus knots, which is the result of the product. Although we define β to be our multiplication operation, when we are referring to torus link multiplication we will simplify our notation slightly denote the torus link (p,q) times the torus link (r,s) as $(p,q)(r,s)$.

Figure 7 shows the multiplication $(0,1)(1,0)$.



(a) A representation of the product $(0,1)(1,0)$

(b) The resolution of the product $(0,1)(1,0)$ under the skein relation

Figure 7: The multiplication of $(0,1)$ times $(1,0)$.

Table 1 (see page 39) gives the results of some more small knot products.

We now must show that the multiplication operation β that we defined on links in S is associative. To do so, note that for three links K_1, K_2, K_3 embedded in individual tori,

$$\beta((K_1, I), \beta((K_2, I), (K_3, I))) =$$

$$\beta((K_1, I), ((K_2, \frac{I}{2}) \cup (K_3, \frac{I+1}{2}))) =$$

$$(K_1 \times \frac{I}{2}) \cup ((K_2 \times (\frac{1}{2} + \frac{I}{4})) \cup (K_3 \times (\frac{3}{4} + \frac{I}{4}))) =$$

$$(K_1 \times \frac{I}{2}) \cup (K_2 \times (\frac{1}{2} + \frac{I}{4})) \cup (K_3 \times (\frac{3}{4} + \frac{I}{4}))$$

An easy isotopy on this last set then gives us

$$(K_1 \times \frac{I}{4}) \cup (K_2 \times (\frac{1}{4} + \frac{I}{4})) \cup (K_3 \times (\frac{1}{2} + \frac{I}{2}))$$

which we see is equal to $\beta(\beta((K_1, I), (K_2, I)), (K_3, I))$, which proves the that β is associative.

Now that we have shown that our operation β is associative, all we must do to show that our multiplication operation $(\cdot)$ is satisfies our requirements for an algebra is to show that it is distributive, associative, and commutes with scalars. To do so, note that $(\cdot)$ is just the linear extension of torus knot multiplication over our vector space, and thus commutes with scalars and is distributive by definition. The associativity of the operation follows from this fact as well, combined with the associativity of the β operation. Thus, we have shown that S is an algebra.

Since S is now an algebra, we can now talk about multiplying torus links. This opens up a number of questions. For example, it is clear that the additive basis of S is the torus links, since every torus link is independent of every other, and every other link is reduced to a linear combination of torus links in S. However, the size of the multiplicative basis is not as clear. We also at this time do not know how to predict the results of a multiplication, and instead must go through the long process of recursive skein resolution. Answers to these questions will be provided in further sections.

1.5 What Comes Next

Up to now we have primarily focused on work previously done. In what comes next we will explore results that our comps group came up with on our own. It should be noted here that while all of our results were done without the consultation of other works, many of our results are identical to results found in [2] and [3].

The main results of following sections are:

- In Section 2.3 we prove the the Decomposition Theorem:

  Theorem 2. *Given a link (p, q) that is not in $\{(1,0), (0,1), (1,1), (1,1)\}$, it is possible to write the link as a sum of products of smaller links.*

  The end of this section gives an example decomposition and illustrates how it can be used to multiply links.

- Using our theorems from Section 2, we begin work on the genus two surface (See Figure 8) - the main goal of our project. We seek a generalization of the Decomposition Theorem, Theorem 2. Specifically:

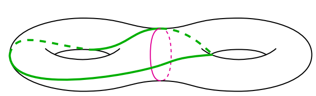


Figure 8: A Double Torus Knot

  1. A decomposition theorem for double torus links similar to the Decomposition Theorem from 2.3,
  2. A way to easily multiply double torus links.

- This problem we address by restricting ourselves to pieces of the double torus and addressing each individually, then considering how re-uniting the pieces works. We give partial results for this problem in Sections 3-4:
  - In Section 3 we show that in the case when the link does not intersect the boundary, our basis is unchanged and our theorems from Section 2 hold.
  - In Section 4 we address the case in which links on the punctured torus do intersect the boundary. Specifically we show that at most three unique knots can exist on the torus with boundary without crossing each other, and also derive a formula for one crossing torus links that intersect the boundary.
- In Section 5 we use results from sections 2, 3, and 4 to give a bijective notation for the double torus.
- Finally, in Section 6 we indicate conjectures and further directions of our research.

2 Knots on Tori without Boundary

We determine the structure of the algebra of torus knots by finding the generators. En route to this, we examine basic properties of knot products, in particular the number of crossings, and then use patterns that appear in knot products with one crossing to define a method of writing knots as the sum of products of the generators.

2.1 Notation and Basic Properties

Consider the product of two torus links (p, q) and (r, s). By applying the skein relation to all of the crossings, we get $(p, q)(r, s) = \Sigma g_i(A)(u_i, v_i)$, where $g_i(A)$ is a polynomial in terms of A and (u_i, v_i) is some link. In this notation, we allow (u_i, v_i) to be $(0, 0)$, which we think of not as a torus knot or link but just a loop bounding a disk. When we write out the terms of the sum, we will replace the $(0, 0)$ with $-A^2 - A^{-2}$. There are many useful properties of the link product.

Theorem 3. *Let (p, q) and (r, s) be two torus links, and let $(p, q)(r, s) = \Sigma g_i(A)(u_i, v_i)$. The following properties hold:*

1. $|u_i| \leq |p| + |r|$ *and* $|v_i| \leq |q| + |s|$
2. $(r, s)(p, q) = \overline{(p, q)(r, s)}$, *where* $\overline{(p, q)(r, s)} = \Sigma g_i(A^{-1})(u_i, v_i)$
3. $(s, r)(q, p) = \Sigma g_i(A)(v_i, u_i)$
4. $(-r, s)(-p, q) = \Sigma g_i(A)(-u_i, v_i)$
5. $(-p, q)(-r, s) = \Sigma g_i(A^{-1})(-u_i, v_i)$

Proof. 1. Recall that $|u_i|$ is the number of times the link (u_i, v_i) goes around the torus longitudinally. Since the skein relation cannot increase the number of times the link $(p, q)(r, s)$ crosses any meridian loop, $|u_i|$ can be no more than that. Since (p, q) crosses any meridian loop $|p|$ times and (r,s) crosses any meridian loop $|r|$ times, the total number of meridian crossings is $|p|+|r|$, and thus $|u_i| \leq |p|+|r|$. Similarly, by counting the number of times we go around meridianally, $|v_i| \leq |q|+|s|$.

2. We observe that the difference between $(p,q)(r,s)$ and $(r,s)(p,q)$ is the order in which they are multiplied, or in other words, which link is on top of the other. This means that every crossing is switched between $\otimes$ and $\otimes$. Since $\otimes = A \circledcirc + A^{-1} \ominus$ and $\otimes = A \ominus + A^{-1} \circledcirc$, each time we perform the skein relation on a crossing we change the sign of the powers of A in our resulting links. Doing this with every crossing means the final signs on the powers of A are all switched, and $(r,s)(p,q) = \overline{(p,q)(r,s)}$.

3. Consider the preimage of the standard torus projection in the x, y -plane. We observe that (q,p) and (s,r) are 180 degree rotations of (p,q) and (r,s) around the line $y=x, z=0$ in $\mathbb{R}^3$, respectively. Furthermore, because the link that was on top is on the bottom after rotating both, $(s,r)(q,p)$ is the rotation of $(p,q)(r,s)$ around $y=x, z=0$. If we look at how the skein relation works under this rotation, we see that the rotation switches $\otimes = A \circledcirc + A^{-1} \ominus$ and $\otimes = A \ominus + A^{-1} \circledcirc$ from one to the other, so rotating and performing the skein relations as usual is the same as performing the skein relations and then rotating. Therefore, we can rotate, perform the skein relations, and rotate back and have the same result as just doing the skein relations, and so we can find $(s,r)(q,p)$ by rotating $\Sigma g_i(A)(u_i,v_i)$, which gives $\Sigma g_i(A)(v_i,u_i)$.

4. We use a similar rotation technique for this property. Observe that $(-p,q)$ and $(-r,s)$ are the 180 degree rotations of (p,q) and (r,s) around the line $y=\frac{1}{2}, z=0$, respectively. As in the proof of the previous property, this means that $(-r,s)(-p,q)$ is the rotation of $(p,q)(r,s)$ around $y=\frac{1}{2}, z=0$. Consider the effect of this rotation on the skein relations: this time $\otimes = A \circledcirc + A^{-1} \ominus$ goes to itself, and $\otimes = A \ominus + A^{-1} \circledcirc$ goes to itself. However, it is still the case that the processes of rotating and performing the skein relations commute with each other, so we can find the result of $(-r,s)(-p,q)$ by rotating $\Sigma g_i(A)(u_i,v_i)$, which in this case gives $\Sigma g_i(A)(-u_i,v_i)$.

5. This property follows directly from properties 2 and 4.

□

In addition to these properties, the knot product has a useful number associated with it, which we define as follows.

Definition 13. *The* **determinant** *of a knot product* $(p,q)(r,s)$ *is*

$$\det((p,q)(r,s)) = ps - qr.$$

2.2 Crossing Number

In this section, we will consider the crossing number of link products and look at link products with one crossing in more depth. In order to prove a theorem about the crossing number of a link product, we need the following lemmas.

Lemma 1. *If a and b are relatively prime integers, and b is positive, then $ca \equiv da \mod b$ if and only if $d - c \equiv 0 \mod b$ (in other words, $d - c$ is an integer multiple of b).*

Proof. Suppose $ca \equiv da \mod b$. Then $da - ca = (d - c)a \equiv 0 \mod b$, so b divides $(d - c)a$. Since a and b are relatively prime, b must divide $d - c$. Suppose $d - c \equiv 0 \mod b$. Then $(d - c)a \equiv 0 \mod b$, so $da \equiv ca \mod b$. □

Lemma 2. *The standard torus projection of (ℓ, m) intersects the image of the x-axis (henceforth referred to as simply the x-axis of $\mathbb{R}^2/_\sim$) m times, at $\frac{1}{m}$ intervals, and the image of the y-axis (henceforth referred to as simply the y-axis of $\mathbb{R}^2/_\sim$) ℓ times at $\frac{1}{\ell}$ intervals.*

Proof. Let us first assume that ℓ and m are relatively prime. To find the locations of the intersections of the standard torus projection with the x-axis of $\mathbb{R}^2/_\sim$, we need to look at all of the points in $\{(x, y)|y = \frac{m}{\ell}x\}$ which have an integer value for y. Similarly, to find the locations of the intersections of the standard torus projection with the y-axis of $\mathbb{R}^2/_\sim$, we need to look at all of the points in $\{(x, y)|y = \frac{m}{\ell}x\}$ which have an integer value for x. First, we will show that we only need to consider the segment of the line $\{(x, y)|y = \frac{m}{\ell}x, x \in [0, l)\} = \{(x, y)|x = \frac{\ell}{m}y, y \in [0, m)\}$ (Note that if ℓ is negative, this is really $x \in (\ell, 0]$ but for simplicity we will simply call it $[0, \ell)$.)

Suppose $(x_1, y_1) \in \{(x, y)|y = \frac{m}{\ell}x\}$ but $(x_1, y_1) \notin \{(x, y)|y = \frac{m}{\ell}x, x \in [0, \ell)\}$. There exists an integer z such that $x_2 = x_1 - z\ell \in [0, 1)$. Let $y_2 = \frac{m}{\ell}x_2$, and consider (x_2, y_2). We observe that $(x_2, y_2) \in \{(x, y)|y = \frac{m}{\ell}x, x \in [0, \ell)\}$, and that $y_2 = \frac{m}{\ell}x_2 = \frac{m}{\ell}(x_1 - z\ell) = \frac{m}{\ell}x_1 - zm = y_1 - zm$. Since $x_2 = x_1 - z\ell$ and $y_2 = y_1 - zm$, and z, ℓ, and m are all integers, $(x_1, y_1) \sim (x_2, y_2)$. Therefore $f((x_1, y_1)) = f((x_2, y_2))$, which means that $f(\{(x, y)|y = \frac{m}{\ell}x, x \in [0, \ell)\}) = f(\{(x, y)|y = \frac{m}{\ell}x\})$, the standard torus projection.

We now focus our attention on finding the image of each point (x, y) such that y is an integer and $(x, y) \in \{(x, y)|x = \frac{\ell}{m}y, y \in [0, m)\}$. Assume that $y \in [0, m)$ is an integer. Define u by $u \equiv \ell y \mod m$, where $u \in [0, m)$. Consider $\frac{u}{m}$. We observe that $x - \frac{u}{m} = \frac{\ell}{m}y - \frac{u}{m} = \frac{\ell y - u}{m}$. Since $u \equiv \ell y \mod m$, $u = \ell y - zm$ for some integer z, so $x - \frac{u}{m} = \frac{\ell y - (\ell y - zm)}{m} = \frac{zm}{m} = z$. Therefore we have $x - \frac{u}{m} \in \mathbb{Z}$ and $y - 0 \in \mathbb{Z}$, so $(x, y) \sim (\frac{u}{m}, 0)$, and $f((x, y)) = f((\frac{u}{m}, 0))$. Observe also that $\frac{u}{m} \in [0, 1)$ because $u \in [0, m)$. Therefore $f((x, y)) = f((\frac{u}{m}, 0)) = (\frac{u}{m}, 0)$, so $(\frac{u}{m}, 0)$ is a point where the standard torus projection and the x-axis of $\mathbb{R}^2/_\sim$ intersect.

This process of finding the image of (x, y) works for each integer $y \in [0, m)$, or in other words each $y \in \{0, 1, ..., m - 1\}$. Suppose y_1 and y_2 are distinct elements of $\{0, 1, ..., m - 1\}$. By Lemma 1, since ℓ and m are relatively prime, and $|y_1 - y_2| < m$, it cannot be the case that ℓy_1 and ℓy_2 are in the same equivalency class mod m. Therefore, each distinct value of y yields a distinct value for u. Since there are clearly m distinct values for y, there are m distinct values for u, and since u must be in $\{0, 1, ..., m - 1\}$, these m distinct values must be $\{0, 1, ..., m - 1\}$. It follows that $\{(\frac{0}{m}, 0), (\frac{1}{m}, 0), ..., (\frac{m-1}{m}, 0)\}$ are all of the intersections of the standard torus projection with the x-axis of $\mathbb{R}^2/_\sim$.

Similarly, we consider all of the points (x, y) such that x is an integer and $(x, y) \in (\{(x, y)|y = \frac{m}{\ell}x, x \in [0, \ell)\}$. For this part of the proof, we will assume that ℓ is positive, and allow m to be either positive or negative. Recall that the standard torus projections of (ℓ, m) and $(-\ell, -m)$ are identical, so this assumption does nothing to change the previous results in this proof, and still covers all cases. Furthermore the proof that we only need to consider a segment of the line is still valid.

The points with which we are concerned are the points $\{(x,y)|y=\frac{m}{\ell}x, x\in\{0,1,...,\ell-1\}\}$. Define v by $v\equiv mx \mod \ell$, where $v\in[0,\ell)$. Consider $\frac{v}{\ell}$. We observe that $y-\frac{v}{\ell}=\frac{m}{\ell}x-\frac{v}{\ell}=\frac{mx-v}{\ell}$. Since $v\equiv mx \mod \ell$, $v=mx-z\ell$ for some integer z, so $y-\frac{v}{\ell}=\frac{mx-(mx-z\ell)}{\ell}=\frac{z\ell}{\ell}=z$. This means that $x-0\in\mathbb{Z}$ and $y-\frac{v}{\ell}\in\mathbb{Z}$, so $(x,y)\sim(0,\frac{v}{\ell})$ and $f((x,y))=f((0,\frac{v}{\ell}))$. Furthermore, $\frac{v}{\ell}\in[0,1)$ because $v\in[0,\ell)$. Thus $f((x,y))=f((0,\frac{v}{\ell}))=(0,\frac{v}{\ell})$, and $(0,\frac{v}{\ell})$ is a point where the standard torus projection and the y-axis of $\mathbb{R}^2/_\sim$ intersect.

This process of finding the image of (x,y) works for each $x\in\{0,1,...,\ell-1\}$. Suppose x_1 and x_2 are distinct elements of $\{0,1,...,\ell-1\}$. By the above lemma, since ℓ and m are relatively prime, and $|x_1-x_2|<\ell$, it cannot be the case that my_1 and my_2 are in the same equivalency class mod ℓ. Therefore, each distinct value of x yields a distinct value for v. Since there are clearly ℓ distinct values for x, there are ℓ distinct values for v, and since v must be in $\{0,1,...,\ell-1\}$, these ℓ distinct values must be $\{0,1,...,\ell-1\}$. It follows that $\{(0,\frac{0}{\ell}),(0,\frac{1}{\ell}),...,(0,\frac{\ell-1}{\ell})\}$ are all of the intersections of the standard torus projection with the y-axis of $\mathbb{R}^2/_\sim$.

We have now shown that if ℓ and m are relatively prime, then the intersections with the x- and y-axis occur at $\frac{1}{m}$ and $\frac{1}{\ell}$ intervals, respectively. Now suppose that ℓ and m are not relatively prime, and let $k=\gcd(\ell,m)$. The standard torus projection is defined to be k evenly-spaced copies of the standard torus projection of $(\frac{\ell}{k},\frac{m}{k})$. Since $\frac{\ell}{k}$ and $\frac{m}{k}$ are relatively prime, each one of the k copies intersects the x-axis $\frac{m}{k}$ times at intervals of $\frac{k}{m}$ and the y-axis $\frac{\ell}{k}$ times at intervals of $\frac{k}{\ell}$. Since the k copies are evenly spaced, when we look at them all together, there are $k\frac{m}{k}=m$ intersections with the x-axis, occuring at intervals of $\frac{k}{km}=\frac{1}{m}$, and $k\frac{\ell}{k}=\ell$ intersections with the y-axis, occuring at intervals of $\frac{k}{k\ell}=\frac{1}{\ell}$. □

In order to continue our discussion of torus links, we first need to prove a lemma involving lattice points in parallelograms that will be useful for determining the number of crossings in a link product.

Lemma 3. *For any parallelogram ADJG with integer-valued vertices, the number of lattice points inside the parallelogram is* $xy-wz+n+m+3$*, where* $AF=EJ=x,\ AB=IJ=y,\ BC=IH=z,\ CE=FH=w$*, and* n *and* m *are the number of lattice points that lie on* AG *and* AD *respectively.*

Proof. Let $AF=EJ=x,\ AB=IJ=y,\ BC=IH=z,\ CE=FH=w.$

Consider the parallelogram $ADJG$ (see Figure 9). Let there be n points with integer coordinates lying on the segments AG and DJ and m such points on the segments AD and GJ; we'll call these points with integer-value coordinates lattice points. We want to show that there are $wz-xy+n+m+3$ lattice points in the interior of this parallelogram, including points lying on the edges AG,AD,DJ, and GJ.

We can divide the rectangle $ACJH$ into two rectangles ($BCED$ and $FGIH$), four triangles (ABD, JIG, AGF, and JED), and the parallelogram $ADJG$. To figure out how many points are inside the parallelogram, we can subtract the points inside the other shapes from the rectangle $ACJH$.

We know $ACJH$ has a base of length $w+x$ and a height of length $y+z$. Therefore, there are $w+x-1$ lattice points along the base of the rectangle (including A and H) and $z+y+1$ points

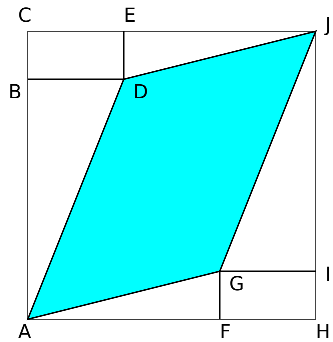


Figure 9: The parallelogram described in Lemma 3.

along the segment AC. Thus, there are $(w + x + 1)(y + z + 1)$ lattice points inside the rectangle $ACJH$, including points on the boundary of the rectangle. From this set of points, we want to subtract all the points that are not in the parallelogram.

We can start by considering the rectangle $BCED$. This rectangle has width w and height z, so it contains $(w + 1)(z + 1)$ lattice points. However, we don't want to subtract the point D, which is a vertex of $ADJG$, so the rectangle covers $(w + 1)(z + 1) - 1$ points, as does $FGHI$. Thus, we should subtract $2(w + 1)(z + 1) - 2$ from our total.

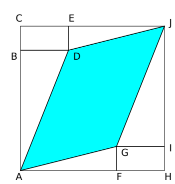


Figure 10: Counting triangles ABD and JIG

Now, consider the triangles ABD and JIG. We can glue these two triangles together by attaching the segment AD to GJ to form the rectangle $ABJI$. Of course, this means that we've replaced $2m$ lattice points, m on each edge, and 4 vertices with only m and the two endpoints A and J. As we will soon explain, the vertices aren't really an issue. As for the $2m$ points along the edges, we know they lie within $ADJG$, so we don't want to subtract them from our total; by combining these two edges, we implicitly remove m of those $2m$ points from the set of points not in $ADJG$. We'll remove the other m soon.

This new rectangle has base w and height y, so it must contain $(w+1)(x+1)$ lattice points. However, we've already counted the edges AB and GI when we were looking at $BCED$ and $FGHI$, so we are actually only counting $(w+1)(y-1)$ points. Notice that this eliminates the problem of double-counting the triangle vertices, since we've already subtracted them from our total. Furthermore, we don't want to include the points on the segment AJ, as those points lie on the boundary of $ADJG$, so we need to subtract m fewer points from our total. So the total number of points we want to subtract is $(w + 1)(y - 1) - m$.

Similarly, since we can complete the same process for the triangles AGF and JDE, we need to subtract another $(x - 1)(z + 1) - n$ points from our total.

This means that the total number of points inside and on the boundary of the parallelogram $ADJG$ is

$$\begin{aligned} ADJG &= (w+x+1)(y+z+1) - 2((w+1)(z+1)-1) \\ &\quad -((w+1)(y-1)-m) - ((x-1)(z+1)-n) \\ &= wy + wz + w + xy + xz + x + y + z + 1 - 2wz - 2w - 2z \\ &\quad -wy - y + w + 1 - xz - x + z + 1 \\ &= xy - wz + n + m + 3 \end{aligned}$$

□

These tools allow us to prove the Crossing Number Theorem.

Theorem 4. Crossing Number Theorem
The number of crossings in the link product $(p,q)(r,s)$, when both are in the standard torus projection is $|det((p,q)(r,s))| = |ps - qr|$.

Proof. We first consider the case when $ps - qr = 0$, and show that $(p,q)(r,s)$ has no crossings. Let us then suppose that $ps - qr = 0$, or equivalently that $ps = qr$. Suppose s is zero. Then either q or r must be zero. If r is zero, then $(r,s) = (0,0)$ is not a knot, so it must be that q is zero. Then $(p,q) = (p,0)$ and $(r,s) = (r,0)$ are just loops around the longitude of the torus, so they can be shifted so that there are no crossings.

Similarly, if q is zero, then p or s must be zero. As before, p cannot be zero, so s must be zero, and again we just have loops around the longitude of the torus. Therefore the number of crossings is zero if either q or s is zero.

So we can assume that neither q nor s is zero, and can divide both sides of $ps = qr$ by qs. This gives us $\frac{p}{q} = \frac{r}{s}$. Since r, and s are both positive, from this we can conclude that $(p,q) = (k_1u, k_1v)$ and $(r,s) = (k_2u, k_2v)$ for some integers k_1, k_2, u, and v, where, with the possible exception of u, all are positive. Then the standard torus projections of (p,q) and (r,s) are made up of parallel lines. If any overlap, they can be shifted slightly sideways to eliminate the overlap and again there are no crossings. Therefore, if $ps - qr = 0$, the number of crossings in $(p,q)(r,s)$ is $|ps - qr|$.

Now let us consider the case when $ps - qr \neq 0$, and study the pre-image of the links (p,q) and (r,s) under the quotient map f. We observe that $[0,1) \times [0,1)$ in the pre-image is identical to the complete set of representatives $[0,1) \times [0,1)$ in the quotient space, and therefore any crossings that appear in one will also appear in the other. Now that we are working in $\mathbb{R}^2$, we consider the effects of invertible linear transformations on the pre-images. The plan is to transform all crossings into integer lattice points, which will make counting them easier. First, we recall some basic properties of invertible linear transformations.

Linear transformations preserve parallelism. Furthermore, if three points are colinear, then the ratios of the distances between the three points will be preserved.

Under an invertible linear transformation, the square $[0,1) \times [0,1)$ will be mapped to a parallelogram, and any intersections of lines within $[0,1) \times [0,1)$ will be mapped to an intersection of lines within the parallelogram. Under the inverse linear transformation any intersection of lines within the parallelogram will be mapped to an intersection of lines within $[0,1) \times [0,1)$. Therefore when we look at $(p,q)(r,s)$, the number of crossings of the two links (which correspond to intersections of the lines representing them) will be equal to the number of crossings of the image of the two links under the invertible linear transformation (which correspond to the intersections of the images of the lines).

Given these properties of linear transformations, consider the linear transformation with the matrix representation

$$\begin{bmatrix} s & -r \\ -q & p \end{bmatrix}$$

in the standard ordered basis for $\mathbb{R}^2$. The determinant of this matrix is $ps - qr$, which is non-zero. Therefore the linear transformation is invertible, so we can calculate the number of crossings of the links by finding the number of intersections of the lines representing them in the image of the linear transformation. Observe that

$$\begin{bmatrix} s & -r \\ -q & p \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ -q \end{bmatrix}$$

$$\begin{bmatrix} s & -r \\ -q & p \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -r \\ p \end{bmatrix}$$

$$\begin{bmatrix} s & -r \\ -q & p \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} ps - qr \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s & -r \\ -q & p \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ ps - qr \end{bmatrix}$$

From the above equations, we have that all of the line segments in the projection of (p,q) are mapped to line segments which are parallel to the vector $(ps - qr, 0)$ and all of the line segments in the projection of (r,s) are mapped to line segments which are parallel to the vector $(0, ps - qr)$. Because of the preservation of distance ratios for colinear points, we also have that the image of (p,q) intersects the vector $(s,-q)$ (the image of $(1,0)$) q times at evenly spaced intervals and the vector $(-r,p)$ p times at evenly spaced intervals. These segments, then, are at multiples of $(\frac{s}{q}, -1)$ and $(\frac{-r}{p}, 1)$. Since the segments are also parallel to the vector $(ps-qr, 0)$, the image of the projection of (p,q) is the union of the integer horizontal lines intersected with the image of $[0,1) \times [0,1)$.

Similarly, the image of the projection of (r,s) intersects the vector $(s,-q)$ in s evenly spaced points, or at multiples of the vector $(1, \frac{-q}{s})$. It also intersects $(-r,p)$ in r evenly spaced points, or at multiples of $(-1, \frac{p}{r})$. Since these line segments are parallel to the vector $(0, ps - qr)$, the image of the projection of (r,s) is the union of the integer vertical lines intersected with the image of $[0,1) \times [0,1)$.

Therefore, in the image, the intersections of (p,q) and (r,s) occur at all of the lattice points in the image of $[0,1) \times [0,1)$, which is the parallelogram formed by the two vectors $(s,-q)$ and $(-r,p)$, not including the other two edges of the parallelogram. By Lemma 3, the number of lattice points in this parallelogram, including all edges, is $|ps - qr| + n + m + 3$, where n and m are the number of lattice points on the two vectors forming the parallelogram, not including the vertices. Then since $|ps - qr| + n + m + 3$ counts the three vertices not in $[0,1) \times [0,1)$ and the n and m lattice

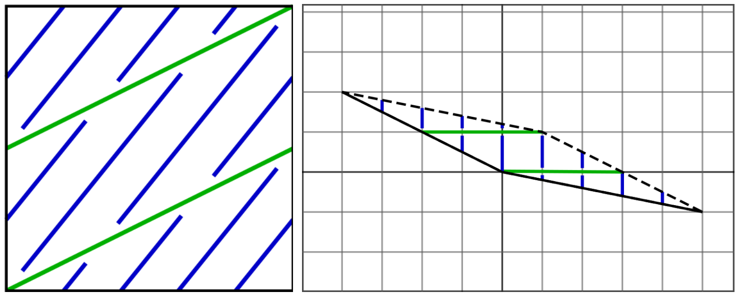


Figure 11: The standard torus projection and the transformation corresponding to the knot product $(2,1)(4,5)$.

points on the opposite edges, the number of lattice points we have that count distinct intersections is $|ps - qr|$.

□

Now we look at link products with only one crossing. Observe that if there is only one crossing, both links will actually be knots since p, q and r, s must be pairs of relatively prime numbers for $|ps - qr|$ to be 1.

Consider two knots (p, q) and (r, s) which when multiplied have one crossing; that is by the Crossing Number Theorem, $|ps - qr| = 1$. We prove a general formula for the result of their product:

Theorem 5. One Crossing Theorem
For a given knot product $K_1K_2 = (p, q)(r, s)$ with $|det(K_1K_2)| = 1$,

$$(p,q)(r,s) = \begin{cases} A(p+r, q+s) + A^{-1}(p-r, q-s) & det(K_1K_2) = 1 \\ A^{-1}(p+r, q+s) + A(p-r, q-s) & det(K_1K_2) = -1 \end{cases}$$

Proof. Our plan is to first show that the resulting knots are of the form $(p+r, q+s)$ and $(p-r, q-s)$, and then show that A and A^{-1} appear as indicated. We divide the proof into two cases. In the first case, we assume that p, q, r, and s are all non-negative.

For this case, we begin by proving a lemma that will simplify our proof greatly:

Lemma 4. *Either $p \geq r$ and $q \geq s$ or $p \leq r$ and $q \leq s$.*

Proof:
Assume not — then two cases are possible. In the first case, $p > r$ and $s > q$. Then, since we are assuming that $p, q, r, s \in \mathbb{Z}$, we see that $p = r + n$ and $s = q + m$, where n, m are in the positive naturals. Then $\pm 1 = ps - qr = (r + n)(q + m) - qr$, which for any values of n, m allowed is impossible unless $r = q = 0$ — so we cannot have $p > r$ and $s > q$ unless $r = q = 0$. If $r = q = 0$, then we must have $(1, 0)(0, 1)$ and this specific case has been shown to be true (see page 39). The proof of the case where $p < r$ and $s < q$ is similar and not included here.

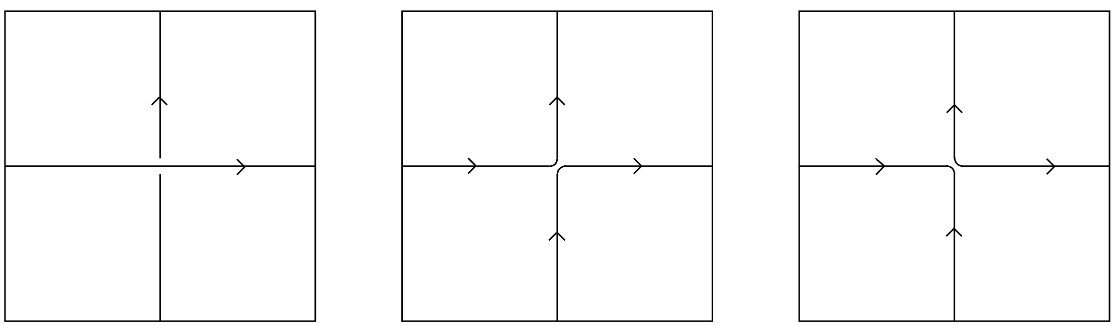
Thus, we see that either $p \geq r$ and $q \geq s$ or $p \leq r$ and $q \leq s$. Furthermore, from property 2 of Theorem 3 we know that $(p, q)(r, s) = \overline{(r, s)(p, q)}$; this combined with the fact that $det((p, q)(r, s)) =$

$-det((r,s)(p,q))$ lets us also ignore the case where $p \leq r$ and $q \leq s$. So, for the remainder of the proof we assume that $p \geq r$ and $q \geq s$.

Continuing the proof, consider our two torus knots $K_1 = (p,q)$, $K_2 = (r,s)$ which have a single crossing in their product, $(p,q)(r,s)$. If both knots are in their standard torus projection, we know that both intersect the origin, so this one crossing must occur at the origin in the standard torus projection. We isotope the knots from the standard torus projection so that their point of intersection occurs at $(\frac{1}{2}, \frac{1}{2})$ while maintaining the straight line properties of the projection.

To show that the resulting knots are of the from $(p+r, q+s)$ and $(p-r, q-s)$,, first give each knot in the product a positive orientation (i.e. an orientation such that each strand is directed up and to the right). Consider the resolution of the this product under the skein relation, which generates two 0-crossing knots, (which we know are torus knots).

Since the results of a knot product resolution are independent of the order in which the two knots were multiplied, (up to coefficients), we may assume that our knot product around $(\frac{1}{2}, \frac{1}{2})$ looks like the crossing illustrated in Figure 12a. Note that because K_1 and K_2 both begin and end at the origin we may consider them as paths in the fundamental group of the torus, and may operate on them with the operation of this group.



(a) The knot product K_1K_2 locally around $(0.5, 0.5)$.

(b) The right resolution of the knot product K_1K_2

(c) The left resolution of the knot product K_1K_2

Figure 12: The knot product and its right and left resolutions.

Figure 12b illustrates the knot resolution of the product K_1K_2 locally around $(\frac{1}{2}, \frac{1}{2})$. Notice that both strands in the figure are still positively oriented. Because orientation is a global property (a knot in the standard projection either has a positive orientation or is does not), this allows us to see that the right resolution of our knot product is exactly the operation of the fundamental group, which indicates that our knot is equal to $(K_1 \star K_2)$,where $\star$ is the operation of the fundamental group. Furthermore, since the fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, (under the isomorphism f), we see that $(K_1 \star K_2) = (p,q) \star (r,s) \rightarrow_f (p+r, q+s) \in \mathbb{Z} \times \mathbb{Z} \rightarrow_{f^{-1}} (p+r, q+s)$ as a torus knot.

Furthermore, as seen in Figure 12c the left resolution also yields a single knot that is the combination of two positively oriented strands, but in this case, the orientations do no match up when the curves are glued together. Because orientation must be a single direction, this forces us to change the orientation of one of the curves. We could choose to switch the orientation of either curve — by convention we choose to keep $K_1 = (p,q)$ positively oriented and change the orientation of orientation of $K_2 = (r,s)$ to be negatively oriented. Our picture is now as in Figure 13. Thus, the knot remaining is the path $(p,q) \star \overline{(r,s)}$. From this we see that $(p,q) \star \overline{(r,s)} \rightarrow_f (p-r, q-s) \in$

$\mathbb{Z} \times \mathbb{Z} \rightarrow_{f^{-1}} (p-r, q-s)$ is our torus knot product of our left resolution.

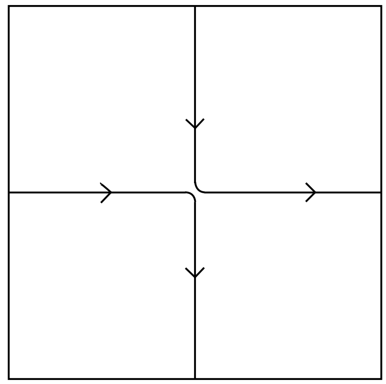


Figure 13: The left resolution of K_1K_2 with the orientation of K_2 switched.

We have shown that for a one crossing knot product with $p, q, r,$ and s non-negative, the torus knots produced will be $(p+r, q+s)$ and $(p-r, q-s)$. We now show how the sign of the determinant $|ps-qr|$ determines the coefficients.

Case 1: $ps-qr=1$.

Assume that r and p are both greater than 0 and note that $ps-qr=1 \Rightarrow ps-qr>0 \Rightarrow \frac{s}{r} > \frac{q}{p}$. Thus, the slope of the strands in the standard projection of K_2 is greater than the slopes of the strands in the standard projection of K_1 — thus locally around (.5, .5) our knot crossing is like the knot crossing in Figure 14a. Now referring to Figures 12b and 12c, notice how the right and left resolutions of this knot give the addition $(p+r, q+s)$ and subtraction $(p-r, q-s)$ terms, respectively. Since the right resolution gives a coefficient of A and the left resolution gives a coefficient of A^{-1}, we see that when $ps-qr=1$, the result of the knot product $(p,q)(r,s)$ is $A(p+r, q+s)+A^{-1}(p-r, q-s)$.

Now, if $r=0$ then K_2 will be a vertical line, and will thus have a greater slope than any K_1, so the above still holds. If $p=0$, then q or r must be less than 0, which we are not allowing. Thus, the above proof is valid for every case when $ps-qr=1$ and $p, q, r,$ and s are non-negative.

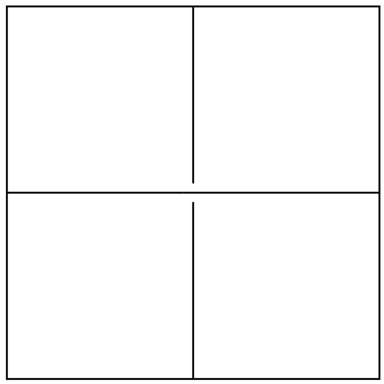
Case 2: $ps-qr=-1$.

For this, we again use property 2 of Theorem 3, which states that $(p,q)(r,s) = \overline{(r,s)(p,q)}$. Since $det((p,q)(r,s)) = -det((r,s)(p,q))$, $(r,s)(p,q)$ falls into case 1, so $(r,s)(p,q) = A(r+p, s+q) + A^{-1}(r-p, s-q)$. Therefore $(p,q)(r,s) = A^{-1}(r+p, s+q) + A(r-p, s-q) = A^{-1}(p+r, q+s) + A(p-r, q-s)$. Thus, we are able to conclude our proof for $p, q, r,$ and s non-negative.

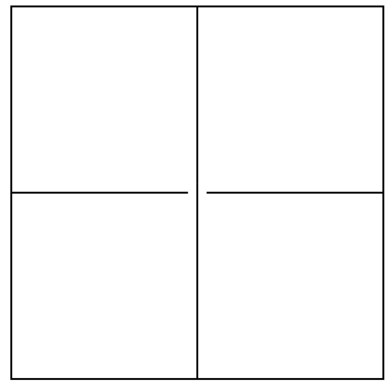
Now that we have proven this for the case of $p, q, r,$ and s all being non-negative, we consider the case where at least one of these numbers is negative. Since q and s must be non-negative, the only possibilities are p and r.

Suppose $p<0$ and $r \geq 0$. This implies that $ps-qr<0$ and therefore $ps-qr=-1$. Since q cannot be zero (or p would have to be positive), either r or s must be 0 to keep $ps-qr$ greater than -2. Consider $(-p,q)(-r,s)$. Since $r=0$ or $s=0$, $(-p,q)(-r,s) = (-p,q)(r,s)$, and $(-p,q)(r,s)$ satisfies the conditions for case 1, since $(-p)s-qr$ is 1 if $r=0$ or -1 if $s=0$.

If $r=0$, then $(-p,q)(-r,s) = (-p,q)(r,s) = A(-p+r, q+s) + A^{-1}(-p-r, q-s)$. Then by



(a) The crossing in K_1K_2 where the slope of K_2 is greater than the slope of K_1 near $(.5, .5)$.



(b) The crossing in K_1K_2 where the slope of K_2 is less than the slope of K_1 near $(.5, .5)$.

Figure 14: The two possibilities for the crossing in the knot product K_1K_2.

property 5 of Theorem 3, $(p, q)(r, s) = A^{-1}(p - r, q + s) + A(p + r, q - s)$, which is the same as $A^{-1}(p + r, q + s) + A(p - r, q - s)$ since $r = 0$.

If $s = 0$, then $(-p, q)(-r, s) = (-p, q)(r, s) = A^{-1}(-p + r, q + s) + A(-p - r, q - s)$. Then by property 5 of Theorem 3, $(p, q)(r, s) = A(p - r, q + s) + A^{-1}(p + r, q - s)$, which is the same as $A(p - r, q - s) + A^{-1}(p + r, q + s)$ since $s = 0$.

So if $p < 0$ and $r \geq 0$, the theorem holds. Now suppose $r < 0$ and $p \geq 0$. Then we have just shown that $(r, s)(p, q) = A^{-1}(r + p, s + q) + A(r - p, s - q)$ so by property 2 of Theorem 3, $(p, q)(r, s) = A(r + p, s + q) + A^{-1}(r - p, s - q) = A(p + r, q + s) + A^{-1}(p - r, q - s)$.

The only remaining possibility to check is $p < 0$ and $r < 0$. In this case, $(-p, q)(-r, s)$ satisfies the first case with determinant equal to $-(ps - qr)$. So if $ps - qr = 1$, $(-p, q)(-r, s) = A^{-1}(-p + -r, q + s) + A(-p + r, q - s)$ and by property 5 of Theorem 3, $(p, q)(r, s) = A(p + r, q + s) + A^{-1}(p - r, q - s)$. Finally, if $ps - qr = -1$, $(-p, q)(-r, s) = A(-p + -r, q + s) + A^{-1}(-p + r, q - s)$ and by property 5 of Theorem 3, $(p, q)(r, s) = A^{-1}(p + r, q + s) + A(p - r, q - s)$. □

2.3 Relationships from the One Crossing Theorem

The One Crossing Theorem results in some elegant relationships among knots. For instance, we find a recursive relationship in knots of the form $(1, q)$ by applying the One Crossing Theorem to $(0, 1)(1, q - 1)$.

$$\begin{aligned} (0,1)(1,q-1) &= A(1,q-2) + A^{-1}(1,q) \\ (1,q) &= A(0,1)(1,q-1) - A^2(1,q-2) \end{aligned}$$

If we look at $(2, q)$, we don't find quite so elegant a relationship, but a clear pattern remains.

$$(2,q) = \begin{cases} (1,\frac{q}{2})(1,\frac{q}{2}) & q \text{ even} \\ A(1,\frac{q+1}{2})(1,\frac{q-1}{2}) - A^2(0,1) & q \text{ odd} \end{cases}$$

Other sequences of numbers have beautiful relationships as well. If F_n denotes the nth Fibonacci number (with $F_0 = 0$ and $F_1 = 1$), then the following equation holds (for $n \geq 3$):

$$(F_n, F_{n+1}) = A^{(-1)^n}(F_{n-2}, F_{n-1})(F_{n-1}, F_n) - A^{2(-1)^n}(F_{n-3}, F_{n-2})$$

Note that the limit of this recursive sequence as n goes to inf is the irrational knot $(1, \Phi)$, where Φ is the golden ratio, which would cover the torus entirely.

2.4 Decomposition

In this section we will show that we can write every link as a sum of products of $(1,0), (0,1)$, and $(1,1)$ if we restrict ourselves to polynomial coefficients, or in terms of just $(1,0)$ and $(0,1)$ if we allow rational coefficients. In other words, $\{(1,0),(0,1),(1,1)\}$ and $\{(1,0),(0,1)\}$ form bases for algebras with polynomial and rational coefficients, respectively.

Definition 14. *We call the process of finding a representation in terms of the basis* **decomposing** *the link, and the representation itself the* **decomposition** *of the link.*

The decomposition of a link is not unique, but the technique we present for decomposing a link will always work, and will always result in the same decomposition. First, we explain the technique, and then we provide an example.

Decomposing a link is a recursive process of writing a link in terms of smaller links, and repeating that with the smaller links, until the link is written in terms of $(1,0)$, $(0,1)$, and $(1,1)$.

Definition 15. *A link (p', q') is smaller than (p,q) if (p', q') goes around the longitude or the meridian fewer times and does not go around either one more times than(p,q). In other words, $|p'| \leq |p|$, $q' \leq q$, and at least one of the inequalities is strict.*

First, we must show that it is possible to do this for all links other than $(1,0), (0,1), (1,1)$, and $(-1,1)$.

Theorem 6. *Given a link (p,q) that is not in $\{(1,0),(0,1),(1,1),(-1,1)\}$, it is possible to write the link as a sum of products of smaller links.*

Proof. We will divide the proof into two cases. First, we will consider (p,q) where both p and q are non-negative.

Suppose p is zero. Then $(p,q) = (0,q)$ is just the product of q copies of $(0,1)$, and so $(p,q) = (0,1)^q$, and we are done. Similarly, if q is zero, then $(p,q) = (1,0)^p$. Now assume that both p and q are strictly greater than 0.

Suppose that p and q are not relatively prime. If $k = \gcd(p,q)$, then (p,q) is k copies of $(\frac{p}{k}, \frac{q}{k})$. This means that $(p,q) = (\frac{p}{k}, \frac{q}{k})^k$, and again we are done. Now assume that p and q are relatively prime. Our method of attack will be to show that we can find two links (r_1, s_1) and (r_2, s_2) which cross once and $(r_1 + r_2, s_1 + s_2) = (p,q)$. We then apply the One Crossing Theorem to get an equation for the link product only containing links that are smaller than or equal to (p,q).

Towards this end, we seek numbers such that the following equations hold:

$$r_1s_2 - s_1r_2 = 1$$
$$r_1 + r_2 = p$$
$$s_1 + s_2 = q$$

A quick algebraic manipulation shows that $r_1 = p - r_2$ and $s_1 = q - s_2$. A substitution of these yields

$$1 = ps_2 - qr_2 \tag{1}$$

Because p and q are relatively prime, we know that a pair of integer values for r_2 and s_2 exist to satisfy this equation, and one such pair can be found through the Euclidean algorithm. However, this pair of numbers may not be the ones for which we are looking.

Recall that we want (r_2, s_2) to be smaller than (p, q), which means that $|r_2| \leq p$ and $|s_2| \leq q$ (and at least one is strictly less than). Furthermore, we want $(r_1, s_1) = (p - r_2, q - s_2)$ to be smaller than (p, q), which means that $|p - r_2| \leq p$ and $|q - s_2| \leq q$. From this, we conclude that we want $-2p \leq -r_2 \leq 0$ and $-2q \leq -s_2 \leq 0$. Combining all of these inequalities tells us that we want $0 \leq r_2 \leq p$ and $0 \leq s_2 \leq q$, again with at least one of the upper bounds on r_2 and s_2 being a strict inequality. Observe that since $pq - qp = 0$, if we find a solution to equation (1), we are guaranteed not to have $r_2 = p$ and $s_2 = q$, so we can ignore the requirement that one of the inequalities is strict.

Let us call the results of the Euclidean algorithm r and s, and we will show that if r and s do not satisfy these inequalities, then there exists an integer n such that $r + np$ and $s + nq$ satisfy both the equation and the inequalities.

Consider the line $py - qx = 1$ in the x, y -plane. We can rewrite this as $y = \frac{q}{p}x + \frac{1}{p}$, and we see that the line hits the three points $(r, s), (0, \frac{1}{p})$ and $(p, q + \frac{1}{p})$. Let $r = r' - np$, where $0 \leq r' \leq p$ and n is an integer, and define $s' = \frac{q}{p}r' + \frac{1}{p}$. Substitution gives us the following:

$$\begin{aligned} s' &= \frac{q}{p}r' + \frac{1}{p} \\ &= \frac{q}{p}(r + np) + \frac{1}{p} \\ &= \frac{q}{p}r + nq + \frac{1}{p} \\ &= s + nq \end{aligned}$$

So we see that $(r + np, s + nq)$ is another solution, and furthermore since $r + np$ is between 0 and p, $s + nq$ is between $\frac{1}{p}$ and $q + \frac{1}{p}$. Since r, s, p, q, and n are all integers, $s + nq$ is between 1 and q. Thus $r_2 = r + np$ and $s_2 = s + nq$ are the numbers we are looking for.

Since $0 \leq r_2 \leq p$, and $0 \leq s_2 \leq q$, it follows that $0 \leq p - r_2 \leq p$ and $0 \leq q - s_2 \leq q$. Recall that we have $r_1 = p - r_2$ and $s_1 = q - s_2$, and since it is not the case that both r_2 and s_2 were zero, at least one of r_1 and s_1 must be strictly less than the upper bound. Therefore (r_1, s_1) is also a smaller link than (p, q).

Since $r_1s_2 - s_1r_2 = 1$, we know from the One-Crossing Theorem that $(r_1, s_1)(r_2, s_2) = A(p, q) + A^{-1}(r_1 - r_2, s_1 - s_2)$, so if we solve for (p, q) we have the equation $(p, q) = A^{-1}(r_1, s_1)(r_2, s_2) - A^{-2}(r_1 - r_2, s_1 - s_2)$.

All we need to show now is that $(r_1 - r_2, s_1 - s_2)$ is a smaller link than (p, q), and we will be done showing that (p, q) can be expressed in terms of smaller links. If $r_1 \geq r_2$, then clearly $0 \leq r_1 - r_2 \leq r_1 \leq p$, and if $r_2 > r_1$, then $|r_1 - r_2| = r_2 - r_1$, and $|r_1 - r_2| \leq r_2 \leq p$. Similarly, if $s_1 \geq s_2$, then $0 \leq s_1 - s_2 \leq s_1 \leq q$, and if $s_2 > s_1$, then $|s_1 - s_2| = s_2 - s_1$, and $|s_1 - s_2| \leq s_2 \leq q$. The only thing remaining to check is that $(r_1 - r_2, s_1 - s_2) \neq (p, q)$ and $(r_1 - r_2, s_1 - s_2) \neq (-p, q)$.

Suppose $(r_1 - r_2, s_1 - s_2) = (p, q)$. Then since $(r_1 + r_2, s_1 + s_2) = (p, q)$, $r_2 = s_2 = 0$, which is a contradiction. Next, suppose $(r_1 - r_2, s_1 - s_2) = (-p, q)$. Then $r_2 = p$, $r_1 = 0$, $s_1 = q$, and $s_2 = 0$, but then $ps_2 - qr_2 = -qp$, and since q and p were both positive, this cannot be 1, so this is also a contradiction. Therefore, $(r_1 - r_2, s_1 - s_2)$ is a smaller link than (p, q). We conclude that $(p, q) = A^{-1}(r_1, s_1)(r_2, s_2) - A^{-2}(r_1 - r_2, s_1 - s_2)$ expresses (p, q) in terms of smaller links.

Finally, consider the case where p or q is negative. Since we can use the property that $(p, q) = (-p, -q)$ to force one of the two to be positive, we will assume that q is positive, and that p is negative since all other possibilities were handled previously. Recalling that the standard torus projection of (p, q) is the rotation of the standard torus projection of $(-p.q)$ around the line $x = \frac{1}{2}, z = 0$ in $\mathbb{R}^3$ and property 5 of the link product, if p is negative and $(-p, q) = A^{-1}(r_1, s_1)(r_2, s_2) - A^{-2}(r_1 - r_2, s_1 - s_2)$, then $(p, q) = A(-r_1, s_1)(-r_2, s_2) - A^2(r_2 - r_1, s_1 - s_2)$.

$\square$

This theorem tells us that we can write any link (p, q) where $|p| > 1$ of $|q| > 1$ in terms of smaller links. By doing this and repeating on the smaller links, we can write any link in terms of $(1, 0)$, $(0, 1)$, $(1, 1)$, and $(-1, 1)$.

We also have, from our table of simple link products, the equations:

$$(1,0)(0,1) = A(1,1) + A^{-1}(-1,1)$$
$$(0,1)(1,0) = A(-1,1) + A^{-1}(1,1)$$

If we solve both of these for $(-1, 1)$, we have

$$\begin{aligned} (-1,1) &= A(1,0)(0,1) - A^2(1,1) \\ (-1,1) &= A^{-1}(0,1)(1,0) - A^{-2}(1,1) \end{aligned}$$

We can substitute either one of these in for $(-1, 1)$ to get the link in terms of just $(1, 0)$, $(0, 1)$, and $(1, 1)$ with polynomial coefficients. Additionally, we can set the two equations equal and solve for $(1, 1)$.

$$\begin{aligned} A(1,0)(0,1) - A^2(1,1) &= A^{-1}(0,1)(1,0) - A^{-2}(1,1) \\ \Rightarrow (-A^2 + A^{-2})(1,1) &= A^{-1}(0,1)(1,0) - A(1,0)(0,1) \\ \Rightarrow (1,1) &= \frac{1}{-A^3 + A^{-1}}(0,1)(1,0) - \frac{1}{-A + A^{-3}}(1,0)(0,1) \end{aligned}$$

Plugging this in for $(1, 1)$ into the expression for the link gives an expression in terms of $(1, 0)$ and $(0, 1)$ with rational coefficients.

Note that this then implies that the generators for our algebra S are $(0, 1), (1, 0), (1, 1)$. Furthermore, this allows us to multiply links without applying the skein relation. Because of the distributivity of the algebra, we can decompose the two links we wish to multiply and then multiply the decompositions.

As an example of how to decompose a link, we will take a look at $(4,5)$.

Example 3. Since 4 and 5 are relatively prime, we want to find a pair of knots (p,q) and (r,s) such that $ps-qr=1$, $p+r=4$ and $q+s=5$ so that we can apply the One Crossing Theorem. From the proof of Theorem 6 we know we want solutions to the equation $4s-5r=1$. By inspection or the method outlined in the proof, we see that $r=3$ and $s=4$ is a solution. This gives us $p=1$ and $q=1$.

Thus we have that $(1,1)$ and $(3,4)$ are a pair of knots with $1\cdot 4-1\cdot 3=1$, so $(1,1)(3,4)=A(4,5)+A^{-1}(2,3)$. Therefore $(4,5)=A^{-1}(1,1)(3,4)-A^{-2}(2,3)$.

To continue decomposing $(4,5)$, we now need to decompose $(3,4)$, $(2,3)$, and $(1,1)$. We'll start with $(3,4)$ and look for a new pair of knots (p,q) and (r,s) such that $ps-qr=1$, $p+r=3$ and $q+s=4$. We now have $3s-4r=1$, and we get $r=2, s=3, p=1$, and $q=1$.

Applying the One Crossing Theorem again, we have that $(1,1)(2,3)=A(3,4)+A^{-1}(1,2)$, so $(3,4)=A^{-1}(1,1)(2,3)-A^{-2}(1,2)$. If we substitute this into our equation for $(4,5)$, we have

$$\begin{aligned}(4,5) &= A^{-1}(1,1)(A^{-1}(1,1)(2,3)-A^{-2}(1,2))-A^{-2}(2,3)\\ &= A^{-2}(1,1)(1,1)(2,3)-A^{-3}(1,1)(1,2)-A^{-2}(2,3)\\ &= \big(A^{-2}(1,1)(1,1)-A^{-2}\big)(2,3)-A^{-3}(1,1)(1,2)\end{aligned}$$

This leaves us with $(1,1)$, $(1,2)$, and $(2,3)$ to decompose. We'll take $(2,3)$ this time. Following the process we used for $(4,5)$ and $(3,4)$, we find that $(1,1)(1,2)=A(2,3)+A^{-1}(0,1)$, so $(2,3)=A^{-1}(1,1)(1,2)-A^{-2}(0,1)$. Making this substitution gives us

$$\begin{aligned}(4,5) &= \big(A^{-2}(1,1)(1,1)-A^{-2}\big)\big(A^{-1}(1,1)(1,2)-A^{-2}(0,1)\big)-A^{-3}(1,1)(1,2)\\ &= \big(A^{-2}(1,1)(1,1)-A^{-2}\big)A^{-1}(1,1)(1,2)-\big(A^{-2}(1,1)(1,1)-A^{-2}\big)A^{-2}(0,1)-A^{-3}(1,1)(1,2)\\ &= \big(A^{-3}(1,1)(1,1)-2A^{-3}\big)(1,1)(1,2)-\big(A^{-4}(1,1)(1,1)-A^{-4}\big)(0,1)\end{aligned}$$

We'll move on to $(1,2)$ now. $(1,2)=A^{-1}(1,1)(0,1)-A^{-2}(1,0)$, so we have

$$\begin{aligned}(4,5) &= \big(A^{-3}(1,1)(1,1)-2A^{-3}\big)(1,1)\big(A^{-1}(1,1)(0,1)-A^{-2}(1,0)\big)-\big(A^{-4}(1,1)(1,1)-A^{-4}\big)(0,1)\\ &= \big(A^{-3}(1,1)(1,1)-2A^{-3}\big)(1,1)A^{-1}(1,1)(0,1)-\big(A^{-3}(1,1)(1,1)-2A^{-3}\big)(1,1)A^{-2}(1,0)-\\ &\quad \big(A^{-4}(1,1)(1,1)-A^{-4}\big)(0,1)\\ &= \big(A^{-4}(1,1)(1,1)(1,1)(1,1)-3A^{-4}(1,1)(1,1)+A^{-4}\big)(0,1)\\ &\quad -\big(A^{-5}(1,1)(1,1)-2A^{-5}\big)(1,1)(1,0)\end{aligned}$$

We now have an expression for $(4,5)$ in terms of the generators of the algebra with polynomial coefficients. We know that $(1,1)=\frac{1}{-A^3+A^{-1}}(0,1)(1,0)-\frac{1}{-A+A^{-3}}(1,0)(0,1)$, so we can substitute that in to get an equation in terms of polynomials in A and $(0,1)$ and $(1,0)$ with rational coefficients.

3 Knots and Links on the Tori with Boundary, not Intersecting

Next, we considered knot multiplication on the punctured torus with knots that do not intersect the boundary. As we quickly discovered, this case was not dramatically more complicated than the ordinary torus, and our previous results generalized quite well. In this section, we show that for a given p and q, all (p, q) knots that do not touch the boundary on the punctured torus are isotopic to one another. In addition, we prove that the only new non-intersecting loop on the punctured torus can be written in term of the generators $(1, 0)$, $(0, 1)$ and $(1, 1)$.

3.1 Isotopy of Non-intersecting Torus Knots

Theorem 7. *All (p, q) torus knots on the torus-with-boundary are isotopic.*

Proof. The standard representation of (p, q) consists of $p + q - 1$ parallel lines in the unit square $[0, 1) \times [0, 1)$. Each of these lines has slope $\frac{q}{p}$, and they are evenly spaced across the torus. However, the square torus is a complete set of representatives of the equivalence classes of $\mathbb{R}^2/\sim$. In fact, the torus knot (p, q) is the set of all lines with slope $\frac{q}{p}$ in $\mathbb{R}^2/\sim$ that intersect $(0, \frac{n}{q})$ for some n in $\mathbb{N}$. Since the square torus is a complete set of representatives under our equivalence relation, we can position the unit square anywhere in $\mathbb{R}^2$ and still have a set of points that represents the torus, with the line segments contained within the square representing the torus knot (p, q).

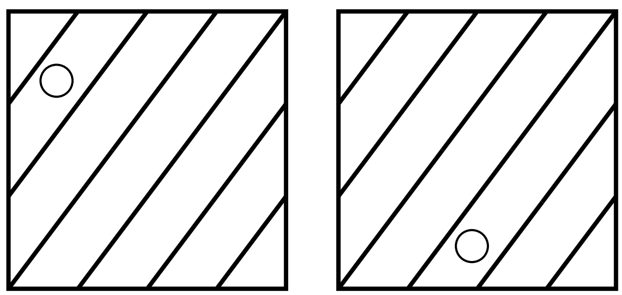


Figure 15: Two different projections of (p, q) on the punctured torus

Suppose we're given two (p, q) torus knots with boundaries in different locations. In both knots, the boundary doesn't intersect with the knot. We can assume that the boundaries are arbitrarily small in each case. Furthermore, assume in both cases that p and q are positive integers. Boundary B_1 is located at (n_1, m_1), while B_2 is located at (n_2, m_2), where $n_1, m_1, n_2, m_2 \in [0, 1)$. We want to show that these two torus knots with boundary are identical despite their different boundary locations.

We start by shifting the torus-square for the second knot, so that the new origin is at $(n_2 - n_1, m_2 - m_1)$. Thus, our new complete set of representatives for the second torus knot is the set of points in $[n_2 - n_1, n_2 - n_1 - 1) \times [n_2 - n_1, n_2 - n_1 - 1)$, but it still represents the same torus knot and boundary. For both squares, the position of the boundary relative to the origin is now (n_1, m_1).

Furthermore, we can define an isotopy for each torus knot that puts each knot in standard position. For each torus, the knot is already a set of parallel straight lines, and the boundary lies between two of them. This means that the knot intersects the base of the unit square q times, so the horizontal distance between any two adjacent lines is $\frac{1}{q}$. If the center of the boundary is closer to the left line, the set of lines can be shifted in the negative x-direction by up to $\frac{1}{2q}$ without intersecting the boundary; likewise, if the center is closer to the line on the right, the lines can be shifted in the positive x-direction by up to $\frac{1}{2q}$. If the center of the boundary is equidistant from both lines, both tori can be moved in the positive x-direction by $\frac{1}{4q}$, then this process can be repeated for each. Since the origin must lie between two of the intersection points along the base of the unit square, we can create an isomorphism that moves the torus knot so it intersects the origin of the square without crossing the boundary. Since we can do this for each torus and end up with identical pictures in $\mathbb{R}^2/\sim$, then clearly any two (p, q) torus knots on the boundary are isotopic.

□

3.2 Knots Around the Boundary

We now show how the knots composed of n concentric circles around the boundary (e.g. for $n = 1$, see Figure 16), the new elements in the additive basis for the punctured torus, are able to be created through our previously found generators - i.e. we show that these elements are not generators for the algebra of torus knot on the torus with boundary.

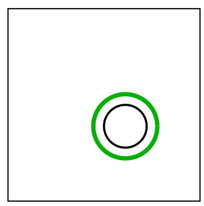


Figure 16: A circle around the boundary

Theorem 8. *The empty loop around the boundary of a punctured torus is not a basis element.*

Proof. We can easily find several examples of knot multiplication that generate an empty loop around the boundary. One of these is $(1, 1)(0, 1)(1, 0)$, as seen in Figure 17.

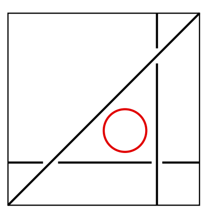
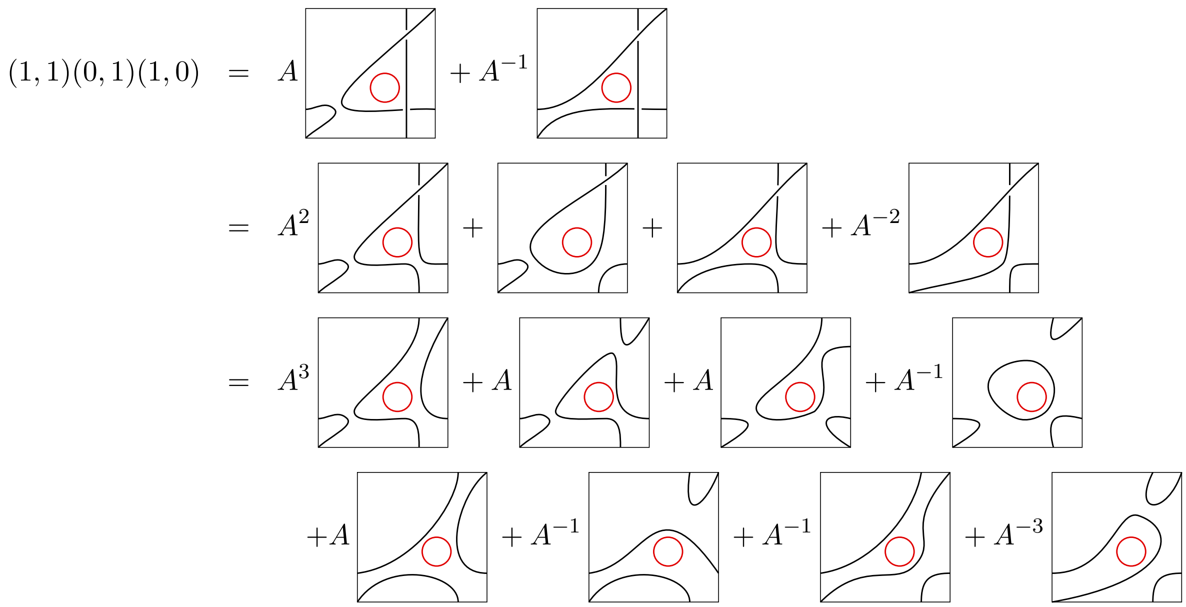


Figure 17: The boundary of a torus surrounded by the product $(1, 1) * (0, 1) * (1, 0)$

We can use skein relations to produce a polynomial for this product as follows, where (loop) represents the empty loop around the boundary:



$$= A^3(0,2) + A(loop) + A(loop) + A^{-1}(-A^2 - A^{-2})(loop) + A(-A^2 - A^{-2}) + A^{-1}(2,0) + A^{-1}(2,2) + A^{-3}(loop)$$

This can be simplified to give us the solution

$$(loop) = A^{-1}(1,1)(0,1)(1,0) - A^2(2,0) + A^2 + A^{-2} - A^{-2}(2,0) - A^{-2}(2,2)$$

It is clear from this that we can create n circles around the boundary by simply surrounding the boundary with n copies of $(0,1)$, $(1,0)$, $(1,1)$. This then implies that we can achieve n concentric circles through these three elements of our algebra.

□

4 Knots and Links Intersecting the Boundary of a Torus

We next attempted to explain multiplication of torus knots that do intersect the boundary on the punctured torus. Ultimately, our goal was to use this understanding of multiplication on the punctured torus to explain multiplication on the double-torus, which is simply two punctured tori glued together. Unfortunately, we were unable to completely generalize our algebra to the punctured torus with boundary intersections, but we did find some interesting results. Multiplying a link that doesn't touch the boundary by one that does turns out to be straightforward, and we developed theorems for other special cases including a method for handling one-crossing resolution of boundary-intersecting knots.

In order to discuss multiplication on the double torus, we needed to develop a notation for describing double torus knots. Developing this notation required several new theorems on the punctured torus. We showed that in a set of non-crossing knots that touch the boundary, there can exist at most three different knots, though there can be more than one copy of each of those knots. This allowed us to determine the relative locations of the boundary intersections for any set of knots, which lead to a notation for sets of boundary-intersecting, non-crossing knots on the punctured torus. Finally, we introduced the idea of twisting on the double torus and why it is necessary to describe links on the double-torus, before finally introducing the Fisher-Phelps-Wells notation for links on the double-torus.

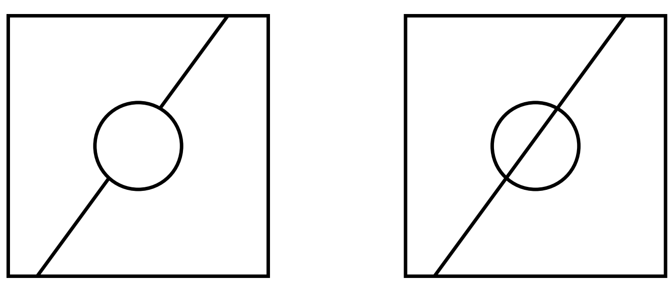
4.1 Multiplication with One Knot Intersecting the Boundary

Definition 16. *On a torus with boundary, we define* $(\mathbf{p}, \mathbf{q}, \mathbf{n})$ *to be the link that goes around the torus* p *times longitudinally and* q *times meridianally (again where the positive directions are up and to the right) and intersects the boundary* $2n$ *times.*

As before, $(-p, -q, n) = (p, q, n)$. Note also that there are at least n components to the link. We now prove several lemmas and theorems that allow us to verify that this notation is in one-to-one correspondence with links without any crossings on the torus with boundary.

Lemma 5. *If* $(p, q, 1)$ *is a knot with no crossings on the torus with boundary, then* p *and* q *are relatively prime.*

Proof. If we were to take the torus with boundary and fill in the boundary with a direct connection between the two points of intersections (see Figure 18), then we would have a (p, q) knot on the torus without boundary. Since this is a knot rather than a link, p and q must be relatively prime. □



(a) A localized view of the boundary with two intersections.

(b) A localized view of the boundary with two intersections filled in.

Figure 18: Filling in a boundary with two intersections.

Theorem 9. *Usual multiplication holds when multiplying together a knot without boundary intersection and a knot with boundary intersection on the punctured torus.*

Proof. Since only one of the knots intersects the boundary, we can treat this multiplication as if neither knot intersected the boundary, treating the boundary as simply a part of the knot that touches it. □

Corollary 1. *If a knot $(p, q, 0)$ on a torus with boundary does not intersect the boundary, then the only knots that do not cross it are $(p, q, 0)$ or $(p, q, 1)$.*

Lemma 6. *Suppose two distinct knots $(p, q, 1)$ and $(r, s, 1)$ on a torus with boundary do not cross each other. Then $|det((p, q)(r, s))| = |ps - qr| = 1$.*

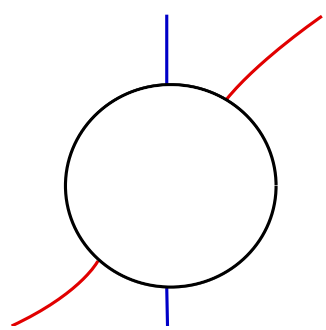
Proof. Suppose we take the torus with boundary and fill in the boundary by directly connecting the two intersections of one knot and the two intersections of the other knot, so that we have a link with two components. Note that the method of doing this will depend on how the two knots intersect the boundary relative to each other (see Figure 19). Since $(p, q, 1) \neq (r, s, 1)$, we know that (p, q) and (r, s) must have at least one crossing on the torus without boundary. Since we connected directly across the boundary, we have at most one crossing, so there must be exactly one crossing. By the Crossing Number Theorem for a torus without boundary, $|ps - qr| = 1$. □

4.2 Multiplication with Two Knots Intersecting the Boundary

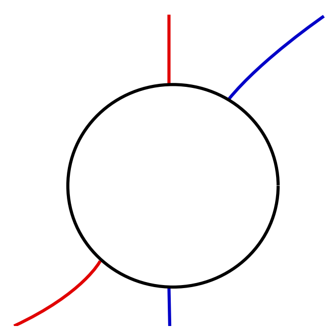
Until this point we haven't considered multiplying two torus knots, both of which touch the boundary. We do so below, and prove a one crossing theorem for torus knots on the torus with boundary, each of which touch once.

Definition 17. *Two torus knots $(p, q, 1)$ and $(r, s, 1)$ are said to be multiplied in* **trans** *if locally their product is as in Figure 19a; this is denoted $(p, q, 1)(r, s, 1)_{trans}$.*

Definition 18. *Two torus knots $(p, q, 1)$ and $(r, s, 1)$ are said to be multiplied in* **cis** *if locally their product is as in Figure 19b; this is denoted $(p, q, 1)(r, s, 1)_{cis}$.*



(a) The product of two knots in trans



(b) The product of two knots in cis

Figure 19: $(p, q, 1)(r, s, 1)$ multiplied in trans and cis, respectively

With these definitions, we give our one-crossing theorem for torus knots with boundary.

Theorem 10. *Given two torus knots $(p, q, 1)$ and $(r, s, 1)$, each of which touches the boundary once,*

$$(p, q, 1)(r, s, 1)_{trans} = \begin{cases} A\left(\frac{p+r}{2}, \frac{q+s}{2}, 1\right)^2_{cis} + A^{-1}\left(\frac{p-r}{2}, \frac{q-s}{2}, 1\right)^2_{cis} & ps - qr = 2 \\ A^{-1}\left(\frac{p+r}{2}, \frac{q+s}{2}, 1\right)^2_{cis} + A^{1}\left(\frac{p-r}{2}, \frac{q-s}{2}, 1\right)^2_{cis} & ps - qr = -2 \end{cases}$$

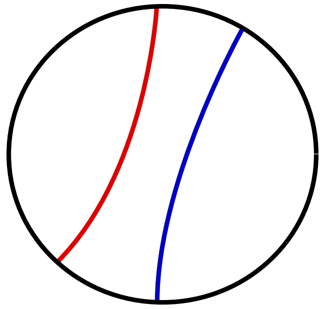


Figure 20: The "missing piece" that would complete the torus without boundary into a torus with boundary, if added.

Proof. First note that the product of two torus knots with boundary in trans can only occur if when boundary is replaced the torus knots have a crossing in this region — i.e. the value of $|ps - qr|$ for a knot product in trans with a single crossing on the torus with boundary must be the same as for a knot product with two crossings on the torus without boundary: two.

Also note that because we are assuming the knots are not parallel, and are also assuming there is only one crossing on our torus with boundary, then we must only have one copy of each knot — $\gcd(p, q) = \gcd(r, s) = 1$. For the case in which $ps - qr > 0$, it follows from the fundamental group of a torus (see the proof of the one crossing theorem on the torus without boundary) and the definition of skein resolution that the right resolution of the single crossing of our product on the torus with boundary must create a knot product in *cis* which has $(p + r)$ meridian crossings and $(q + s)$ longitude crossings.

Similarly, it follows from the same rationale that the left resolution of the trans product on the torus with boundary must create a knot product in *cis* which has $(p - r)$ meridian crossings and $(q - s)$ longitude crossings. It then remains to show how these crossings are distributed amongst the two knots in both the right and left resolution. The proof of this is similar to other proofs in this section: imagine that we were to fill the boundary with a circle that contained the "missing" section of torus (see Figure 20). With this piece filled in on the torus, we would get two torus knots which do not intersect — the only way this is possible is if we have two parallel torus knots. This then implies that on the torus with boundary the result of our product is $A(\frac{p+r}{2}, \frac{q+s}{2})^2_{cis}$ for the right resolution and $A^{-1}(\frac{p-r}{2}, \frac{q-s}{2})^2_{cis}$ for the left resolution. This proves the first half of our theorem. The second half of our theorem, when $ps - qr = -2$, again follows from the proof of the one crossing theorem for tori without boundary — when the crossing number of a knot product is negative, the sign on the powers on the coefficients A are reversed. The rationale in that proof also works for this one, (since the power of A is a local property and would thus effectively ignore the boundary), and completes the second half of our proof.

□

4.3 Three Distinct Knots on the Punctured Torus

Theorem 11. *There can be at most three distinct knots on a torus with boundary with no crossings.*

Proof. We will show this by supposing that there are at least four distinct knots on a torus with boundary without any crossings and showing that this results in a contradiction. We will denote these four knots (p_1, q_1, n_1), (p_2, q_2, n_2), (p_3, q_3, n_3), and (p_4, q_4, n_4). Note that since these are knots rather than links, each n_i must be 0 or 1.

From corollary 1, lemma 5, and lemma 6 we know that the following must be true:

- Each of the knots must intersect the boundary. If one or more knots did not intersect the boundary, lemma 5 states that each of the knots would be parallel, with $p_i = p_j$ for each i and j. Since each knot intersects the boundary once, every n_i must be 1.
- For each $1 \leq i \leq 4$, p_i and q_i must be relatively prime. Since the four knots must be distinct, each $\frac{q_i}{p_i}$ must then be distinct from the others.
- For any pair of these two knots, $|p_i q_j - q_i p_j| = 1$.

Without loss of generality, we will assume that $\frac{q_1}{p_1} < \frac{q_2}{p_2} < \frac{q_3}{p_3} < \frac{q_4}{p_4}$. The following equations must hold:

$$p_1 q_2 - q_1 p_2 = 1 \tag{2}$$
$$p_1 q_3 - q_1 p_3 = 1 \tag{3}$$
$$p_1 q_4 - q_1 p_4 = 1 \tag{4}$$
$$p_2 q_3 - q_2 p_3 = 1 \tag{5}$$
$$p_2 q_4 - q_2 p_4 = 1 \tag{6}$$
$$p_3 q_4 - q_3 p_4 = 1 \tag{7}$$

Furthermore, only one of the p_i may be 0, and if any is, it must be p_4 since $\frac{q_i}{0} = \infty$ and ∞ is greater than every other possible value for $\frac{q_j}{p_j}$. Therefore, we know that none of p_1, p_2, and p_3 are 0. We can therefore solve equations (4), (6), and (7) for q_4 without running the risk of dividing by 0.

$$q_4 = \frac{1 + q_1 p_4}{p_1}$$
$$q_4 = \frac{1 + q_2 p_4}{p_2}$$
$$q_4 = \frac{1 + q_3 p_4}{p_3}$$

From this, we have that

$$\frac{1 + q_1 p_4}{p_1} = \frac{1 + q_2 p_4}{p_2} \tag{8}$$
$$\frac{1 + q_1 p_4}{p_1} = \frac{1 + q_3 p_4}{p_3} \tag{9}$$

We now rearrange equations (8) and (9) to get the following:

$$p_4 \left(\frac{q_1 p_2 - q_2 p_1}{p_1 p_2}\right) = \frac{1}{p_2} - \frac{1}{p_1} \tag{10}$$
$$p_4 \left(\frac{q_1 p_3 - q_3 p_1}{p_1 p_3}\right) = \frac{1}{p_3} - \frac{1}{p_1} \tag{11}$$

From equations (2) and (3), we know that $q_1p_2 - q_2p_1$ and $q_1p_3 - q_3p_1$ are both -1, so we can simplify equations (10) and (11) to the following:

$$p_4 = p_2 - p_1$$
$$p_4 = p_3 - p_1$$

Thus $p_2 = p_3$. We now apply this to equation (5) and see that

$$\begin{aligned} 1 &= p_2q_3 - q_2p_3 = p_2q_3 - q_2p_2 \\ \frac{1}{p_2} &= q_3 - q_2 \end{aligned}$$

Since q_3 and q_2 are integers, $\frac{1}{p_2}$ must be an integer, so p_2 must be either 1 or -1.

Suppose $p_2 = 1$, and therefore that $p_3 = 1$. Let's look at what effects this has on equations (2) and (3).

$$1 = p_1q_2 - q_1p_2 = p_1q_2 - q_1$$
$$1 = p_1q_3 - q_1p_3 = p_1q_3 - q_1$$

This, however, implies that $q_2 = q_3$, which is a contradiction to $q_3 - q_2 = \frac{1}{p_2}$. Suppose instead that $p_2 = -1$. Similarly, this results in $p_3 = -1$ and in $q_2 = q_3$. These contradictions show that we cannot have 4 distinct knots with no crossings.

□

4.4 A Bijective Notation for Punctured Torus Knots

Theorem 12. *A set of boundary-intersecting torus knots that share no crossings can be represented uniquely with three numbers (p, q, n), where p is the number of times a knot crosses the meridian, q the number of times a knot crosses the longitude, and n is half the number of boundary intersections.*

Proof. For a given set of boundary-intersecting, non-crossing torus knots on the punctured torus, we can arrange the knots in standard projection and count how many times a knot in the set crosses the meridian or longitude to determine p and q. Likewise, we can find n by simply counting the boundary intersections, so for any set of non-crossing torus knots that intersect the boundary, a triplet (p, q, n) can be found to describe it.

However, if we are given a triplet (p, q, n), does it determine a unique set of non-crossing torus knots on the punctured torus? Suppose not. Suppose we're given two different sets of knots, A and B, both of which cross the meridian and longitude p and q times respectively, and each of which has a total of $2n$ boundary intersections. So, we can draw a standard projection on the punctured torus for each set. Now, consider the possibility of "filling in" the boundary by connecting pairs of boundary intersections without adding crossings. Every possible way of filling in the boundary results in a knot or a set of knots on the torus without boundary, and one possible knot that would result from this process is the (p, q) knot on the un-punctured torus. Thus, the standard projection for either set must be isotopic to the knot (p, q) on the standard torus with a boundary added to intersect this knot n times. By the transitive property, the standard projections for the two sets are isotopic to each other, so the two sets of knots must be the same, causing a contradiction.

□

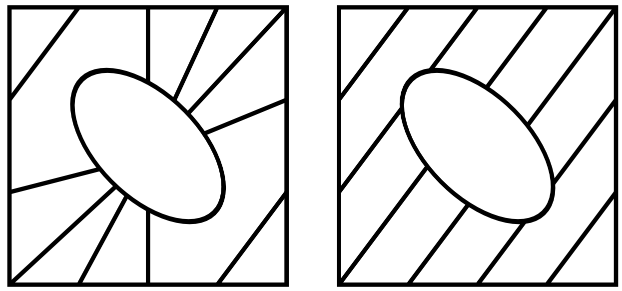


Figure 21: Two punctured tori glued together into a double torus

5 The Double Torus

Using the results from each of the previous section, here we prove our final result: a bijective notation for the double torus. We developed this notation as a means of describing double-torus knots. We used our knowledge of knots on the punctured torus to explore knots on the double torus, since the double torus is simply two punctured tori with their boundaries (in pink) glued together.

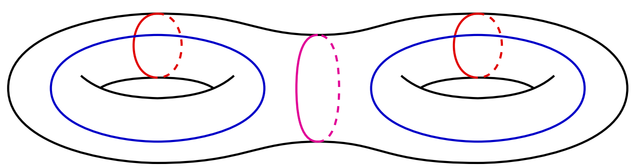


Figure 22: Two punctured tori glued together into a double torus

However, there is a problem. When gluing together two punctured tori with knots on them, we need to include instructions on how to connect the knots and properly line up the boundary intersections. Without such instructions, it's possible to connect knots in different ways so they twist around the center of the double torus, resulting in different knots.

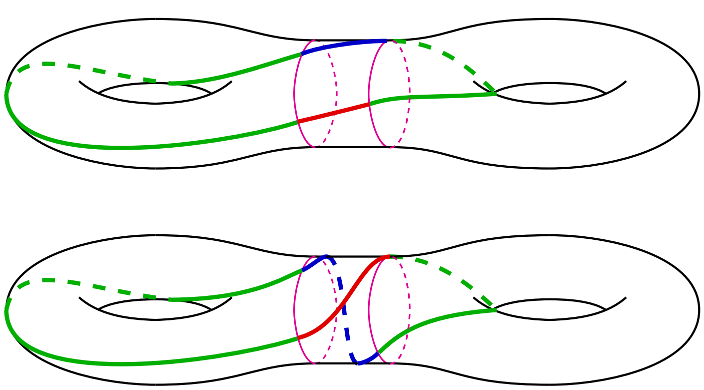


Figure 23: Two different ways of gluing together knots on the punctured torus

A better way to depict this is to use a different model of the double torus. In this model, the meridians, longitudes, and equator from figure 22 are represented in the same red, blue and pink as before. Any knot on the double torus can be redrawn on this model of the torus by crossing the same meridians and longitudes as before. However, this model is better for representing twisting.

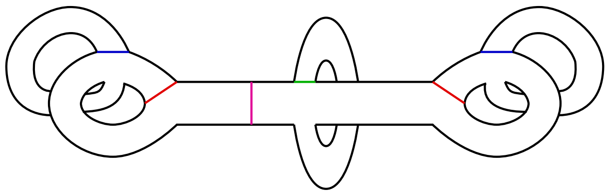


Figure 24: Alternate representation of the double torus

Before going any further, we need to introduce the idea of a zero-point on a boundary. On the punctured torus with $2n$ boundary intersections, the zero-point is simply one of the boundary intersections, but for a given set of knots, the zero-point for that set will be unique. To find the zero-point for a given set of knots, we first determine what knot or knots have the smallest ratio $\frac{q}{p}$. If there is more than one knot with the same ratio $\frac{q}{p}$, then those knots are identical and must exist in parallel on the punctured torus, as shown before. We orient each of these knots so they are circling the center-hole of the torus in a clockwise direction, and select the endpoints of this orientation, which should be immediately adjacent to each other as was previously shown. Of these adjacent boundary intersections we've selected, the one farthest clockwise is the zero-point.

We account for twisting using a twisting constant k. When $k = 0$, we simply connect the two zero-points; by connecting those two points, the rest of the gluing is determined by avoiding any knot crossings. If $k > 0$, then we force the top-most strand on the left to cross the green "twisting line" in figure 24 and loop back around to the other side. We repeat this process k times so that ultimately, the knot or set of knots crosses the green line k times. Note that if $|k| > 2n$, some lines will cross the green twisting line more than once. If $k < 0$, we repeat the process as before, but this time start with the bottom-most strand on the left, twisting it down and around the loop before crossing the green line.

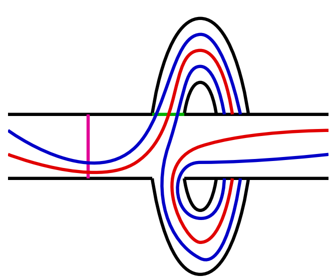


Figure 25: Depiction of twisting for $k = 3,\ n = 1$

With the help of this twisting constant, we are now ready to introduce a new notation for knots on the double torus.

Theorem 13. *We can use six numbers, $(p_1, q_1, n, k, p_2, q_2)$, to describe knots on the double-torus, where: p_1 and q_1 describe the number of times the knot crosses the meridian and longitude of the* *left hole of the double torus, p_2 and q_2 describe the number of times the knot crosses the meridian and longitude of the right hole of the double torus, n is half the total number of times the knot crosses the equator, and k is the number of times the knot crosses the twisting line. This notation is well-defined and describes every knot on the double torus.*

Proof. We have already shown that (p_1, q_1, n) and (p_2, q_2, n) uniquely define the sets of knots on the left and right punctured tori. Furthermore, k is defined to identify a unique way to glue these two sets of knots together, so clearly the notation determines a unique knot on the double torus.

Likewise, for any knot on the double torus, there exists an isomorphism to move the knot onto the alternate model of the double torus in figure 24. At that point, you simply need to count the number of times the knot intersects the meridians, latitudes, equator, and twisting line to figure out your values for p_1, p_2, q_1, q_2, n, and k respectively. $\square$

6 Conjectures and Further Directions

In this paper we have shown a number of results for knots on the single, punctured and double torus, but there is still a lot more that we can do. To begin, we still do not have a clear idea how multiplication works on the double torus (or in general the n-genus surface), and only have a few theorems for the torus with boundary. Even more, we do not as of yet have a clear idea what the generators are for either of these algebras. We believe that solving this latter question will give us direction on the former, but as of yet finding the answers to both has proved difficult. In the case of the torus with boundary intersections, we do have a one crossing theorem, but this theorem does not allow for a similar decomposition result as in the torus without boundary: the results of the products are not smaller or larger, but often in between, and furthermore are not single knots but the product of two knots in cis. For these two reasons, (and others), any sort of decomposition theorem for the torus with boundary intersections or the double torus will most likely be quite different in structure and formulation.

We also have several conjectures and ideas that we believe will give us direction, but did not have time to pursue. The first is concerning multiplication on the double torus. We investigated an alternate formulation of this problem by considering the double torus not as two punctured tori glued together but as two punctured tori glued to either side of a cylinder. That way all of the crossings from both tori and be pushed onto the cylinder, effectively reducing the problem to understanding knots on a cylinder. Given that the cylinder is equivalent to a square with two sides identified, this formulation quickly leads toward an investigation of Jones-Wenzel idempotents, which we believe will play an important part in the the answer.

An alternate research question alluded to above is the multiplication of torus knots on the cylinder $\mathbb{R}^2/\mathbb{Z}$. As mentioned above, we believe the answer to the basic questions for knots on this surface may yield valuable information for knots on the double torus. Furthermore, the surface is just interesting in its own right.

In section 2.2 a recursive formula is given for knots of the form $(1, [1]_n)$, where $[1]_n$ denotes the n^{th} element of the continued fraction sequence $[1, 1, \dots]$. It is also mentioned that because this sequence $[1]_n$ converges to ϕ, this gives us a formulation for the irrational knot $(1, \Phi)$ as a limit of the knots $(1, [1]_n)$. The question that we investigated but were not able to answer is "are there any other recurrence relations among torus knots based around other sequences of continued fractions or irrational numbers?"

We mentioned above the idea of a sequence of knots limiting on another knot, but it seems unclear whether such a limit makes sense, since there is no metric or even topology on our space S. Another project would then be to formulate an idea of "open set" in our space S, or even more, formulate a notion of distance (i.e. a metric). Such a formulation (although seemingly far removed from the current project) may give valuable information on torus knots.

While investigating the one crossing theorem for knots with boundary intersection, it became clear that there was a strong connection between one crossing knot products on this surface, $(p, q, 1)(r, s, 1)$, and two crossing knot product on $\mathbb{T}$. (Since a one can imagine filling in the boundary on a one crossing knot product with boundary to get a two crossing knot on $\mathbb{T}$. Given the theorem 10, it seems possible that an explicit formula for such a product could be found be exploiting the relationships between theses two surfaces.

Throughout this project we worked toward developing a bijective notation for torus knots. While we do believe we have found one, (the so called F-P-W notation), we have yet to complete the proof that this notation is bijective. Furthermore, this notation only holds for the genus-2 surface, and does not seem to allow for an easy generalization. It would be interesting and important to develop a general notation for the n-genus surface.

7 Appendix

	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,2)$	$A(-1,1)+A^{-1}(1,1)$	$A(1,0)+A^{-1}(1,2)$
$(1,0)$	$A(1,1)+A^{-1}(-1,1)$	$(2,0)$	$A(2,1)+A^{-1}(0,1)$
$(1,1)$	$A(1,2)+A^{-1}(1,0)$	$A(0,1)+A^{-1}(2,1)$	$(2,2)$
$(1,2)$	$A(1,3)+A^{-1}(1,1)$	$A^2(0,2)+2(-A^2-A^{-2})+A^{-2}(2,2)$	$A(0,1)+A^{-1}(2,3)$
$(2,1)$	$A^2(2,2)+2(-A^2-A^{-2})+A^{-2}(2,0)$	$A(1,1)+A^{-1}(3,1)$	$A(3,2)+A^{-1}(1,0)$
$(1,3)$	$A(1,2)+A^{-1}(1,4)$	$A^3(0,3)-3A^3(0,1)+A^{-3}(2,3)$	$A^2(0,2)+2(-A^2-A^{-2})+A^{-2}(2,4)$
$(-1,1)$	$A(1,0)+A^{-1}(-1,2)$	$A(-1,2)+A^{-1}(0,1)$	$A^2(2,0)+2(-A^2-A^{-2})+A^{-2}(0,2)$

	$(1,2)$	$(2,1)$
$(0,1)$	$A(1,1)+A^{-1}(3,1)$	$A^2(2,0)+2(-A^2-A^{-2})+A^{-2}(2,2)$
$(1,0)$	$A^2(2,2)+2(-A^2-A^{-2})+A^{-2}(0,2)$	$A(3,1)+A^{-1}(1,1)$
$(1,1)$	$A(2,3)+A^{-1}(0,1)$	$A(1,0)+A^{-1}(3,2)$
$(1,2)$	$(2,4)$	$A^3(-1,1)-3A^{-3}(1,1)+A^{-3}(3,3)$
$(2,1)$	$A^3(3,3)-3A^3(1,1)+A^{-3}(-1,1)$	$(4,2)$
$(1,3)$	$A(0,1)+A^{-1}(2,5)$	$A^3(0,2)(1,0)+(-2A^3+A^{-1})(1,0)-3A^{-2}(0,1)(1,1)$ $+(-A+A^{-3})(1,2)+A^{-5}(3,4)$
$(-1,1)$	$A^3(2,1)-3A^{-3}(0,1)+A^{-3}(0,3)$	$A^3(3,0)-3A^3(1,0)+A^{-3}(1,2)$

Table 1: The results of some small knot products.

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