Groups, Graphs, and Symmetry: Cayley Graphs and the Cayley Isomorphism Property[1]

Gregory Michel

Carleton College

November 3, 2013

[1] Joint work with Christopher Cox (Iowa State University) and Hannah Turner (Ball State University) as a part of the 2013 REU at Iowa State University (NSF DMS 0750986) under the guidance of Sung Y. Song (Iowa State University) and Kathleen Nowak (Iowa State University)

Definition (Group)

A *group* G is a set that is closed under some binary associative operation $*$ where

1. There is an *identity element* e for which $a * e = e * a = a$ for all $a \in G$.
2. Every element $a \in G$ has an *inverse* a^{-1} for which $aa^{-1} = a^{-1}a = e$

Cayley Graphs

Definition (Cayley Graph)

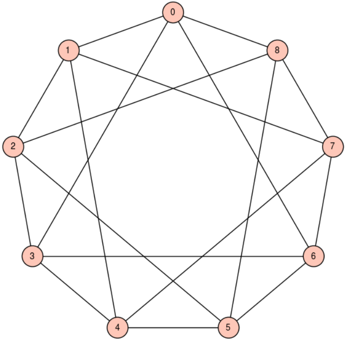
Given a finite group G and a symmetric connector set $S \subseteq G \setminus \{e\}$, the *Cayley graph*, denoted $Cay(G, S)$, is the graph with $V = G$ and $E = \{(x, y) \in V \times V : x^{-1}y \in S\}$ (i.e $y = xs$ for some $s \in S$.)

Cayley Graphs

Definition (Cayley Graph)

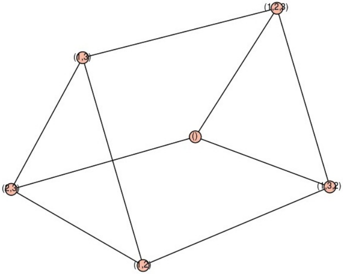
Given a finite group G and a symmetric connector set $S \subseteq G \setminus \{e\}$, the *Cayley graph*, denoted $Cay(G, S)$, is the graph with $V = G$ and $E = \{(x, y) \in V \times V : x^{-1}y \in S\}$ (i.e $y = xs$ for some $s \in S$.)

$Cay(\mathbb{Z}_9, \{1, 3, 6, 8\})$

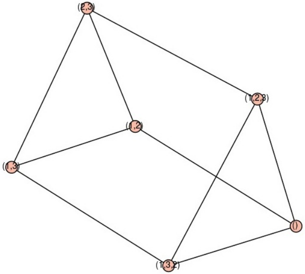


Motivating Example: $\mathcal{S}_3$

$Cay(\mathcal{S}_3, \{(2\ 3), (1\ 2\ 3), (3\ 2\ 1)\})$

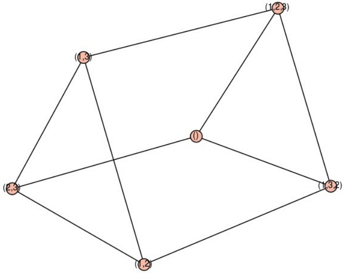


$Cay(\mathcal{S}_3, \{(1\ 2), (1\ 2\ 3), (3\ 2\ 1)\})$

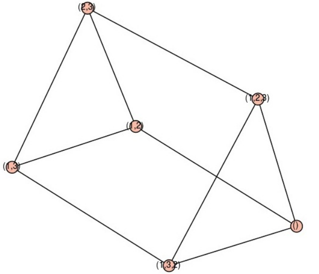


Motivating Example: $\mathcal{S}_3$

$Cay(\mathcal{S}_3, \{(2\ 3), (1\ 2\ 3), (3\ 2\ 1)\})$



$Cay(\mathcal{S}_3, \{(1\ 2), (1\ 2\ 3), (3\ 2\ 1)\})$

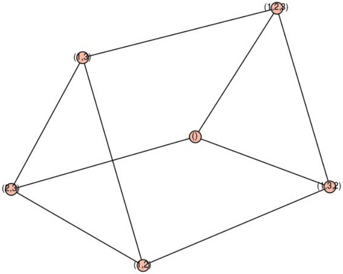


Remark

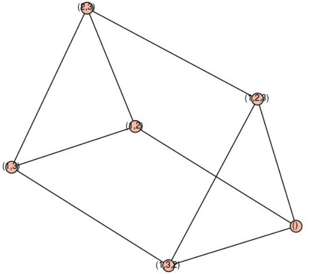
These graphs are isomorphic!

Motivating Example: $\mathcal{S}_3$

$Cay(\mathcal{S}_3, \{(2\ 3), (1\ 2\ 3), (3\ 2\ 1)\})$



$Cay(\mathcal{S}_3, \{(1\ 2), (1\ 2\ 3), (3\ 2\ 1)\})$



Remark

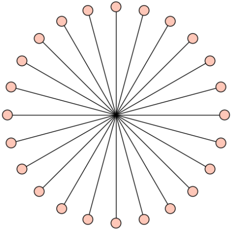
These graphs are isomorphic!

Remark

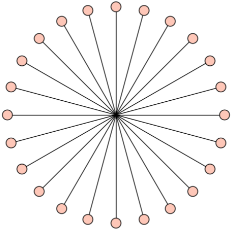
If we let α be the inner automorphism defined by conjugating by the element $(1\ 3)$, then $\alpha(\{(2\ 3), (1\ 2\ 3), (3\ 2\ 1)\}) = \{(1\ 2), (1\ 2\ 3), (3\ 2\ 1)\}$.

Motivating Example: $\mathcal{S}_4$

$Cay(\mathcal{S}_4, \{(1\ 2)\})$

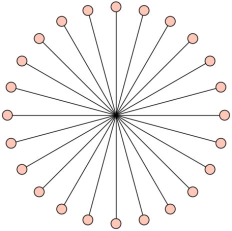


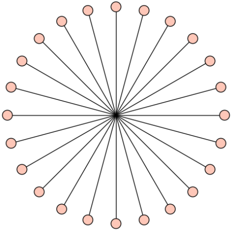
$Cay(\mathcal{S}_4, \{(1\ 2)(3\ 4)\})$



Motivating Example: $\mathcal{S}_4$

$Cay(\mathcal{S}_4, \{(1\ 2)\})$ $Cay(\mathcal{S}_4, \{(1\ 2)(3\ 4)\})$



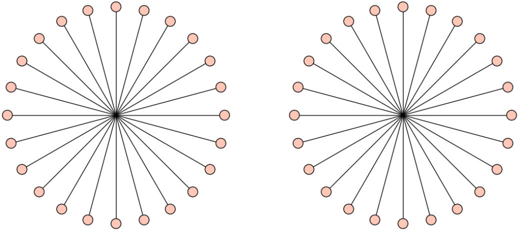


Remark

These graphs are isomorphic!

Motivating Example: $\mathcal{S}_4$

$Cay(\mathcal{S}_4, \{(1\ 2)\})$ $\quad$ $Cay(\mathcal{S}_4, \{(1\ 2)(3\ 4)\})$



Remark

These graphs are isomorphic!

Remark

Automorphisms in $\mathcal{S}_4$ are all inner automorphisms, which preserve cycle structure. Thus, there is no automorphism of $\mathcal{S}_4$ that sends $(1\ 2)$ to $(1\ 2)(3\ 4)$.

The Cayley Isomorphism Property

Definition (The Cayley-Isomorphism (CI) Property)

A Cayley graph of a group G with a symmetric subset $S \subseteq G$, $Cay(G, S)$, is said to be a *CI-graph* if, for any T such that $Cay(G, S) \cong Cay(G, T)$, there exists an $\alpha \in Aut(G)$ such that $\alpha(S) = T$. A group is said to be a *CI-group* if every Cayley graph of this group is a CI-graph.

The Cayley Isomorphism Property

Definition (The Cayley-Isomorphism (CI) Property)

A Cayley graph of a group G with a symmetric subset $S \subseteq G$, $Cay(G, S)$, is said to be a *CI-graph* if, for any T such that $Cay(G, S) \cong Cay(G, T)$, there exists an $\alpha \in Aut(G)$ such that $\alpha(S) = T$. A group is said to be a *CI-group* if every Cayley graph of this group is a CI-graph.

Remark

Given a two sets $S, T \subseteq G$ for which $\alpha(S) = T$ for some automorphism $\alpha \in Aut(G)$, $Cay(G, S)$ and $Cay(G, T)$ will be graphically isomorphic. However, the converse does not necessarily hold.

The Cayley Isomorphism Property

Definition (The Cayley-Isomorphism (CI) Property)

A Cayley graph of a group G with a symmetric subset $S \subseteq G$, $Cay(G, S)$, is said to be a *CI-graph* if, for any T such that $Cay(G, S) \cong Cay(G, T)$, there exists an $\alpha \in Aut(G)$ such that $\alpha(S) = T$. A group is said to be a *CI-group* if every Cayley graph of this group is a CI-graph.

Remark

Given a two sets $S, T \subseteq G$ for which $\alpha(S) = T$ for some automorphism $\alpha \in Aut(G)$, $Cay(G, S)$ and $Cay(G, T)$ will be graphically isomorphic. However, the converse does not necessarily hold.

Classifying Groups based on the CI-property

Claim

$\mathcal{S}_3$ is CI and $\mathcal{S}_4$ is not CI.

Classifying Groups based on the CI-property

Claim

$\mathcal{S}_3$ is CI and $\mathcal{S}_4$ is not CI.

Conjecture (Ádám's Conjecture)

$\mathbb{Z}_n$ is a CI group for all n.

Classifying Groups based on the CI-property

Claim

$\mathcal{S}_3$ is CI and $\mathcal{S}_4$ is not CI.

Conjecture (Ádám's Conjecture)

$\mathbb{Z}_n$ is a CI group for all n.

Theorem (Muzychuk (2003))

The cyclic group $\mathbb{Z}_n$ is a CI-group if and only if $8 \nmid n$, and $p^2 \nmid n$ for any odd prime p, or $n \in \{8, 9, 18\}$.

Classifying Groups based on the CI-property

Claim

$\mathcal{S}_3$ is CI and $\mathcal{S}_4$ is not CI.

Conjecture (Ádám's Conjecture)

$\mathbb{Z}_n$ is a CI group for all n.

Theorem (Muzychuk (2003))

The cyclic group $\mathbb{Z}_n$ is a CI-group if and only if $8 \nmid n$, and $p^2 \nmid n$ for any odd prime p, or $n \in \{8, 9, 18\}$.

Question

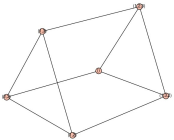
Which groups are CI?

Cayley Graphs and Subgroups

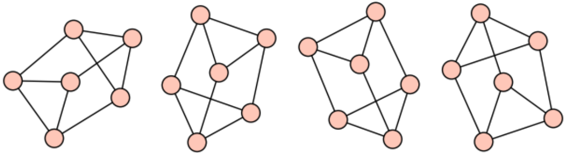
Claim

For a finite group G, let $H = \langle S \rangle,\ H \leq G$. $\Gamma = Cay(G, S)$ has a connected component $Cay(H, S)$ with $|H|$ vertices and is composed of $[G : H]$ many disjoint isomorphic copies of this component.

$Cay(\mathcal{S}_3, \{(2\ 3), (1\ 2\ 3), (3\ 2\ 1)\})$



$Cay(\mathcal{S}_4, \{(2\ 3), (1\ 2\ 3), (3\ 2\ 1)\})$



Cayley Graphs and Subgroups, cont.

Corollary

$Cay(G, S)$ is connected if and only if $\langle S \rangle = G$.

Cayley Graphs and Subgroups, cont.

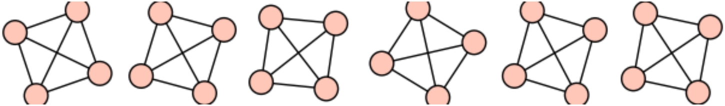
Corollary

$Cay(G,S)$ is connected if and only if $\langle S \rangle = G$.

Corollary

$\langle S \rangle = S \cup \{e\}$ if and only if each connected component is $K_{|S|+1}$

$$Cay(\mathcal{S}_4, \{(1\ 2), (3\ 4), (1\ 2)(3\ 4)\}) \cong 6 \cdot K_4$$



Subgroups of CI Groups

Theorem

Every subgroup of a CI-group is also CI.

Theorem

If G has a subgroup that is non-CI, then G is also non-CI.

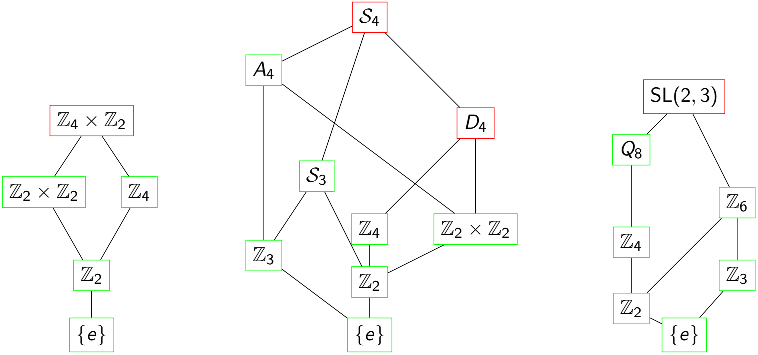
Claim

$\mathcal{S}_n$ is non-CI for $n \geq 4$.

Irreducibly Non-CI

Definition

A finite group G is called *irreducibly non-CI* (INCI) if G is a non-CI group and every proper subgroup of G is CI.



Two Non-Isomorphic Subgroups of the Same Order

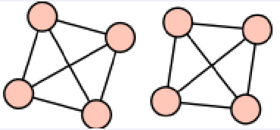
Theorem

If G has two non-isomorphic subgroups of the same order, then G is non-CI.

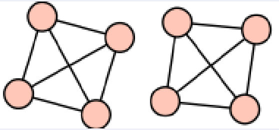
Claim

$\mathbb{Z}_4 \times \mathbb{Z}_2$ is non-CI.

$Cay(\mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_4 \setminus \{e\})$

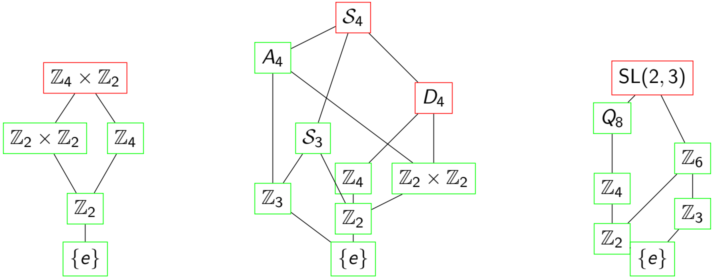


$Cay(\mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{e\})$



These graphs are isomorphic, and no automorphism of $\mathbb{Z}_4 \times \mathbb{Z}_2$ will send $\mathbb{Z}_4$ to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

What's Left?

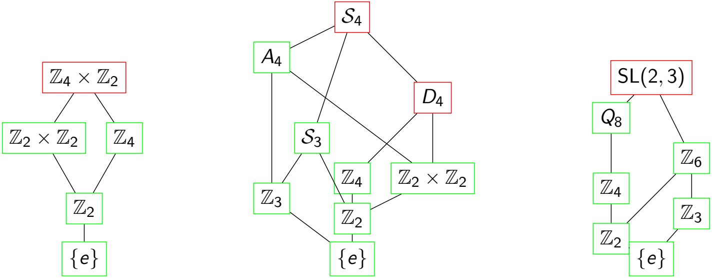


Theorem

G is non-CI if

1. *If G has two non-isomorphic subgroups of the same order.*
2. *If G has a subgroup that is non-CI.*

What's Left?



Theorem

G is non-CI if

1. *If G has two non-isomorphic subgroups of the same order.*
2. *If G has a subgroup that is non-CI.*

Claim

$Cay(G, S)$ is connected if and only if $\langle S \rangle = G$.

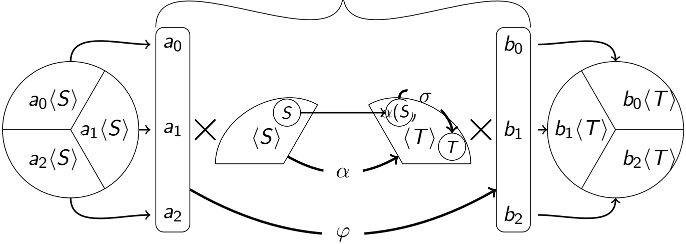
Classification of Abelian Non-CI-Groups

Theorem

If an abelian group G is non-CI then one of the following two mutually exclusive conditions hold:

1. 1. *G has a proper subgroup that is non-CI (i.e. G is not INCI) AND/OR*
   2. *G has two non-isomorphic subgroups of the same order*
2. *All non-CI Cayley graphs of G are connected.*

$$\gamma(g) = \gamma(a_k s) = \varphi(a_k)\sigma(\alpha(s))$$

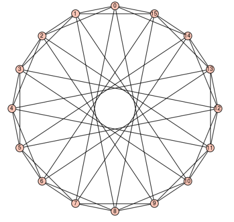


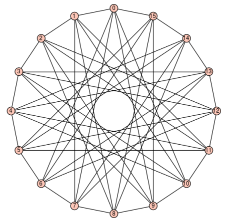
Remaining Groups

Z_{16} and Z_{24}

For $\mathbb{Z}_n$ where $8|n$ and $8 \neq n$

$S = \{1, 2, \frac{n}{2} - 1, \frac{n}{2} + 1, n - 2, n - 1\}$ $T = \{1, \frac{n}{2} - 2, \frac{n}{2} - 1, \frac{n}{2} + 1, \frac{n}{2} + 2, n - 1\}$





SL(2, 3)

$$S = \left\{ \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \right\}$$

$$T = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \right\}$$

Open Problems

Open Problems

1. Extend the classification of subgroup group structures of INCI groups to the non-abelian case (specifically Dedekind groups).

Open Problems

1. Extend the classification of subgroup group structures of INCI groups to the non-abelian case (specifically Dedekind groups).
2. Why are INCI groups non-CI?

Open Problems

1. Extend the classification of subgroup group structures of INCI groups to the non-abelian case (specifically Dedekind groups).
2. Why are INCI groups non-CI?
3. For what values of k is $\mathbb{Z}_p^k$ a CI group?

References

- M. Conder and Li C.H. "On isomorphisms for finite Cayley graphs". In: *European Journal of Combinatorics* 19 (1998), pp. 911–919.
- C.H. Li, Z.P. Lu, and Palfy P.P. "Further restriction on the structure of finite CI-groups". In: *Journal of Algebraic Combinatorics* 26 (2007), pp. 161–181.
- M. Muzychuk. "Ádám's conjecture is true in the square-free case". In: *Journal of Combinatorial Theory* 72 (1995), pp. 118–134.
- M. Muzychuk. "An elementary abelian group of large rank is not a CI-group". In: *Discrete Mathematics* 264 (2003), pp. 167–185.
- M. Muzychuk. "On Ádám's conjecture for circulant graphs". In: *Discrete Mathematics* 167 (1997), pp. 495–510.
- M.Y. Xu. "Automorphism groups and isomorphisms of Cayley digraphs". In: *Discrete Mathematics* 182 (1998), pp. 309–320.

Acknowledgments

- Thank you to Eric Egge, Adam Berliner, St. Olaf College, and the the NUMS Conference.

Acknowledgments

- Thank you to Eric Egge, Adam Berliner, St. Olaf College, and the the NUMS Conference.
- Thank you to my mentors Dr. Sung Y. Song and Kathleen Nowak and to my co-researchers Christopher Cox and Hannah Turner.

Acknowledgments

- Thank you to Eric Egge, Adam Berliner, St. Olaf College, and the the NUMS Conference.
- Thank you to my mentors Dr. Sung Y. Song and Kathleen Nowak and to my co-researchers Christopher Cox and Hannah Turner.
- Thank you to the Iowa State University for hosting this research and the NSF (NSF DMS 0750986) for funding.

Acknowledgments

- Thank you to Eric Egge, Adam Berliner, St. Olaf College, and the the NUMS Conference.
- Thank you to my mentors Dr. Sung Y. Song and Kathleen Nowak and to my co-researchers Christopher Cox and Hannah Turner.
- Thank you to the Iowa State University for hosting this research and the NSF (NSF DMS 0750986) for funding.

- Thank you all for coming!

Questions?