Weighted Voting Systems

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Voting systems can be deceptive. For instance, a voting system might consist of four people, in which three of the people have 2 votes, one has 1 vote and the number of votes needed to pass a proposal is 4. It turns out that the person with 1 vote will never have an effect on the outcome, despite the fact that he or she has a vote. In this paper, we analyze weighted voting systems using Banzhaf's definition of power in order to find out how power can be divided among n voters, a question posed in a paper by John Tolle. Tolle counted the number of ways power can be distributed among groups of three and four voters (though his conditions are different from ours). At the end of the paper, we provide an upper bound for the number of ways power can be distributed to n voters.

1 Terms

A *weighted voting system* consists of a group of voters. Each voter is given a number of votes, which is known as a *weight*. A weighted voting system also includes the number of votes needed to pass a proposal, or a *quota*.

We call the voters *players* and denote the players by $P_1, \ldots, P_n$. We write the weights given to each player as $v_1, \ldots, v_n$, where v_i corresponds to P_i. Furthermore, $v_1 \geq v_2 \geq \ldots \geq v_n$. In other words, P_1 signifies the player with the largest weight, whereas P_n stands for the player with the smallest weight.

The quota is represented by q, and must be greater than half of the total number of votes. If the quota were exactly half (or less than half), then many problems arise.

All of the information about a weighted voting system can be captured by the notation $[q; v_1, \ldots, v_n]$.

We define a *coalition* as any subset of the set of players $\{P_1, \ldots, P_n\}$.

We say a coalition *wins* (or is winning) when the sum of the weights of each player in the coalition meets or exceeds the quota. We say a coalition *loses* (or is losing) when the sum of the weights of each player in the coalition does not meet the quota.

An *assignment* is a function that assigns to each coalition the label "W" or "L."

A *valid assignment* is an assignment in which there are no contradictions arising from the labels given to each coalition. That is, there exists some $[q; v_1, \ldots, v_n]$ such that all coalitions the assignment labels "W" are winning and all coalitions labeled "L" are losing.

We say a coalition is *winnable* if there exists a valid assignment in which the coalition is labeled "W." We day it is *non-winnable* if there is no valid assignment in which it is labeled "W." That is, it is labeled "L" in all valid assignments.

2 Two Players

Suppose we have two players. We know that there are four coalitions, since coalitions are simply subsets of the set of two players, $\{P_1, P_2\}$. Thus, the four coalitions are $\emptyset$, $\{P_1\}$, $\{P_2\}$ and $\{P_1, P_2\}$. But we don't know any of the specifics of the voting system. That is, we don't know the weights of the players or the quota. So, instead, we have to find out which of the following assignments are possible for a 2-player voting system.

Assignment	1	2	3	4	5	6	7	8
$\emptyset$	L	L	L	L	L	L	L	L
$\{P_1\}$	W	W	W	W	L	L	L	L
$\{P_2\}$	W	W	L	L	W	W	L	L
$\{P_1, P_2\}$	W	L	W	L	W	L	W	L

It is easy to see that some assignments are not valid. Assignments where $\emptyset$ is winning are not included in the table for this reason. We know these assignments are invalid because the sum of the weights of the players in $\emptyset$ is 0. 0 cannot be greater than or equal to the quota because the quota is positive in all cases. There are more invalid assignments in the table. For example, assignments 1 and 2 say that both $\{P_1\}$ and $\{P_2\}$ are winning. However, this is impossible because we know that $q > \frac{1}{2}\sum_{i=1}^{n} v_i$. If both of these coalitions are winning, then each player possesses more than $\frac{1}{2}$ of the total votes. Assignment 4 is invalid because it says that $\{P_1\}$ wins but $\{P_1, P_2\}$ loses. This is impossible since $\{P_1, P_2\}$ must have at least as many votes as $\{P_1\}$. Assignment 6 is invalid because it says that $\{P_2\}$ wins and $\{P_1\}$ loses. This is again impossible since $v_1 \geq v_2$.

Thus, there are only three valid assignments for a 2-player voting system. These are represented by assignments 3, 7 and 8: $\langle L, W, L, W\rangle$, $\langle L, L, L, W\rangle$ and $\langle L, L, L, L\rangle$.

3 Valid Assignments

When we decide whether an assignment is valid, we apply one or more of the following concepts.

Theorem 3.1. *If the sum of the weights of the players in coalition A is greater than or equal to the sum of the weights of the players in coalition B and B is a winning coalition, then A is a winning coalition.*

Proof. B is a winning coalition if and only if the sum of the weights of the players in B is greater than or equal to the quota. By hypothesis, the sum of the weights of the players in A is greater than or equal to the sum of the weights of the players in B. Therefore, the sum of the weights of the players in A is greater than or equal to the quota. Thus, A is a winning coalition. □

Theorem 3.2. *If $\{P_{i_1}, P_{i_2}, \ldots, P_{i_k}\}$ is a winning coalition, then any superset of this coalition is winning.*

Proof. The hypothesis states that $\sum_{j=0}^{k} v_{i_j} \geq q$. Suppose a player joins the coalition $\{P_{i_1}, P_{i_2}, \ldots, P_{i_k}\}$. When a player joins a coalition, the weight of that player, x, is added to the number of votes already held by the coalition. Since the weight of every player is greater than or equal to 0, $\sum_{j=0}^{k} v_{i_j} + x \geq q$. Therefore, the coalition remains winning. □

Theorem 3.3. *If $\{P_{i_1}, P_{i_2}, ..., P_{i_k}\}$ is a winning coalition, then any coalition that is disjoint from this coalition is a losing coalition.*

Proof. Suppose $\{P_{i_1}, P_{i_2}, ..., P_{i_k}\}$ is a winning coalition. Suppose there is a coalition d that contains none of $P_{i_1}, P_{i_2}, ..., P_{i_k}$, but is also winning. This implies that the sum of the weights of the players in both $\{P_{i_1}, P_{i_2}, ..., P_{i_k}\}$ and the coalition d meet or exceed the quota. Since $q > \frac{1}{2}\sum_{i=1}^{n} v_i$, both coalitions have more than half of the total number of votes. This means that together these coalitions have more votes than the total number of votes. This is clearly impossible. Therefore, only one of two disjoint coalitions can be a winning coalition. If $\{P_{i_1}, P_{i_2}, ..., P_{i_k}\}$ is winning, any coalition d that is disjoint is losing. □

4 Winnables

To understand winnable coalitions, it is enough to understand non-winnable coalitions. As an example of non-winnable coalitions, consider the following lemma.

Lemma 4.1. *All coalitions of size* 1 *that do not contain* P_1 *are losing in all valid assignments.*

Proof. If $\{P_i\}$, where $i \neq 1$, wins, then P_1 wins because $v_1 \geq v_i$. However, we know by 3.3 that two disjoint coalitions cannot both win. Since assuming $\{P_i\}$ where $i \neq 1$ wins produces a contradiction, $\{P_i\}$ must be losing in all valid assignments. □

This means that all coalitions of size 1 besides $\{P_1\}$ are non-winnable. Here is a more general description of non-winnables.

Theorem 4.2. *If two coalitions A and B are disjoint and the sum of the votes of the players in A must be greater than or equal to the sum of the votes of the players in B, then B is always losing. Therefore, it is a non-winnable.*

Proof. There are two cases to consider. First, suppose A is a winning coalition. Then, by 3.3, B is not a winning coalition. Second, suppose A is a losing coalition. If B were to be winning, then A would also be a winning coalition by 3.1. Again, this is a contradiction of 3.3. Therefore, B is losing when A is losing. Since B is losing regardless of whether A is winning or losing, it is a non-winnable. □

Based on this theorem, we can determine how many coalitions are winnable. It will be convenient for the following proofs to convert coalitions to *player lists.* To form a player list, simply take all of the players' subscripts and put these numbers in a sequence ordered from smallest to largest. For example, the coalition $\{P_1, P_3\}$ becomes $1, 3$. If a player list is named A, call the first number of the player list A_1. In general, call the i^{th} number in the list A_i.

Lemma 4.3. *In an n-player voting system, consider a coalition c of size s that produces player list C. c is winnable $\iff \nexists$ coalition a of size s with player list A such that $A_i < C_i \forall i \in \{1, \ldots, s\}$ and $A_i \neq C_j \forall i, j \in \{1, \ldots, s\}$.*

Proof. $\Leftarrow$ Suppose not. Suppose that c is winnable and $\exists$ coalition a of size s such that $A_i < C_i \forall i \in \{1, \ldots, s\}$ and $A_i \neq C_j \forall i, j \in \{1, \ldots, s\}$. Since $A_i < C_i \forall i$, then $v_{A_i} \geq v_{C_i} \forall i$. Thus, $\sum_{i=1}^{s} v_{A_i} \geq \sum_{i=1}^{s} v_{C_i}$. We know that a and c are disjoint since $A_i \neq C_j \forall i, j \in \{1, \ldots, s\}$. Thus a has at least as many votes as c and they are disjoint. By 4.2, c never wins. This is a contradiction of the statement that c is winnable, because a winnable coalition wins in at least one valid assigment. □

To establish the number of winnable coalitions in a n-player voting system, we convert player lists into paths through a lattice grid. For player list C of size s in an n-player system, the path will run from $(0, 0)$ to $(n-s, s)$ on a $s \times (n-s)$ grid. Think of the construction of the lattice path from C as a series of n moves. On move i, the move is up if $i \in C$ and the move is right if $i \notin C$. It is easy to see that there is a bijection from all player lists of size s in an n-player system (and thus all coalitions of size s) to all lattice paths of length n with s "up" moves.

Lemma 4.4. *A coalition is non-winnable if and only if the lattice path which represents that coalition has no points that are above the $y = x$ line.*

Proof. $\Rightarrow$ Consider a non-winnable coalition c with player list C. Since c is non-winnable, by 4.3, there is a coalition a with player list A such that $A_i < C_i \forall i \in \{1, \ldots, s\}$ and $A_i \neq C_j \forall i, j \in \{1, \ldots, s\}$. In the lattice path that represents c, there is at least one right move before every up move. This is because for each up move expressing the fact that $C_i \in C$, there is a unique right move expressing the fact that $A_i \notin C$. The right move happens before the up move because $A_i < C_i \forall i$. If there is at least one right move before every up move in the lattice path representing c, then this lattice path cannot cross the line $y = x$. The lattice path at most touches the line $y = x$.

$\Leftarrow$ Consider a lattice path that does not go above the $y = x$ line. Let C be the player list that corresponds to this lattice path. Let c be the coalition that corresponds to player list C. Construct player list A such that it contains the numbers that correspond to the first $|c|$ right moves of the lattice path. Because the lattice path does not goes above the line $y = x$ at any point, there are at least as many right moves as up moves. Furthermore, there is a right move before every up move. So for each i, $C_i > A_i$ because the i^{th} right move happens before the i^{th} up move. Thus, A is a player list such that $A_i < C_i \forall i \in \{1, \ldots, |c|\}$ and $A_i \neq C_j \forall i, j \in \{1, \ldots, |c|\}$. This establishes by 4.3 that c is non-winnable. □

With these proofs, we can at last find the number of winnable coalitions.

Theorem 4.5. *In an n-player system, there are $\binom{n}{s-1}$ winnable coalitions of size s, where $s \leq \frac{n}{2}$.*

Proof. We know that there is a bijection from all winnable coalitions of size s to all lattice paths the run from $(0, 0)$ to $(n-s, s)$ through an $(s) \times (n-s)$ grid and have at least one point above the line $y = x$. Therefore, a count of these paths provides a count of the winnable coalitions. In order to do this, we convert these paths into other paths that are easier to count.

Here is how the conversion works. Consider such a lattice path. Find the first point that is above $y = x$. Change every edge after this point from an up move to a right move or from a right move to an up move. This action will result in a new path that goes right $s-1$ times and up $n-s+1$ times. This is because the path must have moved up i times and to the right i times to reach the line $y = x$, for some integer i. Once it reached the line $y = x$, it moved up. Therefore, the unaltered part of the path travels up $i+1$ times and to the right i times. It must go another $(n-s)-i$ moves to the right and $s-(i+1)$ moves up to reach the point $(n-s, s)$. However, these moves have been switched. The new path goes up $(i+1)+(n-s-i) = n-s+1$ times and to the right $(i)+(s-1-i) = s-1$ times. This means that we now have a path on an $(n-s+1)\times(s-1)$ grid.

This process establishes a bijection between paths that have at least one point over $y = x$ on the $(s)\times(n-s)$ grid and paths that have at least one point over $y = x$ on the $(n-s+1)\times(s-1)$ grid.

So, we count the paths through the new grid. In fact, all paths through the new grid must go above $y = x$ because we know that $s \leq \frac{n}{2}$, so the grid is taller than it is long. There are $\binom{n}{s-1}$ paths in this new grid. Every path must go to the right $(s-1)$ times and the path has a total n moves. Thus, constructing a path in this grid is equivalent to choosing which $(s-1)$ positions of a sequence with n positions will be up moves. Therefore, there are $\binom{n}{s-1}$ winnable coalitions of size s. $\square$

Theorem 4.6. *In an n-player voting system, there are $\binom{n}{s}$ winnable coalitions of size s, where $s > \frac{n}{2}$.*

Proof. Since $s > n-s$, the $(s)\times(n-s)$ grid is taller than it is long. Therefore, all paths that go from $(0,0)$ to $(n-s, s)$ go above the $y = x$ line. This is because $(n-s, s)$ is above the line $y = x$. Thus, all coalitions of size s, where $s > \frac{n}{2}$, are winnable. By the same combinatorial argument as above, there are $\binom{n}{s}$ such paths. Since these paths correspond to coalitions of size s, there $\binom{n}{s}$ winnable coalitions of size s. $\square$

These proofs tell us how many winnable coalitions of a given size there are. To determine how many winnable coalitions there are in total, we sum over s.

Theorem 4.7. *The number of winnable coalitions in an n-player system is $2^n - \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. We know that there are $\displaystyle\sum_{i=0}^{n} \binom{n}{i} = 2^n$ coalitions in an n-player system. When $s = \lfloor \frac{n}{2} \rfloor$, the expression for the number of winnables of size s is $\binom{n}{s-1}$. When $s = \lfloor \frac{n}{2} \rfloor + 1$, the expression for the number of winnables of size s is $\binom{n}{s}$. Thus, the missing term is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. This term, then, counts the non-winnables. Therefore, the total number of winnable coalitions is $2^n - \binom{n}{\lfloor \frac{n}{2} \rfloor}$. $\square$

Although the following proof does not relate to our later work, we include this additional result, which makes a connection with the Catalan numbers.

Theorem 4.8. *For $k < \frac{n}{2}$, $\binom{n}{k} = \displaystyle\sum_{i=0}^{k} C_i \binom{n-i-2i}{k-i}$, where C_i is the i^{th} Catalan number.*

Proof. As was shown previously, there are $\binom{n}{s-1}$ lattices paths in the $(s) \times (n-s)$ grid that go from $(0,0)$ to $(n-s,s)$ and go above the line $y=x$ when $s < \frac{n}{2}$.

We can count these paths in another way. Again we count the paths by finding the first point on each path which is over $y = x$. All of these points lie on $y = x+1$. There are two ways that a path can include $(i, i+1)$: the path includes $(i-1, i)$ or it includes (i,i). Since we only want to count paths where $(i, i+1)$ is the first point above $y = x$, we only count the paths that include (i,i). Additionally, we only count paths containing (i,i) if they have been below $y = x$ prior to this point. There are C_i such paths. Since all relevent paths to $(i, i+1)$ came directly from (i,i), there are C_i relevent paths to $(i, i+1)$. Now we must find the number of paths from $(i, i+1)$ to $(s, n-s)$. This is simply $\binom{n-(1+2i)}{s-(1+i)}$ since the path to $(i, i+1)$ has length $1+2i$ and is $1+i$ from the bottom. So the number of paths where $(i, i+1)$ is the first point above $y = x$ is: $C_i\binom{n-1-2i}{k-1-i}$. A sum over all i results in the following formula.

$$\sum_{i=0}^{s-1} C_i \binom{n-1-2i}{s-1-i}$$

The two methods of counting these paths are equal. Therefore,

$$\binom{n}{s-1} = \sum_{i=0}^{s-1} C_i \binom{n-1-2i}{s-1-i}$$

or

$$\binom{n}{k} = \sum_{i=0}^{k} C_i \binom{n-1-2i}{k-i}$$

□

5 Banzhaf Power Index

We have explained a lot about valid assignments, but we have not addressed the question of how much power does a player in a given weighted voting system have. To begin, consider the following example.

Suppose we have a committee consisting of a president, a vice president, a treasurer and a peon. During meetings, the committee needs to make decisions. In order to reflect the importance of each member, they make a weighted voting system. The president is assigned 2 votes. The vice president and treasurer are also each assigned 2 votes, while the peon gets 1. They decide that a simple majority of 4 votes is needed to pass a motion.

The president and the vice president realize that they would like more bacon snacks at committee meetings. As they have 4 votes between them, their votes meets the quota, so the opinions of the other members do not matter. If, however, the president or vice president has a change of heart, then the motion for more bacon snacks will not pass. Therefore their opinions do matter. This makes us notice something: all it takes to make a decision is two people who are not the peon voting in agreement. In fact, the peon's vote never affects the outcome of a vote.

When we gave the peon 1 vote and the rest of the committee 2 votes, we thought we were giving them twice as much power as the peon. Instead, we have created a system where three people have an equal amount of power and one person has none.

Clearly, the number of votes that a player has is not an accurate indicator of power. So, what is a good way to define power? In order to figure out how much power a player in a voting system has, we use the *Banzhaf Power Index*. Simply put, this is a measure of how many times a player's opinion is essential, or critical, to the success of a coalition that he or she is in.

We saw that in the coalition consisting of both the president and vice president both players are essential to the success of the coalition. Should we remove either one of them, there would no longer be enough votes to meet the quota. This situation is called a *critical instance*. A critical instance for a player occurs when removing that player from a coalition causes that coalition to go from winning to losing. In the Banzhaf Power Index, a player's power is defined as the ratio of the player's critical instances to the total number of critical instances. A *power distribution* is a vector of these ratios.

5.1 Finding Critical Instances

In Section 2, we saw that there are only three valid assignments for a two-player system:

1. $\{P_1\}$ and $\{P_1, P_2\}$ win.
2. Only $\{P_1, P_2\}$ wins.
3. No coalition wins.

In the first assignment, P_1 is critical in both coalitions. Furthermore, P_2 is not critical in the coalition $\{P_1, P_2\}$. Therefore, P_1 has power of $\frac{2}{2}$ while P_2 has power of $\frac{0}{2}$. Thus, the power distribution is $\langle 1, 0 \rangle$.

In the second assignment, both players are critical to the winning coalition. So, of the two critical instances, each player claims one. This produces a power distribution of $\langle \frac{1}{2}, \frac{1}{2} \rangle$.

In the last assignment no one wins. Therefore, there are no critical instances, yielding a power distribution of $\langle 0, 0 \rangle$.

6 Examples of Power Distributions

We see that a power distribution depends entirely on the assignment of a given voting system. To find all of the power distributions for an n-player system, we need to find all of the valid assignments. To do this, we analyze the coalitions of an n-player system with the following method.

1. Label all non-winnables "L."
2. Assign c_i, an unlabeled (and therefore winnable) coalition, to winning.
3. Remember the partial assignment as this point.
4. If it is possible to label other coalitions winning or losing based on the theorems of Section 3, then do it.

5. If there are any remaining unlabeled coalitions, recursively run this process by returning to step 2.
6. This will produce all valid assignments that contain c_i as winning.
7. Return to the partial assignment in step 3 and set c_i as losing.
8. If there are any unlabeled coalitions go back to step 2.

With this method, we are able to find all of the valid assignments for the three-player and four-player systems. From here, we can calculate the power distributions. These results follow.

Showing the details of the valid assignment method is difficult in this format and of little value. However, in order to provide some insight into the production of these results, the neccessary elements of the assignments are included in the charts. That is to say, if all the coalitions in the "Coalitions Set to Winning" cell are set to winning and all of the coalitions in the "Coalitions Set to Losing" cell are set to losing, then by applying the rules in section 3, the entire valid assignment is constructed. Also, these listings have the potential to illuminate the general n-player system.

6.1 Three Players

There are six valid assignments and five unique power distributions.

Table 1: Valid Assignments in a 3-Player System

Assignment	Coalitions Set to Winning	Coalitions Set to Losing	Resulting Winning Coalitions
1	$\{P_1\}$	None	$\{P_1\}$, $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_2, P_3\}$
2	$\{P_2, P_3\}$	$\{P_1\}$	$\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$, $\{P_1, P_2, P_3\}$
3	$\{P_1, P_3\}$	$\{P_1\}, \{P_2, P_3\}$	$\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_2, P_3\}$
4	$\{P_1, P_2\}$	$\{P_1, P_3\}$	$\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$
5	$\{P_1, P_2, P_3\}$	$\{P_1, P_2\}$	$\{P_1, P_2, P_3\}$
6	None	$\{P_1, P_2, P_3\}$	None

Table 2: Power Distributions in a 3-Player System

Assignment	Power Distribution
1	$\langle 1, 0, 0 \rangle$
2	$\left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$
3	$\left\langle \frac{3}{5}, \frac{1}{5}, \frac{1}{5} \right\rangle$
4	$\left\langle \frac{1}{2}, \frac{1}{2}, 0 \right\rangle$
5	$\left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$
6	$\langle 0, 0, 0 \rangle$

6.2 Four Players

There are 15 valid assignments and 13 unique power distributions.

Table 3: Valid Assignments in a 4-Player System

Assignment	Coalitions Set to Winning	Coalitions Set to Losing	Resulting Winning Coalitions
1	$\{P_1\}$	None	$\{P_1\}$, $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_4\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
2	$\{P_2, P_3\}$	$\{P_1\}$	$\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_2, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
3	$\{P_1, P_4\}, \{P_2, P_3, P_4\}$	$\{P_1\}$	$\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_4\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_2, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
4	$\{P_1, P_4\}$	$\{P_1\}, \{P_2, P_3, P_4\}$	$\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_4\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
5	$\{P_1, P_3\}, \{P_2, P_3, P_4\}$	$\{P_1, P_4\}, \{P_2, P_3\}$	$\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_2, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
6	$\{P_1, P_3\}$	$\{P_1, P_4\}$, $\{P_2, P_3\}$, $\{P_2, P_3, P_4\}$	$\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
7	$\{P_1, P_2\}$, $\{P_2, P_3, P_4\}$	$\{P_1, P_3\}$	$\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_2, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
8	$\{P_1, P_2\}$, $\{P_1, P_3, P_4\}$	$\{P_1, P_3\}$, $\{P_2, P_3, P_4\}$	$\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
9	$\{P_1, P_2\}$	$\{P_1, P_3\}$, $\{P_2, P_3, P_4\}$, $\{P_1, P_3, P_4\}$	$\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
10	$\{P_2, P_3, P_4\}$	$\{P_1, P_2\}$	$\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_2, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
11	$\{P_1, P_3, P_4\}$	$\{P_1, P_2\}$, $\{P_2, P_3, P_4\}$	$\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
12	$\{P_1, P_2, P_4\}$	$\{P_1, P_2\}$, $\{P_1, P_3, P_4\}$	$\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_2, P_3, P_4\}$
13	$\{P_1, P_2, P_3\}$	$\{P_1, P_2, P_4\}$	$\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_3, P_4\}$
14	$\{P_1, P_2, P_3, P_4\}$	$\{P_1, P_2, P_3\}$	$\{P_1, P_2, P_3, P_4\}$
15	None	$\{P_1, P_2, P_3, P_4\}$	None

Table 4: Power Distributions in a 4-Player System

Assignment	Power Distribution
1	$\langle 1, 0, 0, 0 \rangle$

2	$\left\langle\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right\rangle$
3	$\left\langle\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right\rangle$
4	$\left\langle\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right\rangle$
5	$\left\langle\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}\right\rangle$
6	$\left\langle\frac{3}{5},\frac{1}{5},\frac{1}{5},0\right\rangle$
7	$\left\langle\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}\right\rangle$
8	$\left\langle\frac{1}{2},\frac{3}{10},\frac{1}{10},\frac{1}{10}\right\rangle$
9	$\left\langle\frac{1}{2},\frac{1}{2},0,0\right\rangle$
10	$\left\langle\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\rangle$
11	$\left\langle\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right\rangle$
12	$\left\langle\frac{3}{8},\frac{3}{8},\frac{1}{8},\frac{1}{8}\right\rangle$
13	$\left\langle\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right\rangle$
14	$\left\langle\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\rangle$
15	$\langle 0,0,0,0\rangle$

We made an attempt at finding the valid assignments for the five player system. However, due to the difficulty of this task, we abandoned it and turned to a more general question.

7 Upper Bound

As we have said, it is relatively easy to determine the valid assignments for a voting system with four or fewer players. With five players, it becomes long and tiresome; the six-player system is even worse. Therefore, we changed our goal from finding the valid assignments to finding a reasonable upper bound on the number of valid assigments.

The simplest upper bound is the size of the power set of the set of the coalitions of an n-player system, 2^{2^n}. Each set of coalitions corresponds to the set of winning coalitions for some assignment. Thus, this expression counts every assignment. However, 2^{2^n} is massive for $n \geq 3$ and many of these assignments are invalid. Tne next step is to remove some of these invalid assignments.

We know, for instance, that $\emptyset$ should be considered losing in a valid assignment. In other words, any assignment that labels $\emptyset$ as winning is invalid. Therefore, we can take $\emptyset$ out of consideration. This makes the upper bound 2^{2^n-1}.

We extend this idea to all non-winnable coalitions. In 2^{2^n-1}, we allow all nonempty coalitions to be considered winning. However, we know that some nonempty coalitions are losing in every valid assignment. A better upper bound only counts assignments in which these coalitions are losing. This means that only winnable coalitions are allowed to be winning, which makes our second upper bound $2^{2^n-\binom{n}{\lfloor\frac{n}{2}\rfloor}}$.

Let's compare these two upper bounds.

n	2^{2^n-1}	$2^{2^n-\binom{n}{\lfloor \frac{n}{2} \rfloor}}$	Actual
3	128	32	6
4	32,768	1,024	15
5	2,147,483,648	4,194,304	≈ 63
6	9,223,372,036,854,775,808	17,592,186,044,416	??

7.1 Incorporating Basic Concepts into the Upper Bound

Unfortunately, the second upper bound still greatly overcounts the number of valid assignments. One big problem is that it does not take into account the basic concepts that were discussed earlier in the paper. By taking these ideas into consideration, we can construct a better upper bound.

Thanks to the basic concepts, if we know the size of c, then we know the status of coalitions that are either supersets of c or disjoint from c. We also know how many coalitions of these types there are. Moreover, we know how many coalitions there are in total. Therefore, we know precisely how many other coalitions could be either winning or losing. Compared with the earlier upper bound, there are fewer coalitions that we are "unsure" of. These are the coalitions that are neither completely disjoint from c nor exactly a superset of c. Of course, if we do know which players are part of c, then we might be able determine the status of some of these coalitions. But, from the standpoint of the algorithm, which only knows the size of c, all of these coalitions are uncertain.

To further reduce the number of uncertain coalitions, we can say that all coalitions that are smaller than c are losing. This is permissible because we count from smaller to larger coalitions. In other words, we start with coalitions of size 1, the smallest nonempty coalitions. We consider each coalition of size 1 in the manner described above. Then, we move to coalitions of size 2. Since we have counted all of the valid assignments (and some invalid assignments) that have coalitions of size 1 labeled as winning, we can say coalitions of this size are losing without running the risk of missing a valid assignment.

So, in order to count the number of uncertain coalitions when we consider a coalition c of size s to be winning, we consider three quantities.

First, the number of coalitions of greater or equal size to c.

$$C(s) = \sum_{i=s}^{n} \binom{n}{i}$$

Second, the number of coalitions of greater or equal size that are disjoint from c.

$$D(s) = \sum_{i=s}^{n-s} \binom{n-s}{i}$$

Third, the number of supersets of c.

$$S(s) = \sum_{i=1}^{n-s} \binom{n-s}{i}$$

Therefore, the number coalitions that could be either winning or losing in a valid assignment where c wins is:

$$F(s) = \sum_{i=s}^{n} \binom{n}{i} - 1 - \sum_{i=1}^{n-s} \binom{n-s}{i} - \sum_{i=s}^{n-s} \binom{n-s}{i}$$

In this equation, -1 represents the coalition c, which is known to be winning. However, we can think of c as being part of the term counting the supersets. In this way, $-1 - \sum_{i=1}^{n-s} \binom{n-s}{i}$ becomes $-\sum_{i=0}^{n-s} \binom{n-s}{i}$. Thus, we can reduce these two terms to -2^{n-s}. So, $F(s)$ becomes:

$$F(s) = \sum_{i=s}^{n} \binom{n}{i} - 2^{n-s} - \sum_{i=s}^{n-s} \binom{n-s}{i}$$

So, there are at most $2^{F(s)}$ valid assignments in which coalition c of size s wins.

Of course, this is not yet the formula that gives an upper bound. In order to explain the complete formula, we need to present the following lemmas.

Lemma 7.1. *There is only one valid assignment in which the coalition consisting solely of P_1 wins.*

Proof. Every coalition either contains P_1 (and is therefore a superset) or does not contain P_1 (and is therefore disjoint). Thus, by the basic concepts, if we say $\{P_1\}$ is winning, we know whether any other coalition is winning or losing. This completely determines the assignment in which $\{P_1\}$ is winning. □

Lemma 7.2. *There is only one valid assigment in which the coalition containing all of the players loses.*

Proof. If the coalition containing all of the players loses, then the sum of the weights of all players does not meet the quota. Every other coalition contains fewer players than this coalition. Therefore, every other coalition will have fewer (or equal) aggregate votes compared to this sum. Thus, no other coalition can meet the quota. So, the assignment in which the coalition of all players loses consists of all coalitions labeled "L." □

This follwoing is the formula that gives an upper bound on the number of valid assignments.

$$3 + \sum_{s=2}^{\lfloor n/2 \rfloor - 1} \binom{n}{s-1} 2^{F(s)} + \sum_{s=\lfloor n/2 \rfloor + 1}^{n-1} \binom{n}{s} 2^{F(s)}$$

The previous two lemmas account for 2 of the 3 assignments at the beginning of the formula. This 3 also counts the case in which the coalition of every player is the only coalition that wins. After these extreme cases are taken care of, we focus on coalitions that are in between size 1 and size n. To reiterate, the formula counts the number of winnable coalitions of size s (which can be found in the section about the winnables) and multiplies this quantity by the maximum number of valid assignments that have a coalition of size s as winning (which was presented a few paragraphs earlier).

Let's compare the upper bounds again.

n	$2^{2^n - \binom{n}{\lfloor \frac{n}{2} \rfloor}}$	New Bound	Actual
3	32	15	6
4	1,024	291	15
5	4,194,304	122,963	≈ 63
6	17,592,186,044,416	135,295,402,179	??

7.2 Improving the Upper Bound

There are two main ways to improve the upper bound. First, we can change the way in which we count the number of indeterminate coalitions. Instead of considering all coalitions of size s or greater, we only consider *winnable* coalitions of size s or greater. Second, we can change the way in which we sum over the coalitions in order to consider a coalition to be losing *immediately* after it is considered winning, rather than waiting until the size of the coalitions changes.

The first task is to change $C(s)$ so that only winnable coalitions are counted in the first place.

$$C(s) = \sum_{i=s}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i-1} + \sum_{i=Max(\lfloor \frac{n}{2} \rfloor + 1, s)}^{n-1} \binom{n}{i}$$

As you can see, if s is less than the floor of $\frac{n}{2}$, then the counting begins with coalitions of size s. If s happens to be greater than the floor of $\frac{n}{2}$, the expression $Max(\lfloor \frac{n}{2} \rfloor + 1, s)$ ensures that we also start counting at s, not at the floor of $\frac{n}{2}$.

We do not need to change the term that counts the supersets of c because c is winnable and therefore every superset of c is winnable. Thus, only winnable coalitions are being counted by $S(s)$. We do, however, need to change the term that counts the disjoints of c. Although c is winnable, a coalition that is disjoint to c may or may not be winnable. If we do not change $D(s)$ then we will have the problem of subtracting coalitions that were never added in the first place.

We do not know which coalitions that are disjoint to c are winnable. But do know that all coalitions that are bigger than the floor of $\frac{n}{2}$ are winnable. Therefore, we count only the disjoint coalitions that are bigger than the floor of $\frac{n}{2}$. This looks like:

$$D(s) = \sum_{i=Max(\lfloor \frac{n}{2} \rfloor + 1, s)}^{n-s} \binom{n-s}{i}$$

Thus, the new way of counting indeterminate coalitions is:

$$G(s) = \sum_{i=s}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i-1} + \sum_{i=Max(\lfloor \frac{n}{2} \rfloor+1,s)}^{n-1} \binom{n}{i} - 2^{n-s} - \sum_{i=Max(\lfloor \frac{n}{2} \rfloor+1,s)}^{n-s} \binom{n-s}{i}$$

Here is a comparison between $F(s)$ and $G(s)$.

n	Bound Using F(s)	Bound Using G(s)	Actual
3	15	15	6
4	291	163	15
5	122,963	61,523	≈ 63
6	135,295,402,179	8,459,649,219	??

The second task is to consider a coalition c to be losing immediately after it has been considered winning. This is accomplished by turning the factor that represents the number of winnables of a certain size into a summation.

$$3 + \sum_{s=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{\binom{n}{s-1}-1} 2^{G(s)-i} + \sum_{s=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \sum_{i=0}^{\binom{n}{s}-1} 2^{G(s)-i}$$

Here is a final comparison of upper bounds.

n	Previous Bound	New Bound	Actual
3	15	10	6
4	163	78	15
5	61,523	16,154	≈ 63
6	8,459,649,219	1,206,353,970	??

8 References

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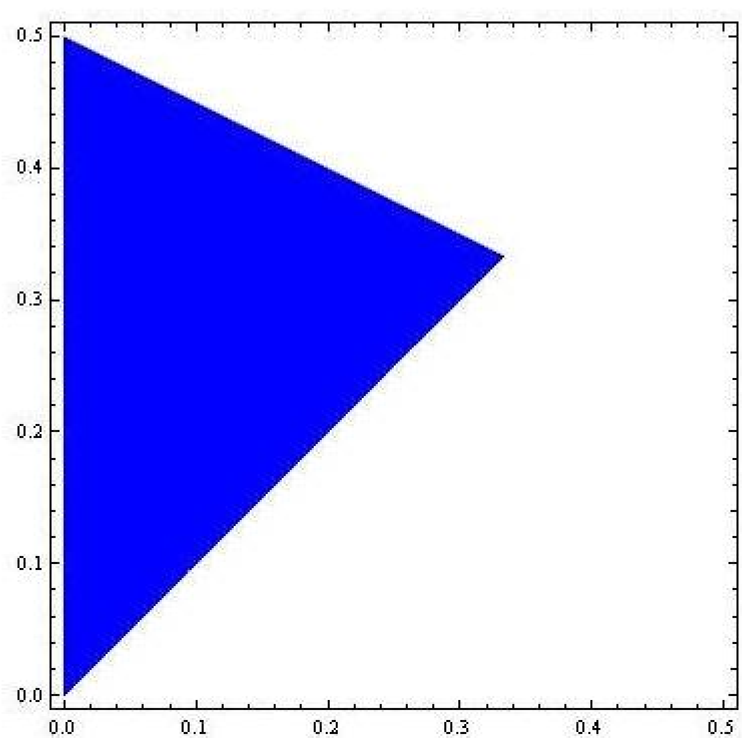
A Graphical Analysis

We also used graphs to analyze weighted voting systems. In fact, we can completely illustrate the cases of two and three players. To do this, we think of the weights and the quota as fractions of the total number of votes, instead of thinking of these numbers in absolute terms.

A.1 Inequalities

Suppose there are three players. Let the x-axis represent the fraction of the total number of votes that belongs to P_3. The value of the x-coordinate is therefore between 0 and $\frac{1}{3}$. Let the y-axis represent the fraction of the total number of votes that belongs to P_2. The value of y-coordinate is therefore between x and $\frac{1-x}{2}$. In other words, P_2 has at least as many votes as P_3 (because $v_2 \geq v_3$) and at most as many votes as P_1, which is equal to half of the remaining fraction of votes. Once the x-coordinate and y-coordinate are determined, the fraction of the votes that belongs to P_1 is known, because this number is $1 - x - y$. Thus, the inequalities $0 \leq x \leq \frac{1}{3}$ and $x \leq y \leq \frac{1-x}{2}$ define the area that represents all possible three-player weighted voting systems.



We can make a graph of the valid assignments in a weighted voting system by finding the conditions under which each of the winnable coalitions is winning. In the three-player system, the conditions are the following inequalities.

Suppose $\{P_1\}$ wins.

$$q \leq 1 - x - y$$

$$y \leq -x + (1 - q)$$

Suppose $\{P_1, P_2\}$ wins.

$$q \leq 1 - x - y + y$$

$$q \leq 1 - x$$

$$x \leq 1 - q$$

Suppose $\{P_1, P_3\}$ wins.

$$q \leq 1 - x - y + x$$

$$q \leq 1 - y$$

$$y \leq 1 - q$$

Suppose $\{P_2, P_3\}$ wins.

$$q \leq x + y$$

$$-x + q \leq y$$

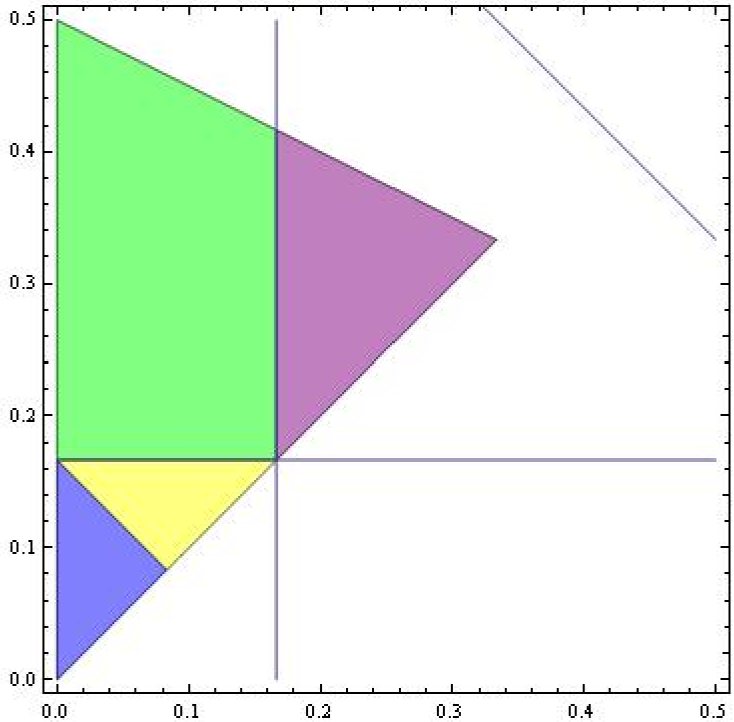
$$y \geq -x + q$$

Suppose $\{P_1, P_2, P_3\}$ wins.

$$q \leq 1 - x - y + y + x$$

$$q \leq 1$$

How do these inequalities appear on a graph? Suppose $q = \frac{5}{6}$.



In this graph, the green area represents the valid assignment in which $\{P_1, P_2\}$ is the smallest coalition that wins. The purple area represents the the valid assignment in which $\{P_1, P_2, P_3\}$ is the smallest (and only) coalition that wins. The yellow area represents the valid assignment in which $\{P_1, P_3\}$ is the smallest coalition that wins. Finally, the blue area represents the valid assignment in which $\{P_1\}$ is the smallest coalition that wins. Two assignments are missing: the one in which $\{P_2, P_3\}$ wins and the one in which no coalition wins. This is because these two assignments do not occur when q is between $\frac{2}{3}$ and 1.

In case you cannot see the colors, here are the boundaries for each valid assignment.
$\{P_1, P_2\}$ is the smallest coalition that wins. The power distribution is $\langle \frac{1}{2}, \frac{1}{2}, 0 \rangle$.
This assignment is bound on the bottom by $y = 1 - q = \frac{1}{6}$. It is bound on the right by $x = 1 - q = \frac{1}{6}$.

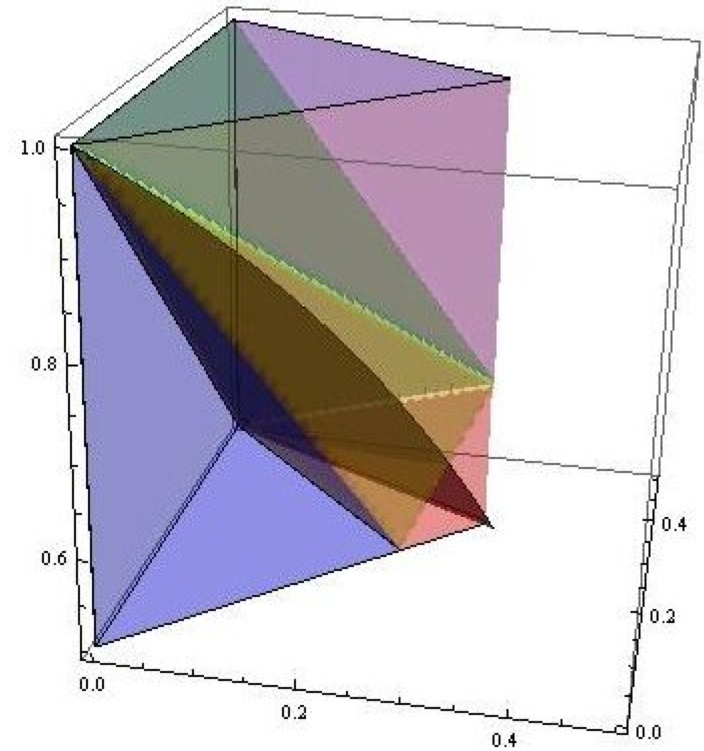
$\{P_1, P_2, P_3\}$ is the smallest (and only) coalition that wins. The power distribution is $\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle$.
This assignment is bound on the left by $x = 1 - q = \frac{1}{6}$.

$\{P_1, P_3\}$ is the smallest coalition that wins. The power distribution is $\langle \frac{3}{5}, \frac{1}{5}, \frac{1}{5} \rangle$.
This assignment is bound on the top $y = 1 - q = \frac{1}{6}$. It is bound on the left by $y = -x + (1 - q) = -x + \frac{1}{6}$.
$\{P_1\}$ wins. The power distribution is $\langle 1, 0, 0 \rangle$.
This assignment is bound on the top-right by $y = -x + (1 - q) = -x + \frac{1}{6}$.

A.2 Three Dimensional Graphs

In this section, the quota will be given a dimension, namely the z-axis, because the quota has a drastic effect on the two-dimensional graph.

The graph below illustrates all possible weighted voting systems with 3-players. Again, the colored regions correspond to different valid assignments.



To reiterate, the blue area corresponds to the assignment with power distribution $\langle 1, 0, 0 \rangle$, red to the one in which every pair of players win, purple to the one in which only the coalition of all players wins, green to the one with power distribution $\left\langle \frac{1}{2}, \frac{1}{2}, 0 \right\rangle$ and yellow to the one with power distribution $\left\langle \frac{3}{5}, \frac{1}{5}, \frac{1}{5} \right\rangle$.