Shadows of the Cantor Set

Matthew Cordes, Becky Patrias, Beatrice White, Zheng Zhu

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Abstract

The Cantor middle third set was created in an attempt to disprove the Continuum Hypothesis. Although it did not accomplish this goal, it has proven to be a useful tool in challenging mathematician's intuition about things such as Lebesgue measure and countability. It is known that if one "shines a flashlight" on the Cartesian product of the Cantor middle third set with itself so that all light rays hit the set at a $45°$-angle, the resulting projection will be the interval $[0, 2]$. It is also clear that shining the light orthogonally to the x-axis will result in the projection being the Cantor middle third set. Our research focused on what happens to the projection in terms of length and Hausdorff dimension when the "flashlight" is at a different angle. We examined what happens in this situation with different Cantor-like sets. We also considered moving the light source to the center of the Cartesian product of Cantor-like sets. We found that there are important differences between projections of fat and thin Cantor-like sets, and that projections range from containing a finite number of intervals to being Cantor-like sets themselves.

Contents

1 Cantor and Cantor-like Sets

1.1 Cantor middle third set and some properties

The Cantor middle third set is constructed by taking out the open middle third of the closed interval $[0,1]$ and repeating this process in each subsequent interval. So the first iteration of the set, $C_1^{\frac{1}{3}}$, is $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. Similarly, the second iteration, $C_2^{\frac{1}{3}}$, is $[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$. In general we denote the n^{th} iteration of $C^{\frac{1}{3}}$ by $C_n^{\frac{1}{3}}$, and $[0,1]$ is $C_0^{\frac{1}{3}}$.

Figure 1: First few iterations of the Cantor middle third set

The resulting set has many surprising properties. We list some below.

Theorem 1.1. *The Cantor middle third set is uncountable.*

Proof. In order to prove this, consider the ternary representation of the numbers in $[0,1]$. If a given point has two representations, choose the one with an infinite number of non-zero digits. For example, $\frac{1}{3} = 0.1 = 0.022222...$, and we take the latter in this case.

It is clear from construction that each infinite string of zeros and twos gives us a unique point in the Cantor set. Note that these representations will give us a subset of the Cantor set. Now by Cantor's Diagonalization argument, the subset is uncountable. Therefore, the Cantor middle third set is uncountable. □

Theorem 1.2. *The Cantor middle third set is totally disconnected.*

Proof. To show that the Cantor set is totally disconnected, we will show that for any two points, p and q, these points are in different connected components of the set. Let p and q be two elements of the set, and let $r = d(p,q)$ where d is the Euclidean metric. Choose n such that $r > \frac{1}{3^n}$. Now consider the n^{th} iteration of the Cantor set. Since $r > \frac{1}{3^n}$, we know that p and q do not lie on the same interval because intervals at this iteration have length $\frac{1}{3^n}$. We can cover each of our 2^n intervals in this iteration using open balls of radius $r' = \frac{1}{3^{n+1}} + \frac{1}{2(3^n)}$ centered on each interval. Denote the ball containing p by $B(p)$. Consider the set U, the union of all of the open balls except for $B(p)$. By construction, these $U \ni q$ and $B(p) \ni p$ will be open, disjoint, non-empty, and their union will contain the Cantor set. Therefore p and q are not in the same connected component for any p and q in the Cantor set.

□

Theorem 1.3. *The Cantor middle third set has Lebesgue measure 0.*

Proof. In order to show that the Lebesgue measure is 0, we calculate the measure of the complement in $[0, 1]$. In the first iteration of the Cantor set, we remove the open middle third, a set with measure $\frac{1}{3}$, and then in the second iteration, we remove an additional two sets of measure $\frac{1}{9}$. At the n^{th} iteration, we remove an additional 2^{n-1} sets of measure $\frac{1}{3^n}$. Since these sets are disjoint, we know that the measure of union is the sum of the measures. Therefore, in total we remove a set of measure $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$. Since the complement is disjoint from the Cantor set, we know that the measure of $[0, 1]$, which is 1, is equal to the measure of the Cantor set plus the measure of the complement. Thus, the Lebesgue measure of the Cantor set is 0. □

1.2 Cantor-like Sets

Cantor-like sets can be constructed using the same method as the Cantor middle third set, but rather than removing the open middle third, remove any length other than $\frac{1}{3}$. When the fraction removed is less than $\frac{1}{3}$, we will call the set as a fat Cantor set. Similarly, for the purpose of this paper, when the fraction removed is greater than $\frac{1}{3}$, we will refer to it as a thin Cantor set.

It is often helpful to label Cantor sets based on the size of the intervals that are left in as opposed to what fraction is taken out. We use λ to denote the relative size of the intervals we keep in the first iteration, and we label our Cantor-like sets as C^λ accordingly. The value λ can also be thought of as the scaling factor at each iteration. For example, in the middle third set we leave in two intervals of length $\frac{1}{3}$, so we use $C^{\frac{1}{3}}$ to represent this set. If instead we are referring to a thin Cantor set where we take out middle halves, the corrsponding value for λ is $\frac{1}{4}$ since we keep in two intervals of length $\frac{1}{4}$. We denote this set by $C^{\frac{1}{4}}$.

Using this new notation, we define a Cantor-like set to be a fat set if $\lambda > \frac{1}{3}$ and to be a thin set if $\lambda < \frac{1}{3}$.

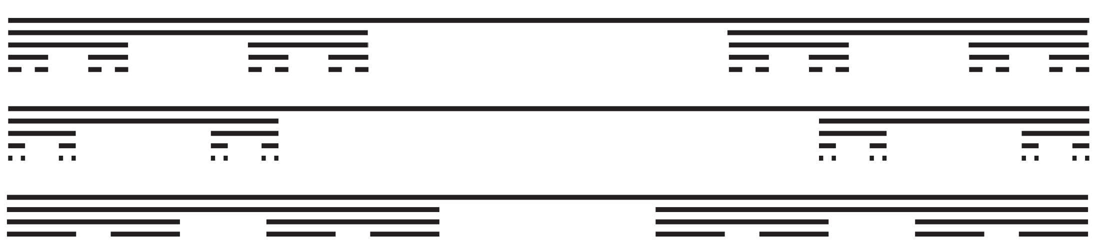

Figure 2: From top to bottom, a middle third set, a thin set, and a fat set

Both fat and thin Cantor sets as we have defined them behave in the same way as the standard middle third Cantor set in terms of the properties previously listed. With slight modifications all of the proofs of Theorems 1.1, 1.2, and 1.3 hold for fat and thin Cantor sets.

For the remainder of this paper, we will mainly consider Cartesian products of a Cantor-like set with itself which we will denote by $C^\lambda \times C^\lambda$. For these sets, we can think of starting with a solid 1×1 square and then removing the middle cross as shown below.

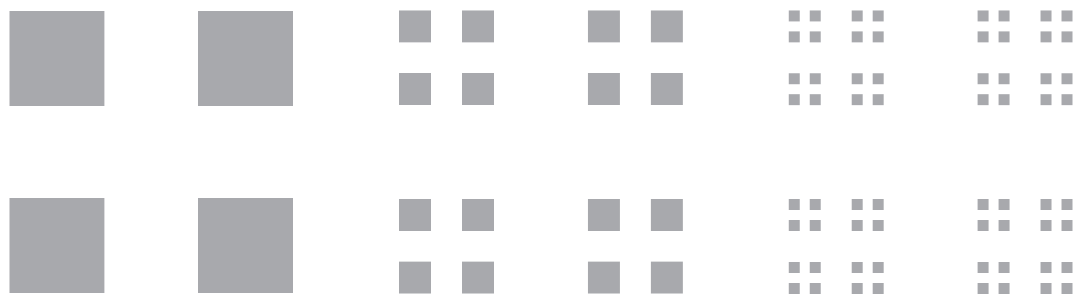

Figure 3: First three iterations of $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$

Through our investigation of Cartesian products of sets, specifically of their projections at different angles, we found differences between the projections of fat and thin sets.

2 Calculating the projections of $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$

In this next section, we move our focus from C^λ, to $C^\lambda \times C^\lambda$. Most basically, $C^\lambda \times C^\lambda$ is equivalent to removing the open middle "plus sign" from a unit square and then repeating this process on the remaining squares.

It is known that if one were to "shine a flashlight" on $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ at a 45 degree angle the shadow cast on the x-axis would be a single interval. This fact inspired our original research question which was to investigate what happens when one shines a flashlight on this set at other angles? To investigate this problem, we need to make the idea of "shining a flashlight" more precise. We will work not in terms of angle, but in terms of the slope of the projection. More specifically, we will think of the light coming from the flashlight, as a family of parallel lines with a given slope.

Definition 2.1. Define μ to be the negative of the slope of the projection. If the family of lines has the equation $y = mx + b$ for some m, then we have $\mu = -m$. For example, for the 45° angle $\mu = 1$, and as our angle approaches 90° μ approaches

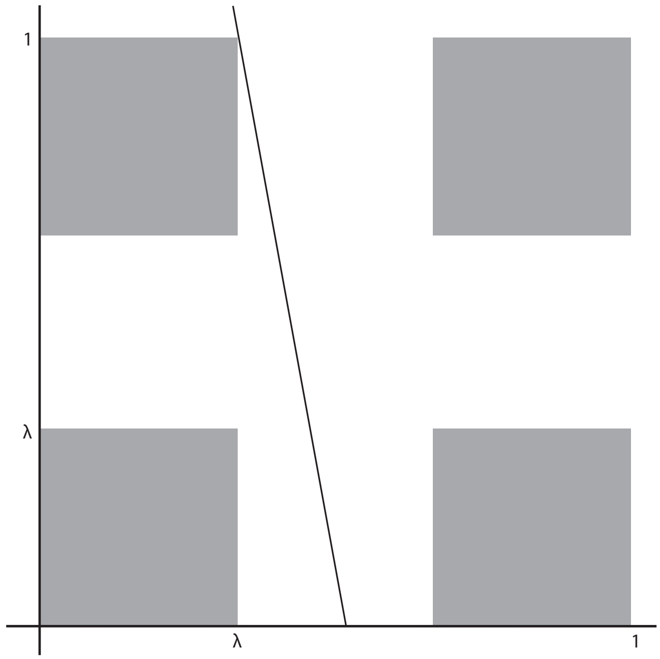

Figure 4: In this figure, $\mu = \frac{2}{1-2\lambda}$

Definition 2.2. For a given μ we say that new light gets through at the n^{th} iteration if the shadow cast on the x-axis by the $n-1^{th}$ iteration is longer than the shadow of the n^{th} iteration. ∞.

We label the blocks of $C^{\lambda} \times C^{\lambda}$ using a, b, c, d as shown in Figure 5. Using these labels, we can describe the paths of light. In the figure, we show light passing through $gap(bc, cd)$. We denote it this way because light comes in between block b and block c and exits between blocks c and d.

Theorem 2.3. *New light will get through the n^{th} iteration of $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ if and only if $\mu > 3^n$.*

Proof. Light will never pass through $gap(ab, ad)$. To see this, consider only the first iteration. Ignoring square a, it is clear that for light to pass through $gap(ab, ad)$, $\mu < 1$. However, for $\mu < 1$ light from $gap(ab)$ will be blocked by square a. This is illustrated in Figure 6

Now we show if the light does not go through iteration n, it will not go through iteration m where $m > n$. Assume light does not go through a given path at iteration n. Then, since Cantor Sets are self similar, the properties described above are preserved for subsequent iterations. And thus the light is blocked.

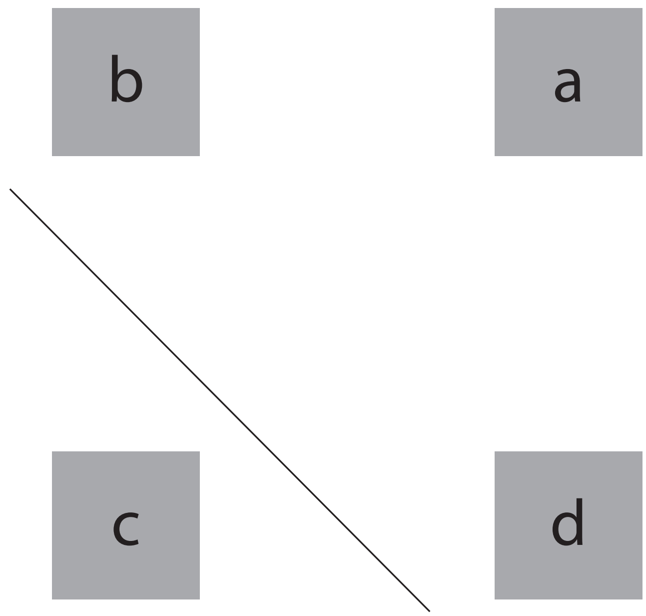

Figure 5: This light passes through $gap(bc, cd)$.

Given what we have shown above, it is sufficient to consider light passing through $gap(ab, cd)$ of the first iteration, the vertical path. Consider $\mu = 3$. The light from the b corner of $gap(ab)$ will hit the d corner of $gap(cd)$ will be blocked because the Cantor Set has height 1 and this gap has width $\frac{1}{3}$.

Consider the second iteration. For light to pass through b $gap(ab, cd)$, it must be that $\mu \geq 3$. However, in order for this light to reach the x-axis, the light entering b $gap(ab)$ must exit through c $gap(cd)$. It is clear for this to happen, we must have $\mu > 9$.

Similarly, for the next iteration, we need $\mu > 27$. And in general, for the n^{th} iteration, we need $\mu > 3^n$. See Figure 7.

Conversely, given $\mu > 3^n$, it is clear from the proportion of the set that the light will reach the x-axis through the n^{th} iteration.

□

The results of Theorem 2.3 tell us that for $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ and a given μ one can determine the last iteration with new light. This information helps us to determine the length of the shadow cast by $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$, because even though there are infinite iterations in $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$, only a finite number

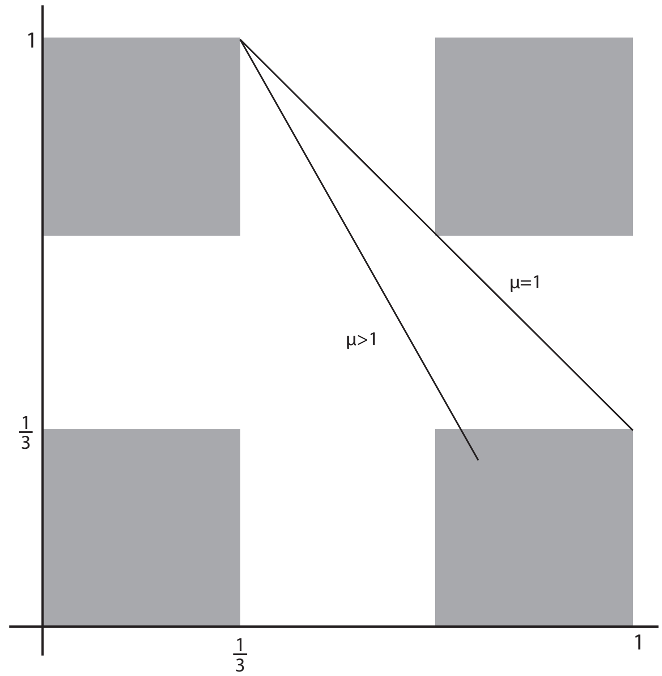

Figure 6: Light will only pass through the vertical columns.

of them have new light. The following two theorems give the formula to calculate the length of the shadow cast on the x-axis by $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ for $\mu \geq 1$ and $\mu \leq 1$.

Theorem 2.4. *The length of the shadow cast by $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ for $\mu \geq 1$ is given by $2^n(\frac{1}{3^n} + \frac{1}{\mu})$ where $n = \lfloor \log_3 \mu \rfloor$.*

Proof. Given μ, there exists n such that $3^n < \mu \leq 3^{n+1}$. We know from Theorem 2.3 that new light will get to the x-axis at the first through n^{th} iterations, but not the j^{th} iteration for $j \geq n+1$. This will give us 2^n intervals of darkness, because at each time new light hits the x-axis, it splits an interval of darkness from the previous iteration into two intervals of darkness.

The length of each individual interval of darkness at the n^{th} iteration can be easily calculated by considering the width of the blocks that make up the n^{th} iteration, $\frac{1}{3^n}$. It is clear from the proof of Theorem 2.3 that light only passes through the vertical paths, and so the intervals of darkness are determined by the width of the shadow cast by the top most and bottom most blocks. Since the height of our Cantor set is 1, each interval of darkness will have width $\frac{1}{3^n} + \frac{1}{\mu}$.

Recalling our previous result, there are 2^n intervals of length $\frac{1}{3^n} + \frac{1}{\mu}$, so the total length of darkness will be $2^n(\frac{1}{3^n} + \frac{1}{\mu})$. □

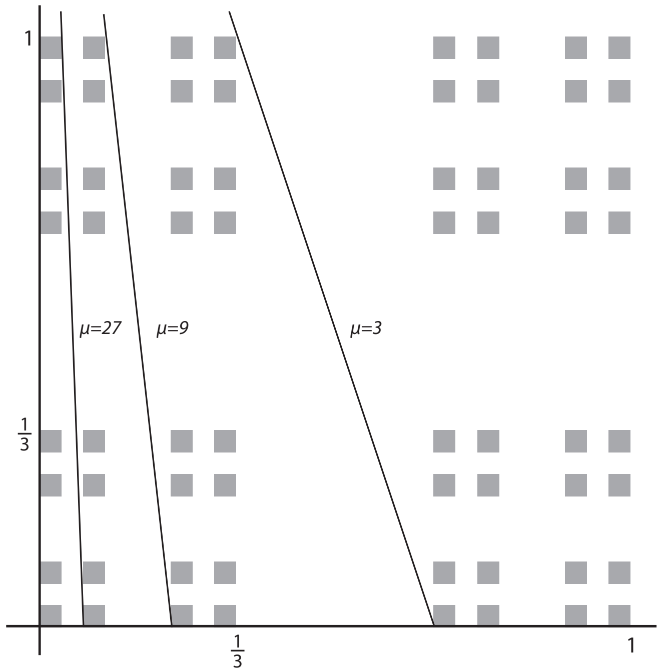

Figure 7: New light gets through the n^{th} iteration when $\mu > 3^n$

Theorem 2.5. *The length of the shadow cast by* $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ *for* $\mu \leq 1$*, which we denote by* $length_\mu$*, is* $\frac{1}{\mu} \cdot length_{\frac{1}{\mu}}$ *and* $n = \lfloor \log_3 \frac{1}{\mu} \rfloor$.

Proof. To find the length of darkness for $\mu \leq 1$, consider instead the intervals of darkness produced by $\frac{1}{\mu}$. It is clear that $\frac{1}{\mu} \geq 1$ and so we may make use of Theorem 2.4. Because of the symmetry of the two-dimensional Cantor set, we can relate the length of darkness given with $\frac{1}{\mu}$ with length of darkness given with μ, by projecting the intervals on the x-axis (given by $\frac{1}{\mu}$) to the y-axis with slope $-\frac{1}{\mu}$. We know from Theorem 2.4, that the length of darkness for $\frac{1}{\mu}$ will be $2^n(\frac{1}{3^n} + \mu)$ for n such that $3^n < \frac{1}{\mu} \leq 3^{n+1}$. Projecting these intervals on the x-axis to the y-axis with slope $-\frac{1}{\mu}$ gives $length_\mu = \frac{1}{\mu} \cdot length_{\frac{1}{\mu}}$. □

Like we saw for C^λ, $C^\lambda \times C^\lambda$ has properties similar to $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ for $\lambda > \frac{1}{3}$. As such, we quickly prove the analogues of Theorems 2.3, 2.4, and 2.5 for $\lambda \geq \frac{1}{3}$.

Theorem 2.6. *Light will get through the* n^{th} *iteration if and only if* $\mu > \left(\frac{1}{\lambda^{n-1}}\right)\left(\frac{1}{1-2\lambda}\right)$.

Proof. Light will never pass through $gap(ab, ad)$. Consider only the first iteration. Ignoring

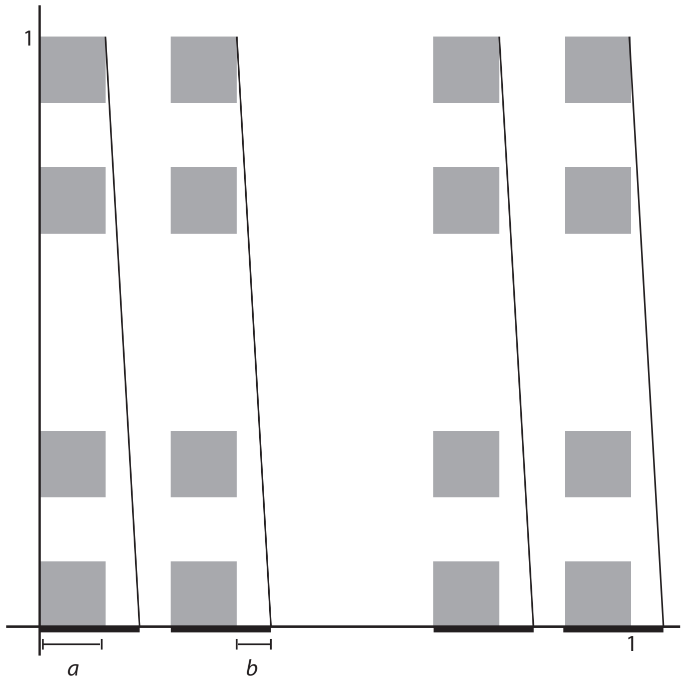

Figure 8: $a = \lambda^2$ and $b = \frac{1}{\mu}$

square a, it is clear that for light to pass through $gap(ab, ad)$, $\mu < 1$. However, for $\mu < 1$ light from $gap(ab)$ will be blocked by square a.

Now we show if the light does not go through iteration n, it will not go through iteration m where $m > n$. Assume light does not go through a given path at iteration n. Then, since Cantor Sets are self similar, the properties described above are preserved for subsequent iterations. And thus the light is blocked.

For the first iteration it is sufficient to consider light passing through $gap(ab, cd)$, the vertical path. Consider $\mu = \frac{1}{1-2\lambda}$. The light from the b corner of $gap(ab)$ will hit the d corner of $gap(cd)$ will be blocked because the Cantor Set has height 1 and this gap has width $1 - 2\lambda$.

Consider the second iteration. For light to pass through b $gap(ab, cd)$, it must be that $\mu \geq \frac{1}{1-2\lambda}$. However, in order for this light to reach the x-axis, the light entering b $gap(ab)$ must exit through c $gap(cd)$. It is clear for this to happen, we must have $\mu \geq \frac{1}{\lambda(1-2\lambda)}$ as $\lambda(1 - 2\lambda)$ is the width of the vertical path.

Similarly, for the third iteration, we need $\mu \geq \frac{1}{\lambda^2(1-2\lambda)}$. And in general, for the n^{th} iteration, we need $\mu \geq \frac{1}{\lambda^{(n-1)}(1-2\lambda)}$.

Conversely, given $\mu \geq \frac{1}{\lambda^{(n-1)}(1-2\lambda)}$, it is clear from the proportion of the set that the light will reach the x-axis through the n^{th} iteration. $\square$

Theorem 2.7. *The length of darkness for* $\lambda \geq \frac{1}{3}$*:* $2^n(\lambda^n + \frac{1}{\mu})$ *when* $n = \lceil \log_{\frac{1}{\lambda}} \mu(1-2\lambda) \rceil$ *for* $\mu \geq 1$.

Proof. Given μ, there exists n such that $\frac{1}{\lambda^{n-1}(1-2\lambda)} < \mu \leq \frac{1}{\lambda^n(1-2\lambda)}$. We know from above that new light will get to the x-axis at the first through n^{th} iterations, but not the j^{th} iteration for $j \geq n+1$. This will give us 2^n intervals of darkness, because at each time new light hits the x-axis, it splits an interval of darkness from the previous iteration into two intervals of darkness.

The length of each individual interval of darkness at the n^{th} iteration can be easily calculated by considering the width of the blocks that make up the n^{th} iteration, λ^n. It is clear from the proof of Theorem 2.6 that light only passes through the vertical paths, and so the intervals of darkness are determined by the width of the shadow cast by the top most and bottom most blocks. Since the height of our Cantor set is 1, each interval of darkness will have width $\lambda^n + \frac{1}{\mu}$.

Recalling our previous result, there are 2^n intervals of length $\lambda^n + \frac{1}{\mu}$, so the total length of darkness will be $2^n(\lambda^n + \frac{1}{\mu})$. $\square$

Theorem 2.8. *The length of darkness for* $\mu \leq 1$*,* $length_\mu$*, will be* $\frac{1}{\mu} \dot{l}ength_{\frac{1}{\mu}}$ *and* $n = \lceil \log_{\frac{1}{\lambda}} \frac{1}{\mu}(1-2\lambda) \rceil$.

Proof. The arguments from Theorem 2.5 hold for $\lambda \geq \frac{1}{3}$. $\square$

Having calculated the length of the shadow cast by $C^\lambda \times C^\lambda$ for $\lambda \geq \frac{1}{3}$ for a set μ, we next investigate how the length of the shadow is effected by changes in μ.

Theorem 2.9. *The length of the shadow cast by* $C^\lambda \times C^\lambda$ *for* $\lambda \geq \frac{1}{3}$ *is continuous with respect to* μ.

Proof. Recall that the length of darkness is given by $2^n(\lambda^n + \frac{1}{\mu})$ when $n = \lceil \log_{\frac{1}{\lambda}} \mu(1-2\lambda) \rceil$ for $\mu \geq 1$ for $\lambda \geq \frac{1}{3}$. This function will be continuous for $\mu \in (\frac{1}{\lambda^{n-1}(1-2\lambda)}, \frac{1}{\lambda^n(1-2\lambda)})$ for any given n. Within each of these ranges of μ, the n in $2^n(\lambda^n + \frac{1}{\mu})$ will be constant. This gives a function of the form $a(b + \frac{1}{\mu})$ where a and b are constant. This is clearly a continuous function of μ because we exclude the case where $\mu = 0$.

Now it remains to show that the length of darkness function is continuous when μ is $\frac{1}{\lambda^k(1-2\lambda)}$ for some $k \in \mathbb{N}$. We do this by looking at the limit of the length as μ approaches from the left, and the limit of the length as μ approaches from the right, and showing they are equal. First consider μ approaching $\frac{1}{\lambda^k(1-2\lambda)}$ from the left. The length will be: $2^k(\lambda^k + \frac{1}{\mu})$. Consider:

$$\begin{aligned}\lim_{\mu \to \frac{1}{\lambda^k(1-2\lambda)}} 2^k(\lambda^k + \frac{1}{\mu}) &= 2^k(\lambda^k + \frac{1}{\frac{1}{\lambda^k(1-2\lambda)}}) \\ &= 2^k(\lambda^k + \lambda^k(1-2\lambda)) \\ &= 2^{k+1}(\lambda^k - \lambda^{k+1})\end{aligned}$$

Now consider approaching from the right; the length will also be $2^k(\lambda^k + \frac{1}{\mu})$:

$$\begin{aligned}\lim_{\mu \to \frac{1}{\lambda^k(1-2\lambda)}} 2^{k+1}(\lambda^{k+1} + \frac{1}{\mu}) &= 2^{k+1}(\lambda^{k+1} + \frac{1}{\frac{1}{\lambda^k(1-2\lambda)}}) \\ &= 2^{k+1}(\lambda^k + \lambda^k(1-2\lambda)) \\ &= 2^{k+1}(\lambda^k - \lambda^{k+1})\end{aligned}$$

□

3 Thin Cantor Sets

Recall C^λ is thin when $\lambda < \frac{1}{3}$. Considering the Cartesian product of these set, $C^\lambda \times C^\lambda$, we see that we will never get sets where the shadow ceases to change after a sufficiently large itereation (see Theorem 2.6). Furthermore, except for some special values of μ, we get projections where the intervals are not all the same length in a given iteration. These special values of μ will produce what we call regular sets. Regular sets are the projections generated when $\mu = (\frac{1}{\lambda})^i$ for some non-negative integer i. These sets will have intervals of shadow distributed in a predictable fashion during construction at each iteration. When $\mu = 1$, these projections turn out to be C^λ where M=5 (see Section 4). Interestingly, for a given λ all of our regular sets will have the same Hausdorff dimension which is detailed in Section 5.

3.1 Regular Sets

Conjecture 3.1. *If $\mu = (\frac{1}{\lambda})^i$ for some non-negative integer i then at the n^{th} itereation of the two-dimensional Cantor set, for $n > i$, we will have $2^i \cdot 3^{n-i}$ intervals of length $2 \cdot \lambda^n$.*

Remark 3.2. We were not able to prove this conjecture. There is an example below showing the first two iterations of a regular projection with $\mu = 4$ and $\lambda = \frac{1}{4}$ in Figure 9. The figure shows how the first iteration has two intervals of darkness, and the second iteration splits each of these intervals into three intervals of equal length. In each subsequent iteration, each interval is split into three intervals of equal length.

Corollary 3.3. *The Lebesgue measure of all regular sets is zero.*

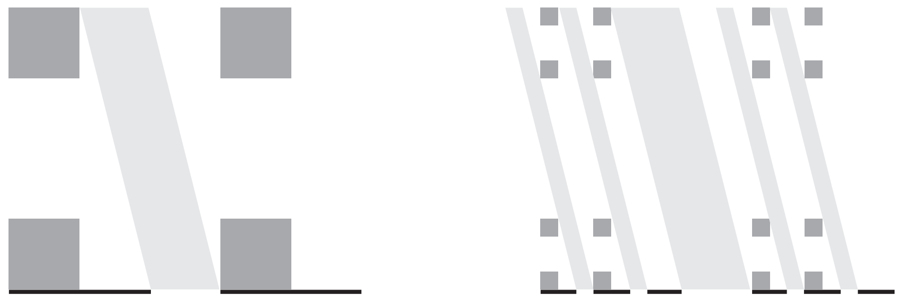

Figure 9: Example of a regular projection given by $\mu = 4$ and $\lambda = \frac{1}{4}$

Proof. Assuming Conjecture 3.1, we know the number of intervals and their lengths. This means that we know the measure of the regular set at each interval: $(2^i \cdot 3^{n-i})(2 \cdot \lambda^n)$. To find the total measure of finished set consider the limit as $n \to \infty$ of that product. Since we have nested intervals and the measure of the 0^{th} iteration is 1 we can say:

$$\begin{aligned}\lim_{n\to\infty}(2^i \cdot 3^{n-i})(2 \cdot \lambda^n) &= \lim_{n\to\infty}\frac{2^{i+1}}{3^i}(3\lambda)^n \\ &= 0 \text{ because } \lambda < \frac{1}{3} \Rightarrow 3\lambda < 1.\end{aligned}$$

□

3.2 Irregular Sets

Now we consider those μ which are not of the form $(\frac{1}{\lambda})^i$. Since these sets do not behave the same way as the other sets we studied, we chose to focus on a single case: $\lambda = \frac{1}{4}$. The construction of irregular sets take out a predictable amount at each iteration, but does not leave intervals of equal length. One of these sets is pictured in Figure 10.

Figure 10: The first few iterations of an irregular set

Projections generated by $C^\lambda \times C^\lambda$ where $\lambda < \frac{1}{4}$ have a more complicated structure due to what we termed the splitting problem. This problem occurs when at a set iteration, the beam of light created from an opening higher in the set hits a block further down the path and gets split into two beams. This is exhibited in Figure 11.

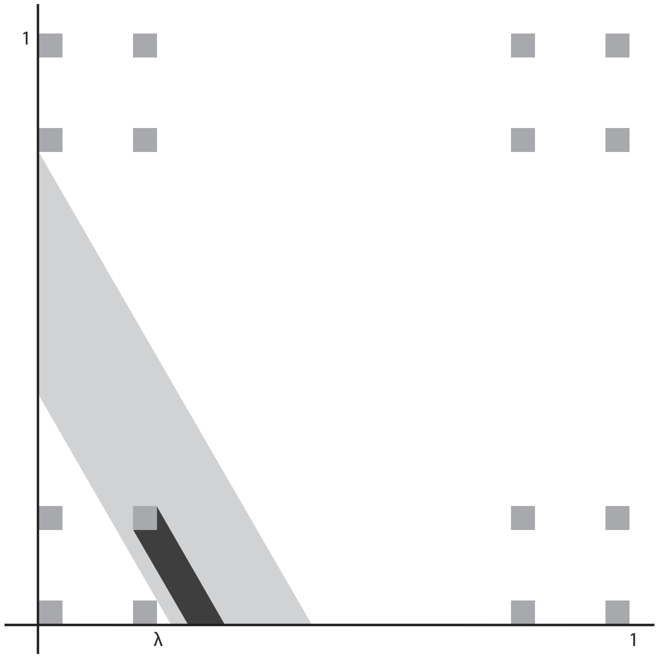

Figure 11: An example of the splitting problem

Although finding a general length formula for projections of thin sets is still an open problem, we were able to generate formulas for $\mu \leq 128$ by direct calculation. These values are given below. Values for higher μ-values can be computed in a similar manner.

Conjecture 3.4. *The length of the projection of the two-dimensional set where $\lambda = \frac{1}{4}$ is:*

$$1 < \mu \leq 8: \frac{\mu+1}{\mu} - \frac{\mu-2}{2\mu} - 2(\frac{8-\mu}{8\mu})\sum_{n=0}^{\infty}(\frac{3}{4})^n$$

$$8 < \mu \leq 16: \frac{\mu+1}{\mu} - \frac{\mu-2}{2\mu} - 2(\frac{8-\mu}{8\mu}) - 4(\frac{8-\mu}{16\mu})\sum_{n=0}^{\infty}(\frac{3}{4})^n$$

$$16 < \mu \leq 32: \frac{\mu+1}{\mu} - \frac{\mu-2}{2\mu} - 2(\frac{8-\mu}{8\mu}) - 8(\frac{32-\mu}{64\mu})\sum_{n=0}^{\infty}(\frac{3}{4})^n$$

$$32 < \mu \leq 128: \frac{\mu+1}{\mu} - \frac{\mu-2}{2\mu} - 2(\frac{8-\mu}{8\mu}) - 4(\frac{32-\mu}{32\mu}) - 8(\frac{32-\mu}{256\mu})\sum_{n=0}^{\infty}(\frac{3}{4})^n$$

Remark 3.5. To find this formula we began with the length of the whole shadow then began subtracting the intervals of light as they came for each iteration. The final sum is $\sigma \sum_{n=0}^{\infty} (\frac{3}{4})^n = 4\sigma$ (where σ is the constant in front of the sum) because from that iteration onward we will get 3^k intervals of light whose length is scaled down by $\lambda = \frac{1}{4}$ at each iteration.

4 M Generalization for Fat Cantor Set

In a previous section we proved that for the fat Cantor set, the length of darkness is $2^n(\lambda^n + \frac{1}{\mu})$ when $n = \lceil \log_{\frac{1}{\lambda}} \mu(1-2\lambda) \rceil$ for $\mu \geq 1$, and the total length of darkness for $\mu \leq 1$, $length_{\mu}$, will be $\frac{1}{\mu} \cdot length_{\frac{1}{\mu}}$ and $n = \lceil \log_{\frac{1}{\lambda}} \frac{1}{\mu}(1-2\lambda) \rceil$. In this section, we will use these theorems to generalize our results.

Consider a Cantor-like set where we begin with the interval $[0,1]$, and at the first iteration we remove the second and forth open fifths of the interval leaving us with $[0, \frac{1}{5}] \cup [\frac{2}{5}, \frac{3}{5}] \cup [\frac{4}{5}, 1]$. Now consider the Cartesian product of this set with itself. Instead of removing a middle cross at the first iteration as we did before, we are now removing a lattice. In order to describe Cantor-like sets such as this one, we need some new terminology.

Definition 4.1. Consider a Cantor-like set C^{λ}. We define M to be the number of intervals in the set at the first iteration. For a given C^{λ} with some M-value, $C^{\lambda} \times C^{\lambda}$ has M^2 blocks in the first iteration. For the purpose of our research, we do not consider the case where $M = 1$ as this is uninteresting.

For example, a set where $M = 3$ is pictured in Figure 12. This Figure could represent a C^{λ} set or a regular projection of a thin $C^{\lambda} \times C^{\lambda}$ set.

Figure 12: The first four iterations of a $M = 3$ C^{λ} set

In the $C^{\lambda} \times C^{\lambda}$ case shown in Figure 13, the set is given by the Cartesian product of C^{λ} and itself. Thus, for C^{λ} where $M = 3$, we have $M^2 = 9$ squares appearing in the first iteration for $C^{\lambda} \times C^{\lambda}$. Note that all given M-values refer to the M corresponding to the C^{λ} set and not that of the $C^{\lambda} \times C^{\lambda}$ set.

All the Cantor-like sets, C^{λ}, we have analyzed in the previous sections are sets where $M = 2$. In this section, we generalize our previous theorems about length of fat Cantor sets and regular projections for all M. In order to do this, we generalize our definitions.

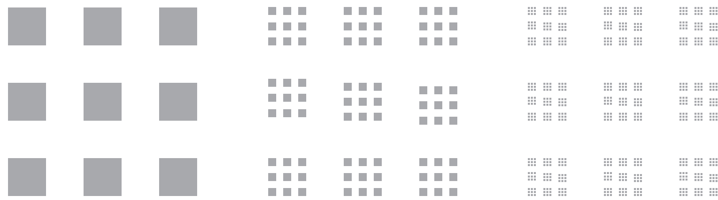

Figure 13: The first three iterations of a $C^\lambda \times C^\lambda$ set where $M = 3$

4.1 Fat Sets

Definition 4.2. A fat Cantor-like set, C^λ, with a given M is one with $\lambda > \frac{1}{2M-1}$ where λ is the scaling factor of each interval. We call $C^\lambda \times C^\lambda$ a fat Cantor-like set if C^λ is fat.

For example, when $M = 3$, a fat set has $\lambda > \frac{1}{2M-1} = \frac{1}{5}$. Figure 14 shows a fat $M = 3$ set with $\lambda = \frac{1}{4}$.

Before generalizing the length of darkness for $M \geq 2$, we need to generalize Theorem 2.6.

Theorem 4.3. *Light will get through the n^{th} iteration of the fat set $C^\lambda \times C^\lambda$ for if and only if $\mu > (\frac{1}{\lambda})^{n-1}\frac{M-1}{1-M\lambda}$.*

Proof. We consider the fat $M \geq 2$ set with $\lambda > \frac{1}{2M-1}$. We want to show that if light gets through the n^{th} iteration of the fat set $C^\lambda \times C^\lambda$, then $\mu > (\frac{1}{\lambda})^{n-1}\frac{M-1}{1-M\lambda}$.

For the first iteration, in order to let the light go through the vertical columns, we need to have a μ with a value ranging from $\frac{1}{\lambda}^{n-1}\frac{M-1}{1-M\lambda}$ and $\mu = \infty$. Notice, the horizontal width of the gap is given by $\frac{1-M\lambda}{M-1}$, and the vertical height of the gap is 1. Thus, we need $\mu > \frac{1}{\frac{1-M\lambda}{M-1}}$, which is equivalent to $\frac{M-1}{1-M\lambda}$.

For the second iteration, there are $M^2 - 1$ vertical columns created. We only need to consider the $M(M-1)$ relatively narrower paths with horizontal width $\frac{\lambda - M\lambda^2}{M-1}$ and vertical height 1, because if the light can get through the narrower columns because that is where new light will be created. Given that, we need $\mu > \frac{1}{\frac{\lambda - M\lambda^2}{M-1}}$, which is equivalent to $\frac{M-1}{\lambda - M\lambda^2}$. Notice that $\frac{M-1}{\lambda - M\lambda^2} = \frac{M-1}{\lambda(1-M\lambda)}$, and we can write it as $(\frac{1}{\lambda})\frac{M-1}{1-M\lambda}$.

In general at the n^{th} iteration, we have narrowest columns of width $\lambda^{n-1}\frac{1-M\lambda}{M-1}$ and height 1. So we need $\lambda > (\frac{1}{\lambda})^{n-1}\frac{M-1}{1-M\lambda}$ for new light at the n^{th} iteration.

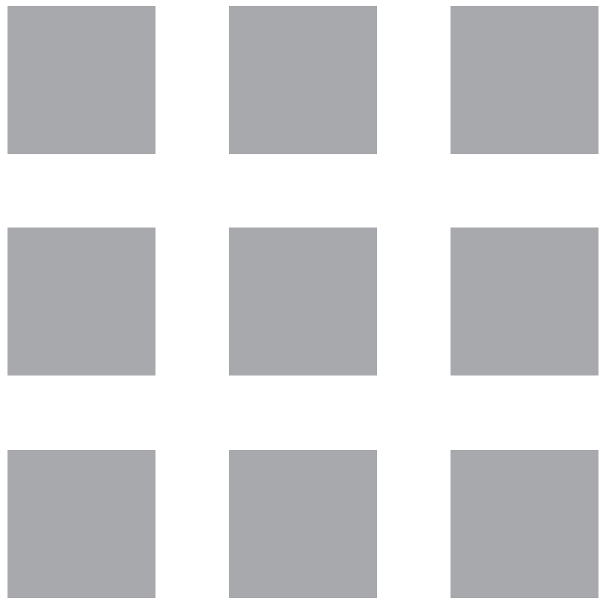

Figure 14: The first iteration of a $M = 3$ $C^\lambda \times C^\lambda$ where $\lambda = \frac{1}{4}$

Conversely, given that $\mu \geq (\frac{1}{\lambda})^{n-1} \frac{M-1}{1-M\lambda}$, it is clear from the proportions of the fat set $C^\lambda \times C^\lambda$ that the light will reach the x-axis through the n^{th} iteration. □

Theorem 4.4. *For fat $C^\lambda \times C^\lambda$, the length of darkness is given by $M^n(\lambda^n + \frac{1}{\mu})$, where $n = \lceil \log_{\frac{1}{\lambda}} \mu \frac{1-M\lambda}{M-1} \rceil$.*

Proof. Consider a fat set $C^\lambda \times C^\lambda$. Given μ, there exists n such that $(\frac{1}{\lambda})^{n-1} \frac{M-1}{1-M\lambda} < \mu \leq (\frac{1}{\lambda})^n \frac{M-1}{1-M\lambda}$. We know from above that new light will get to the x-axis at the first through n^{th} iterations, but not the j^{th} iteration for $j \geq n+1$. This will give us M^n intervals of darkness, because at each time new light hits the x-axis, it splits an interval of darkness from the previous iteration into M intervals of darkness.

The length of each individual interval of darkness at the n^{th} iteration can be easily calculated by considering the width of the blocks that make up the n^{th} iteration, λ^n. It is clear from the proof of the previous theorem that light only passes through the vertical paths, and so the intervals of darkness are determined by the width of the shadow cast by the top most and bottom most blocks. Since the height of our set $C^\lambda \times C^\lambda$ is 1, each interval of darkness will have width $\lambda^n + \frac{1}{\mu}$.

Recalling our previous result, there are M^n intervals of length $\lambda^n + \frac{1}{\mu}$, so the total length of darkness will be $M^n(\lambda^n + \frac{1}{\mu})$.

In the Theorem 4.3, we proved that light will get through the n^{th} iteration of the fat set $C^\lambda \times C^\lambda$ for if and only if $\mu > (\frac{1}{\lambda})^{n-1}\frac{M-1}{1-M\lambda}$. We let $\mu = (\frac{1}{\lambda})^{n-1}\frac{M-1}{1-M\lambda}$, then we have:

$$\mu\frac{1-M\lambda}{M-1} = (\frac{1}{\lambda})^{n-1},$$
$$\text{and so, } \log_{\frac{1}{\lambda}} \mu\frac{1-M\lambda}{M-1} = n-1$$

Thus, we have $n = \lceil \log_{\frac{1}{\lambda}} \mu\frac{1-M\lambda}{M-1} \rceil$.

Consequently, for fat $M \geq 2$ set $C^\lambda \times C^\lambda$, the length of darkness is given by $M^n(\lambda^n + \frac{1}{\mu})$, where $n = \lceil \log_{\frac{1}{\lambda}} \mu\frac{1-M\lambda}{M-1} \rceil$.

□

Recall that for $M = 2$, the length of darkness is given by $2^n(\lambda^n + \frac{1}{\mu})$, where $n = \lceil \log_{\frac{1}{\lambda}} \mu(1-2\lambda) \rceil$, and it is interpreted as the sum of 2^n intervals of length $\lambda^n + \frac{1}{\mu}$. For a fat $M > 2$ set, $C^\lambda \times C^\lambda$, the length of darkness is the sum of M^n intervals of length $\lambda^n + \frac{1}{\mu}$.

4.2 Regular Projections for Thin Sets

Definition 4.5. A thin Cantor-like set, C^λ, with a given M is one with $\lambda < \frac{1}{2M-1}$ where λ is the scaling factor of each interval. We call $C^\lambda \times C^\lambda$ a thin Cantor-like set if C^λ is thin.

For example, Figure 15 shows a thin $C^\lambda \times C^\lambda$ set with $M = 3$.

In a previous section, we analyzed both regular and irregular projections for thin $C^\lambda \times C^\lambda$ sets. Regular projections are generated when $\mu = (\frac{1}{\lambda})^i$ for some non-negative integer i, and we know that they will have intervals of darkness distributed in a predictable fashion during construction at each iteration. Generally speaking, new intervals are created in a evenly-distributed fashion at each iteration. Notice that it is not the case that if two intervals are created in the n^{th} iteration, then two intervals will be created the $(n+1)^{th}$ iteration. Now we generalize this for all M:

Conjecture 4.6. *For regular projections of a thin $M \geq 2$ set $C^\lambda \times C^\lambda$, the projections are $M^i(2M-1)^{n-i}$ intervals of length $2\lambda^n$.*

Figure 15: The first iteration of $C^\lambda \times C^\lambda$, where $M = 3$ and $\lambda = \frac{1}{6}$

To better visualize this conjecture, we can test it on an example. Consider a thin $C^\lambda \times C^\lambda$ with $M = 2$ and $\lambda = \frac{1}{4}$. Then at the n^{th} iteration where $n = 3$ and $i = 1$, we have $2^1(2 \times 2 - 1)^{(3-1)}$ $= 18$ intervals of length $2 \times (\frac{1}{4})^3 = \frac{1}{32}$. This is true because in the first iteration, there are only two intervals of length $\frac{1}{2}$. In the second iteration, the two intervals created in the first iteration are splitted into three evenly distributed intervals, and in the third iteration, each of the six intervals are splitted into three evenly distributed intervals, and each newly-created interval has length $2\lambda^3$ where $\lambda = \frac{1}{4}$. This gives us the final answer that there are eighteen intervals of length $\frac{1}{32}$.

As it is mentioned in the section on thin sets, projections on thin sets when $M = 2$ may encounter light-splitting problem. This can happen to the $M > 2$ cases as well for $C^\lambda \times C^\lambda$ sets. If it happens, more complications are added to the generalization of the length formula for a thin $M \geq 2$ set $C^\lambda \times C^\lambda$.

Irregular projections for thin sets are generated when μ is not in the form $(\frac{1}{\lambda})^i$. The projections are unlike the sets we have been working with, so we did not study them deeply enough to generalize a length formula for them. However, we have generalized the length of darkness formula for the regular projection as a conjecture.

5 Hausdorff Dimension

Due to the number of intervals removed from the Cantor set, one might guess that the dimension of the set is not 1. As the set is simply a collection of points, it is tempting to say that it must, then, have dimension 0, but this is not the case. The Cantor set, as well as many other well known fractals, has a non-integer Hausdorff dimension. We use this concept of Hausdorff dimension to show the dimension of the C^λ and $C^\lambda \times C^\lambda$ variations of the Cantor set that we have discussed. A brief overview is given below.

Hausdorff dimension on an intuitive level is simply an extension of what we colloquially think of as dimension. That is, an n-dimensional object will have Hausdorff dimension n. In general, we define the Hausdorff dimension to be

$$\inf \{\alpha \mid \text{the } \alpha\text{-dimensional Hausdorff measure of the set is zero}\}.$$

For any dimension σ greater than α, our the Hausdorff measure of dimension σ will be zero, and for $\sigma < \alpha$ the Hausdorff measure of dimension σ will be infinite. This idea is well illustrated in Figure 16 below. For more details, see Fractal Geometry [**2**].

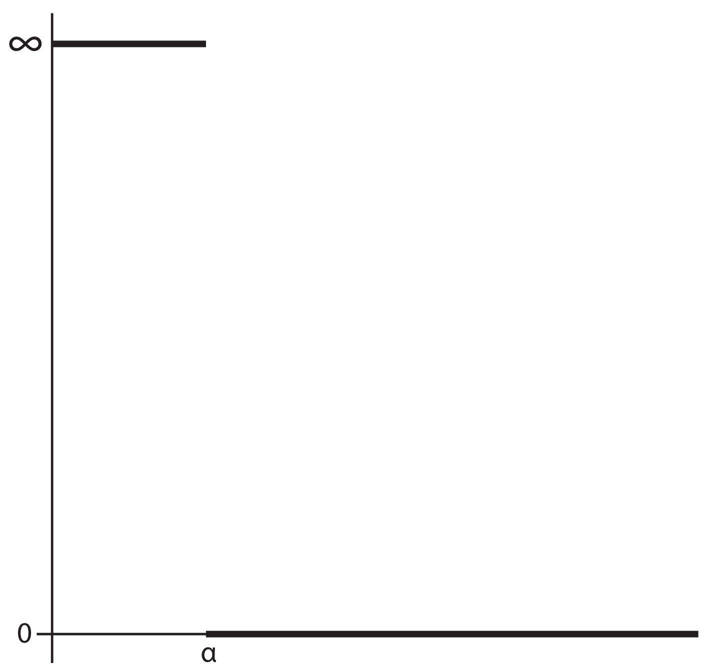

Figure 16: This represents the definition of Hausdorff dimension

It helps to think about this with a more familiar, and simpler, example. Take a closed interval on the x-axis, say $[0, 2]$. The Lebesgue measure of this line segment in $\mathbb{R}^2$ is 0. This tells us that the dimension of the line segment must be less than 2. The Lebesgue measure of the segment in $\mathbb{R}$ is 2, a finite measure. This suggests that the dimension of the line segment is 1 which is exactly what we expect.

For each of the theorems below, we calculate an upper bound for the desired Hausdorff dimension using a cover of the set. In all cases given, the upper bound for the dimension is equal to the lower bound of the dimension. This is explained in Chapter 4 of Fractal Geometry by Falconer, and the proof will not be given here.

5.1 Dimensions for the Middle Third Set

Theorem 5.1. *The Hausdorff dimension of the standard middle third Cantor set is* $\frac{\ln 2}{\ln 3}$.

Proof. At the n^{th} iteration of this Cantor set, we can cover the set using 2^n circles of diameter $(\frac{1}{3^n})$. An upper estimate of the α-dimensional Hausdorff measure of $C^{\frac{1}{3}}$ is then

$$\sum_{k=1}^{2^n} w_\alpha (\frac{1}{3^n})^\alpha = \frac{2^n w_\alpha}{3^{n\alpha}}.$$

This will give an upper bound for the n^{th} iteration of the set, so we want to consider the limit as n goes to infinity. We see that as $n \to \infty$ this fraction will be finite and non-zero when $\alpha = \frac{\ln 2}{\ln 3}$, so $\frac{\ln 2}{\ln 3}$ is the upper bound we are looking for.

We can now use the mass distribution principle, [**2**], to say that this is also a lower bound. Thus $\alpha = \frac{\ln 2}{\ln 3}$ is indeed the true value. □

Theorem 5.2. *The Hausdorff dimension of* $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ *is* $\frac{\ln 4}{\ln 3}$.

Proof. Similarly to above, we cover the set with 4^n circles of diameter $\frac{\sqrt{2}}{3^n}$. We see that the upper estimate of the α-dimensional Hausdorff measure,

$$\sum_{k=1}^{4^n} w_\alpha (\frac{\sqrt{2}}{3^n})^\alpha = \frac{4^n w_\alpha (\sqrt{2})^\alpha}{3^{n\alpha}},$$

will be finite and non-zero as n goes to infinity for $\alpha = \frac{\ln 4}{\ln 3}$. □

5.2 Generalizations to Fat Sets, Thin Sets, and Sets with Different M-values

These two results are easily generalized to fat and thin Cantor sets.

Theorem 5.3. *The Hausdorff dimension of a Cantor-like set,* C^λ, *is* $\frac{\ln 2}{-\ln \lambda}$, *and the Hausdorff dimension of* $C^\lambda \times C^\lambda$ *is* $\frac{\ln 4}{-\ln \lambda}$.

Proof. At the n^{th} iteration of C^λ, we need 2^n circles of diameter $(\lambda)^n$ to cover the set. Following the examples above, it is easy to see that $\sum_{k=1}^{2^n} w_\alpha(\lambda^n)^\alpha$ as $n \to \infty$ is finite and non-zero for $\alpha = \frac{\ln 2}{-\ln \lambda}$. For the 2-dimensional Cantor set, we need 4^n circles of diameter $\sqrt{2}\lambda^n$, and the desired result follows. □

Theorem 5.4. *The Hausdorff dimension of a Cantor-like set with $M \geq 2$ is $\frac{\ln(M)}{-\ln \lambda}$.*

Proof. At the n^{th} iteration of the set, we need m^n circles of diameter $\sqrt{2}\lambda^n$ to cover the set. We have

$$\sum_{k=1}^{M^n} w_\alpha(\sqrt{2}\lambda^n)^\alpha = M^n w_\alpha(\sqrt{2}\lambda^n)^\alpha,$$

which is finite and non-zero as $n \to \infty$ for $\alpha = \frac{\ln(M)}{-\ln \lambda}$. Similarly to the previous theorems, this is the Hausdorff dimension. □

5.3 Other Findings

Theorem 5.5. *For $\lambda < \frac{1}{3}$, the Hausdorff dimsion of $C^\lambda \times C^\lambda$ for $\mu = 1$ is $\frac{\ln(2M-1)}{-\ln \lambda}$.*

Proof. For $\mu = 1$ and $\lambda < \frac{1}{3}$, we get light through each of the diagonal pathways as shown below:

From this, it is clear that there are $2M-1$ intervals of darkness at the first iteraion, and more generally, $(2M-1)^n$ intervals of darkness at the n^{th} iteration. Each of these intervals has length $2\lambda^n$ as can be seen by construction, so we need $(2M-1)^n$ cirlces of diameter $2\lambda^n$ to cover the projected set at the n^{th} iteration.

As we know from previous arguments, we want

$$\sum_{k=1}^{(2M-1)^n} w_\alpha 2\lambda^{n\alpha} = w_\alpha(2M-1)^n\lambda^{n\alpha}$$

to be non-zero and finite which happen for $\alpha = \frac{\ln(2M-1)}{-\ln \lambda}$. □

The previous result becomes much more interesting when one considers a theorem about projections of Borel sets in Fractal Geometry, Projection Theorem 6.1. We will state the theorem here without proof as it is proven in Fractal Geometry and requires a fair amount of background not otherwise needed for this paper.

Theorem 5.6. *Let $F \subset \mathbb{R}^2$ be a Borel set. If the Hausdorff dimension of F is less than or equal to one, then the Hausdorff dimension of the projection will equal the Hausdorff dimension of F for almost all $\mu \in [0, \infty)$.*

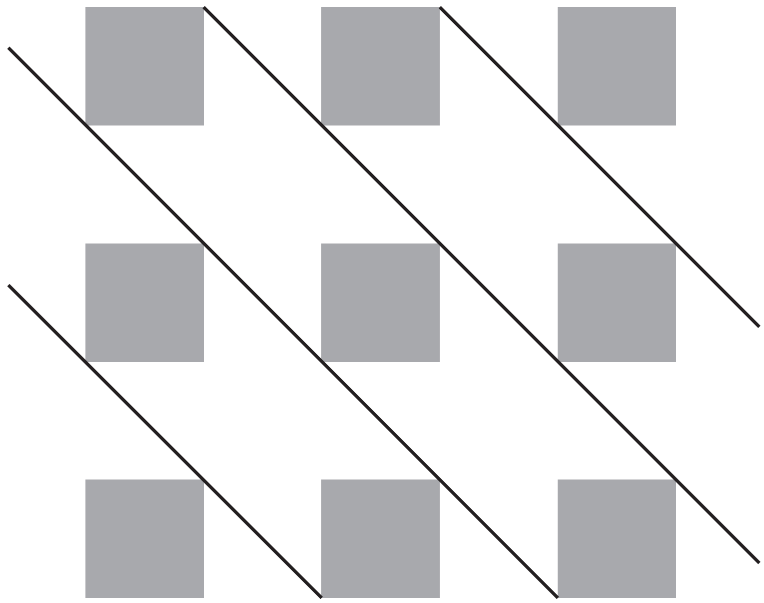

Figure 17: Light will get through diagonal pathways when $\mu = 1$

Using Theorem 5.6 we can say that the set of projections with Hausdorff dimension not equal to the Hausdorff dimension of $C^\lambda \times C^\lambda$ has measure 0 for $\lambda \leq \frac{1}{4}$. However, we proved in Theorem 5.5 that at $\mu = 1$ the dimension of the projection is not equal to that of $C^\lambda \times C^\lambda$. Thus we have found infinitely many of the"few" exceptions to this rule.

After looking at numerous examples of regular sets formed by projections as discussed in Section 3 and computing their Hausdorff dimensions, we came to the following conjecture:

Conjecture 5.7. *All projections of $C^\lambda \times C^\lambda$ where $\mu = (\frac{1}{\lambda})^i$ for non-negative integer i will have Hausdorff dimension $\frac{\ln 3}{-\ln \lambda}$.*

Proof. Assuming Conjecture (3.1) holds, the proof is as follows: At the n^{th} iteration these sets consist of $2^i \cdot 3^{n-i}$ intervals of length $2 \cdot \lambda^n$ for $n > i$. So we want

$$\sum_{k=1}^{2^i \cdot 3^{n-i}} w_\alpha \lambda^{n\alpha} = 2^i \cdot 3^{n-i} w_\alpha \lambda^{n\alpha}$$

to be non-zero and finite as $n \to \infty$. This requires that $\alpha = \frac{\ln 3}{-\ln \lambda}$. □

It follows from this that any projection of $C^\lambda \times C^\lambda$ where $\lambda \leq \frac{1}{4}$ and $\mu = \frac{1}{\lambda^i}$ will not have the same Hausdorff dimension as $C^\lambda \times C^\lambda$. Based on experience, we conjecture that these are the only angles for which this will be true. Limited by time, we were not able to prove that this as we struggled to learn anything about the projections at other angles.

Theorem 5.8. *For light to reach the x-axis when $\mu = 1$, it must be that $\lambda \leq \frac{1}{2M-1}$.*

Proof. Consider the case when $M = 2$. In order for light to reach the x-axis when $\mu = 1$, we need that the path shown below does not hit the top right corner of block C. (Note this is equivalent to the parallel path above it not hitting the bottom left corner of block A.)

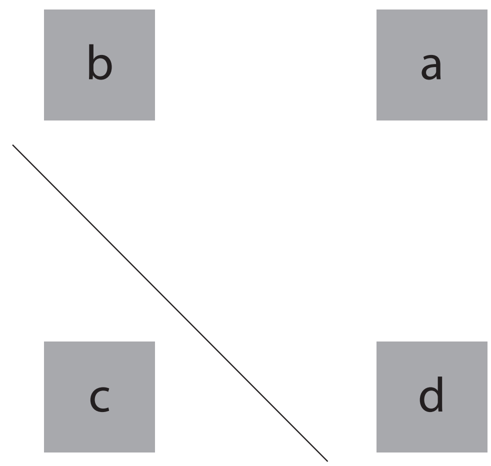

Figure 18: For light to reach the x-axis, it must not hit block c in the path shown.

Taking the bottom left corner of the set to be the origin, the equation of this path is $y = -x + (1 - \lambda)$. The coordinates for the point we are trying to avoid are (λ, λ). Using this information, we want $\lambda < -\lambda + (1 - \lambda)$ or $\lambda < \frac{1}{3}$. Thus the desired value for $M = 2$ is $\lambda = \frac{1}{3} = \frac{1}{2M-1}$.

To find this value for M in general, we only need to consider the square of four blocks in the bottom lefthand corner. See Figure 19.

We can think of this as a mini-($M = 2$) set. This tells us that the cut-off value we are looking for is when this mini-($M = 2$) set is a mini-middle third set, that is, when $\lambda = (1 - 2\lambda)$. Since there are M blocks of length λ and $M - 1$ spaces of length $(1 - 2\lambda) = \lambda$ in the first iteration, we have $1 = M\lambda + (M - 1)\lambda$ or $\lambda = \frac{1}{2M-1}$. □

Note that this is why we define a Cantor-like set with given M to be a thin set if $\lambda \leq \frac{1}{2M-1}$.

The λ values we found above have an additional interpretation. When the projection of a $C^\lambda \times C^\lambda$ set with $\mu = 1$ is the solid interval $[0, 2]$, this means that the addition of C^λ with itself

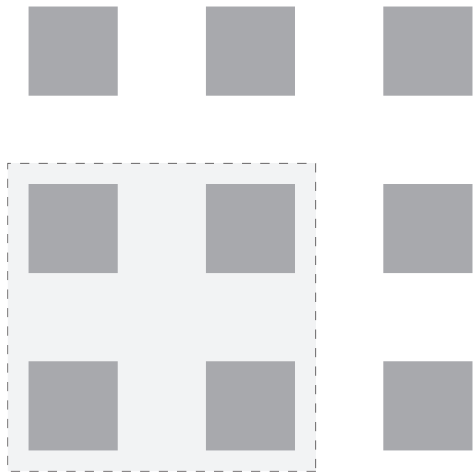

Figure 19: We only need to consider the four blocks highlighted.

gives us every number between 0 and 2. That is, for any $x \in [0,2]$, there exist $c_1, c_2 \in C^\lambda$ so that $c_1 + c_2 = x$. This is a remarkable property considering the intuitive scarcity of points in C^λ. This critical λ value, $\lambda = \frac{1}{2M-1}$, represents the most we can take out of a set for any M so that it retains this property. For any Cantor-like set, C^λ, with $\lambda \leq \frac{1}{2M-1}$, $C^\lambda + C^\lambda$ gives us the entire interval $[0,2]$.

After playing with Hausdorff dimension, one may notice that certain patterns arise. After finding $\lambda_M = \frac{1}{2m-1}$ to be a critical value as above, then for any fixed M the Hausdorff dimension of C^λ divided by the Hausdorff dimension of the projection of $C^\lambda \times C^\lambda$ using the same λ will always equal the Hausdorff dimension of C^{λ_M} for $\mu = 1$, $\frac{\ln(M)}{\ln(2M-1)}$.

Specifically, consider Cantor sets with $M = 2$. The Hausdorff dimension of C^λ set is $\frac{\ln 2}{\ln \lambda}$. The Hausdorff dimension of any projection with $\mu = 1$ of $C^\lambda \times C^\lambda$ with $M = 2$ is $\frac{\ln 3}{-\ln \lambda}$. Then the value we get when we divide the Hausdorff dimension of C^λ by the Hausdorff dimension of $C^\lambda \times C^\lambda$ with $\mu = 1$ is $\frac{\ln 2}{\ln 3}$. We recognize this as the Hausdorff dimension of the middle third Cantor set where $\lambda = \frac{1}{3}$, $C^{\frac{1}{3}}$. As we found before, $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ is the thinnest set whose projection is a solid interval when $\mu = 1$.

This idea is formalized below:

Corollary 5.9. *Let s denote the Hausdorff dimension of C^λ, $s = \frac{\ln(M)}{-\ln \lambda}$. Let p denote the Hausdorff dimension of the $\mu = 1$ projection of $C^\lambda \times C^\lambda$, $p = \frac{\ln(2M-1)}{-\ln \lambda}$. Finally, let c denote the Hausdorff dimension of $C^{\frac{1}{2M-1}} \times C^{\frac{1}{2M-1}}$, $c = \frac{\ln(M)}{\ln(2M-1)}$. Then for any λ and M, $\frac{s}{p} = c$.*

Proof. The result follows directly from calculation. □

Although the previous Corollary is easy to prove by calculation, we would like to know if there is a deeper meaning to the result. This is a question that we explored a bit, but it is still an open problem.

6 Mixed Sets

We also considered sets that were the Cartesian product of two Cantor sets that did not have the same λ-value; call them C^{λ_1} and C^{λ_2}. The projection of these sets when $\mu = 1$ corresponds to the addition of the two sets: $C^{\lambda_1} + C^{\lambda_2} = \{x + y : x \in C^{\lambda_1}, y \in C^{\lambda_2}\}$. Recall we showed that for $\mu = 1$ the projection of $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ is the interval from zero to two. This means that $C^{\frac{1}{3}} + C^{\frac{1}{3}} = [0, 2]$. Consider the value $\sqrt{2} \in [0, 2]$. Since the projection is the interval from zero to two it means that there are two values in the Cantor set that add to $\sqrt{2}$. Curiously, for any two sets where $\lambda_1 \neq \lambda_2$, even if $\lambda_1, \lambda_2 \geq \frac{1}{3}$, we will not get a solid interval for the sum of the two sets due to the following theorem.

Theorem 6.1. *Given C^{λ_1} and C^{λ_2} where $\lambda_1 > \lambda_2$ the projection of $C^{\lambda_1} \times C^{\lambda_2}$ will have an infinite number of intervals not in the shadow.*

Proof. Consider the ratio of the height of a base block of iteration $n - 1$ to the size of the segment removed from the block in the next iteration that is parallel to the x-axis. In Figure 20 we can see that for the first iteration we have the ratio of a to b. Since $a = 1$ and $b = 1 - 2\lambda_1$, the ratio for the first iteration this is $\frac{1}{1-2\lambda_1}$. The second iteration ratio is $\frac{\lambda_2}{\lambda_1(1-2\lambda_1)}$ because λ_2 and λ_1 are the scaling factors for 1 and $1 - 2\lambda_1$ respectively. In general, for n^{th} iteration the ratio is $\frac{1}{(1-2\lambda_1)}(\frac{\lambda_2}{\lambda_1})^{n-1}$. If this ratio is less than one for some n, then light will get through the vertical path. We see that $(\frac{\lambda_2}{\lambda_1})^{n-1} \to 0$ as $n \to \infty$ because $\lambda_1 < \lambda_2$ so there is some $N \in \mathbb{N}$ such that the ratio will be less than 1 for all $n \geq N$.

Consider the top right block. Light from that corner will definitely get through to the x-axis throught the path described above for all $n \geq N$ because the set below will not have a chance to block the light. Thus there will be an infinite number of intervals not in the shadow. □

The addition of Cantor sets with different λ-values is an open problem. Theorem 6.1 alludes the difficulty in solving this problem because the addition of the sets will be not be a finite number of intervals. In fact, the addition will be a very complex set with an infinite number of parts. The construction by itereation of the projection is very irregular because the self-similar properties of sets with the same λ-value are lost.

In order to approach the mixed sets in a different manner, we tried to map a more complicated set, $C^{\frac{1}{4}} \times C^{\frac{1}{3}}$, to a less complicated set that we understood much better, $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$. After the mapping the set, we would also have to consider how the map affects our straight rays of light, or, in other words, how it affects the family of lines $y = -x + k$ for $k \in [0, 2]$. We defined a

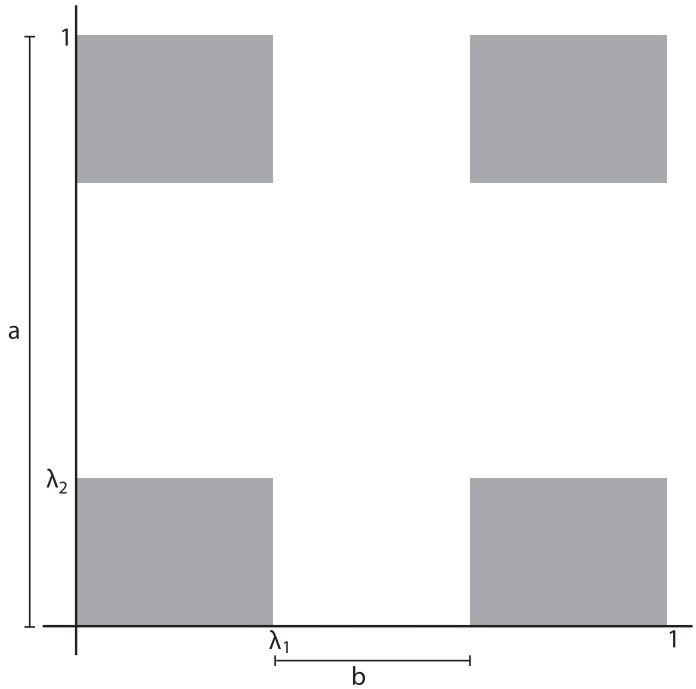

Figure 20: Consider the ratio of a to b

recursive equation to define a map from $C^{\frac{1}{4}}$ to $C^{\frac{1}{3}}$ beginning from $f_1(x)$:

$$f_1(x) = \begin{cases} \frac{4}{3}x, & \text{if } x \in [0, \frac{1}{4}], \\ \frac{2}{3}x + \frac{1}{6} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ \frac{4}{3}x - \frac{1}{3} & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

$$f_n(x) = \begin{cases} \frac{1}{3}f_{n-1}(4x) & \text{if } x \in [0, \frac{1}{4}], \\ f_{n-1}(x) & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ \frac{1}{3}f_{n-1}(4x-3) + \frac{2}{3} & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

So our map from $g : C^{\frac{1}{4}} \times C^{\frac{1}{3}} \to C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ will be $g(x, y) = (f(x), y)$. So our family of lines are now $y = f(-x + k)$ for $k \in [0, 2]$.

How the bent light interacts with the standard $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ is still an open problem because many of the properties that made this set easy to work with were no longer helpful with bent light. This is an area one could explore more deeply in future research.

7 Circle Problem

Up until now, we have considered how the shadow that a Cantor-like set casts changes as μ changes, and as λ changes. In this section, we will move the light source. Specifically, we will consider the shadow made by a Cantor-like set if the light source is located at the center of the set. In this case, instead of a single μ, the light will be hitting the Cantor-like set at a range of μ. We call this section the circle problem, because of the way we originally framed the question: What does the shadow of a Cantor-like set look like if the light is projected from the center onto a circle.

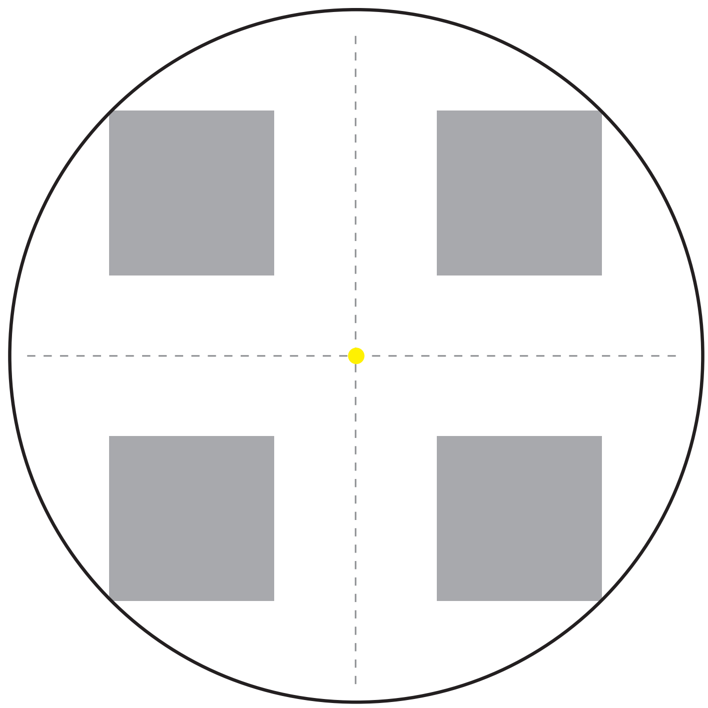

Figure 21: The first iteration of $C^{\lambda} \times C^{\lambda}$ with circle and light at center

For the sake of convenience in calculation, we let the origin be the center of a 2×2 square, then the four corners of the square are (1,1), (-1,1), (-1,-1), and (1,-1) as seen in Figure 22.

Since the light source is fixed at the origin, the symmetry of the set allows us to focus only on the quarter of the set in the first quadrant, that is, all parts of the set that lie within the upper right-hand block of the first iteration. Where, in our previous case where the light was positioned at infinity and we considered the shadow cast onto the x-axis, in this case where the light is positioned at the center we consider the shadow cast by our set on the line $x = 1$.

For consistency, we refer to the squares and gaps created in the first iteration and after using

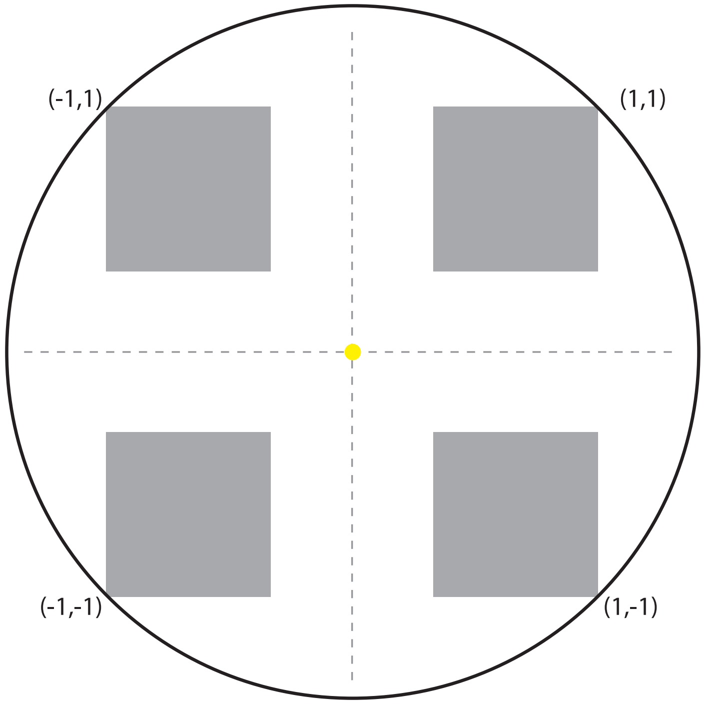

Figure 22: The first iteration of $C^\lambda \times C^\lambda$ with circle and light at center, coordinates labeled

the following system. We name the four squares contained in one square by a, b, c, and d, which stand for upper right square, upper left square, bottom left square, and bottom right square respectively. Thus, ad^2, which is equivalent to add stands for the square d contained in ad, which is contained in square a of the first iteration. Since we always analyze gaps between squares contained in one particular square, we use the form, for example $ad^2 - gap(bc, ad)$, to show the gaps between bc and ad in ad^2 as seen in Figure 23.

7.1 Sets That Yield a Single Interval

Theorem 7.1. *For Cantor-like sets with $\lambda \geq \frac{1}{3}$, the shadow cast by square a onto the line $x = 1$ will be an interval.*

Proof. To show that the shadow of a Cantor-like set with $\lambda \geq \frac{1}{3}$ will be an interval, consider the range of μ that will be hitting the set. At its shallowest, μ will be equal to $1 - 2\lambda$, and at its steepest, $\mu = \frac{1}{1-2\lambda}$. From (Theorem about length from light at infinity case), it is clear that for this range of μ, light will not get through this Cantor-like set. □

We can say, then, that for $\lambda \geq \frac{1}{3}$, the shadow of square a of our Cantor-like set will be an

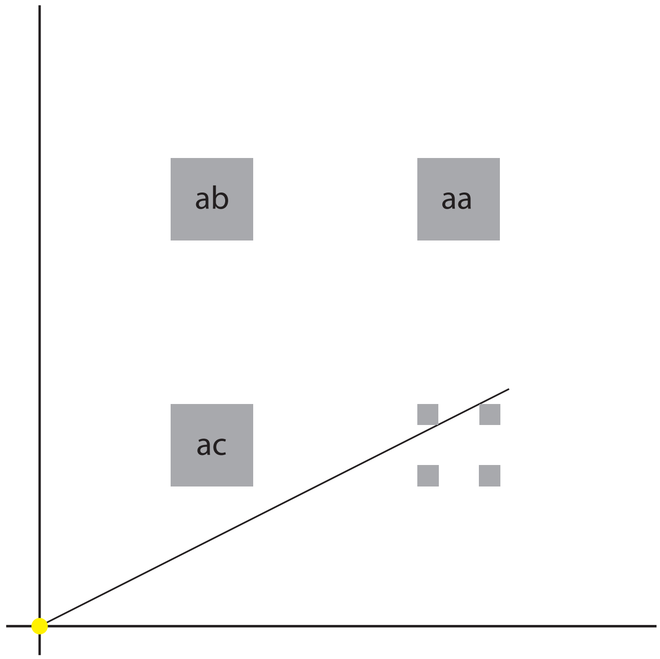

Figure 23: $ad^2 - gap(bc, ad)$

interval. The set and it's shadow as cast on a circle is pictured below. The next question we consider is for which Cantor-like sets will the shadow of square a be not one interval, but multiple intervals?

7.2 Other Sets

Unfortunately, when considering thin Cantor-like sets, we cannot use the reverse of the argument for Theorem 7.1, as in that case we only know that in the light at infinity case *some* of the rays of light will pass through a thin Cantor-like set, which does not guarantee that any will pass through the set with the light at the origin. It turns out that it is useful, when considering these light at the center problems, to think of our Cantor-like sets not in terms of λ, the proportion of the set that we keep at each iteration, but in terms of what is removed.

Definition 7.2. For a Cantor-like set with a given λ, let δ denote the proportion that is removed from each interval at each iteration. Then, in the case where $m = 2$, $\delta = 1 - 2\lambda$. Figure 25 shows the second iteration of square a with corners labeled in term of δ.

Definition 7.3. We say that a value of δ, δ' is a *tipping point* at the n^{th} iteration for a gap if for $\delta \leq \delta'$ no additional light gets through at the n^{th} iteration at the gap in question, and, for

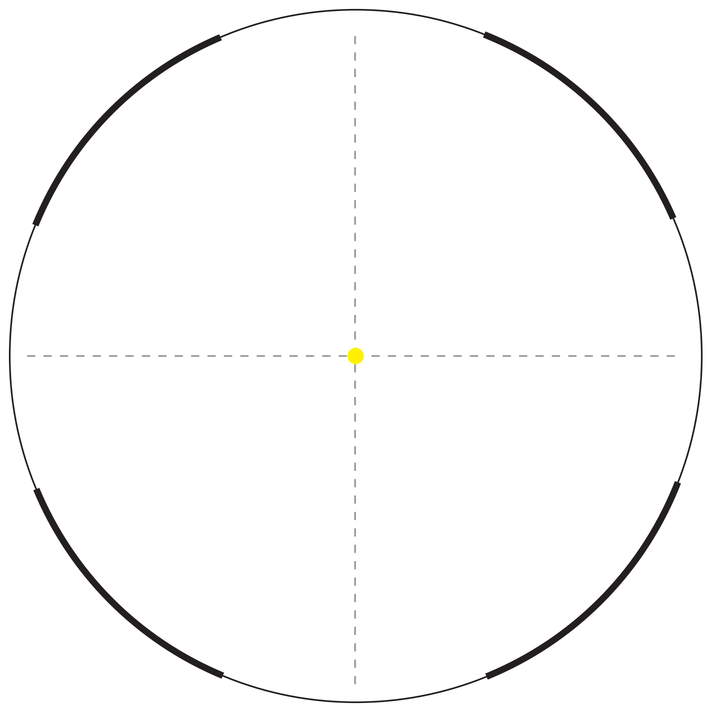

Figure 24: Light source at the center of a fat Cantor-like set.

$\delta > \delta'$, additional light passes through the gap.

For light to pass through a gap, δ must be large enough so that the light that passes between the two squares that define the entrance of the gap is not blocked by the squares that define the exit of the gap. To do this we can draw a line from the origin to the leading side of the gap entrance, and let this line continue to the line $x = 1$. We draw a similar line to the exit of the gap. For light to pass through the gap, the line from the entrance must intersect the line $x = 1$ lower than the line through the exit point. Because of how we have defined our sets, this is equivalent to comparing the slopes of the two lines, and, because our lines go through the origin, the slope of each line will be the y coordinate of the point over its x coordinate. We use this process to prove the next theorem.

Theorem 7.4. *The tipping point for the second iteration is* $\sqrt{2} - 1$.

Proof. At the second iteration, there are two gaps through which light may pass, and, by the symmetry of the set, they are equivalent. So, it suffices to find the tipping point for *a gap*(cd, ad), which, we do by comparing the two lines pictured in Figure 26.

The tipping point will be when δ is such that the slopes of the two lines are equal, or, the roots of the tipping point function $t(\delta)$

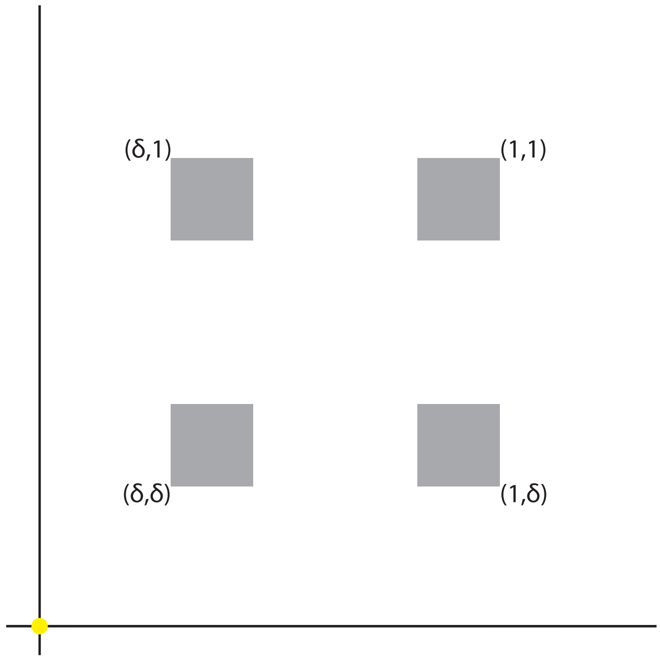

Figure 25: Square a with coordinates labeled in terms of δ.

$$t(\delta) = \frac{\delta}{\delta + 2\left(\frac{1-\delta}{2}\right)^2} - \frac{\delta + 2\left(\frac{1-\delta}{2}\right)^2}{1 - 2\left(\frac{1-\delta}{2}\right)^2}$$

Where $\frac{\delta}{\delta+2\left(\frac{1-\delta}{2}\right)^2}$ is the slope of line a and $\frac{\delta+2\left(\frac{1-\delta}{2}\right)^2}{1-2\left(\frac{1-\delta}{2}\right)^2}$ is the slope of line b. Calculating the roots results in one relevant root, that is, a root that is an element of the reals between zero and one, $\sqrt{2} - 1$. □

Using a similar process to calculate the tipping points for other gaps, we conjectured that the limit of the tipping points of adb^n $gap(bc, ab)$ will be $\sqrt{2}-1$. Theorem 7.5 proves this conjecture.

Theorem 7.5. *As n goes to infinity, the limit of the tipping point of adb^n $gap(bc, ab)$ will be $\delta = \sqrt{2} - 1$.*

Proof. By construction, at the n^{th} iteration, the tipping point of the gap in question will be the relevant root of

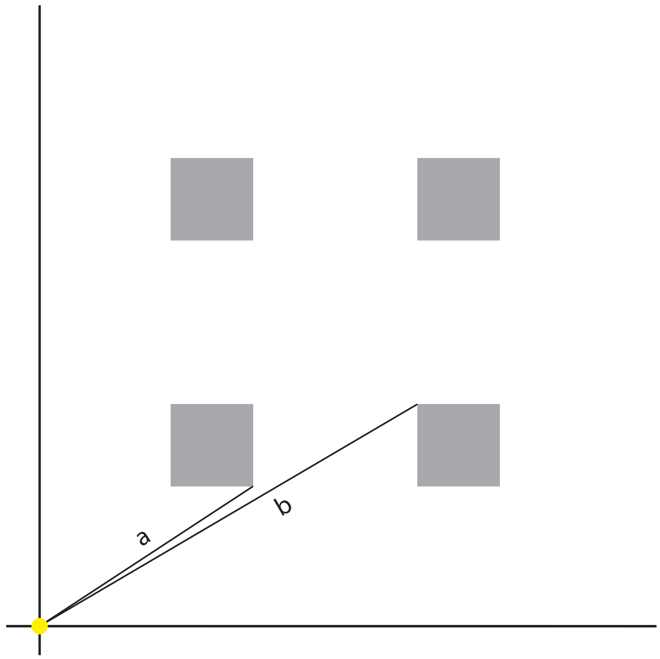

Figure 26: When lines a and b are are the same, δ is the tipping-point for the second iteration. The δ pictured is too large to be the tipping point.

$$t_n(\delta) = \frac{\delta + 2\left(\frac{1-\delta}{2}\right)^2 - 2\left(\frac{1-\delta}{2}\right)^n}{1 - 2\left(\frac{1-\delta}{2}\right)^2 + 2\left(\frac{1-\delta}{2}\right)^n} - \frac{\delta + 2\left(\frac{1-\delta}{2}\right)^2}{1 - 2\left(\frac{1-\delta}{2}\right)^2 + 2\left(\frac{1-\delta}{2}\right)^{n-1} - 2\left(\frac{1-\delta}{2}\right)^n},$$

which, after setting equal to zero and factoring out irrelevant roots, reduces to finding the roots of $\tau_n(\delta) = 2^n(\delta^2 + 2\delta - 1) - 4(1 - \delta)^n$. First we must establish that for all n, there exists a relevant root of $\tau_n(\delta)$. We do this via the Intermediate Value Theorem. For each individual n, $\tau_n(\delta)$, is the difference of two continuous functions $2^n(\delta^2+2\delta-1)$ and $4(1-\delta)^n$ and so is itself a continuous function of δ. Next, $\tau_n(0) = 2^n(-1) - 4$, which is negative. Similarly, $\tau_n(1) = 2^n(2)$ which is positive. So, by the Intermediate Value Theorem, τ_n must have a relevant root for all iterations n. We know that there will be only one relevant root because τ_n is monotone decreasing.

Having shown that a single relevant root must exist, call the relevant root of τ_n, δ_n. Returning to our function, we can further factor our tipping point function, τ_n, to be

$$2^n\left(\delta - (\sqrt{2} - 1)\right)\left(\delta - (-\sqrt{2} - 1)\right) - 4(1 - \delta)^n.$$

Next, we evaluate the limit of $\delta_n - (\sqrt{2} - 1)$ as n goes to infinity, recalling the fact that $0 < \delta_n < 1$

$(\delta_n - (-\sqrt{2}-1)) > 1$. Showing that the limit of $(\delta_n - (\sqrt{2}-1))$ as n goes to infinity is zero is equivalent to showing that δ_n goes to $\sqrt{2}-1$ as n goes to infinity.

$$\begin{aligned}\lim_{n\to\infty} \frac{(\delta_n - (\sqrt{2}-1))}{4} &= \lim_{n\to\infty} \frac{(1-\delta_n)^n}{2^n((\delta - (-\sqrt{2}-1)))} \\ &\leq \lim_{n\to\infty} \frac{(1-\delta_n)^n}{2^n} \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} = 0\end{aligned}$$

□

Theorem 7.6. *As n goes to infinity, the limit of the tipping point of ad^n gap(cd, ad) will be $\delta = \frac{1}{2}$.*

Proof. In the ad^n gap(cd, ad) case, applying similar methods to the proof of Theorem 7.5 leads us to finding the roots of $2(1-\delta)^n + 2^{n+1}\delta - 2^n = 0$. And, while it is clear that $\delta = \frac{1}{2}$ will be a root of the limit of $2(1-\delta)^n + 2^{n+1}\delta - 2^n$ as n goes to infinity, it is not immediately clear that the limit of the roots of $2(1-\delta)^n + 2^{n+1}\delta - 2^n$ will be $\delta = \frac{1}{2}$. As in Theorem 7.5, we can show that there exists a single relevant root that we will call δ_n by making use of the Intermediate Value Theorem by considering when $\delta = 0$ and when $\delta = 1$, and, noticing that our function is monotone increasing because its derivative is positive for relevant values of δ. Then we manipulate our function so that we can solve for $(\delta_n - \frac{1}{2})$.

$$\begin{aligned}2(1-\delta_n)^n + 2^{n+1}\delta_n - 2^n &= 0 \\ 2(1-\delta_n)^n + 2^{n+1}(\delta_n - \frac{1}{2}) &= 0 \\ (\delta_n - \frac{1}{2}) &= \frac{-2(1-\delta_n)^n}{2^{n+1}}\end{aligned}$$

Then, we will find the limit of $(\delta_n - \frac{1}{2})$

$$\lim_{n\to\infty} (\delta_n - \frac{1}{2}) = \lim_{n\to\infty} \frac{-2(1-\delta_n)^n}{2^{n+1}} = 0$$

Showing that the limit of $(\delta_n - -\frac{1}{2})$ as n goes to infinity is zero is equivalent to showing that δ_n goes to $\frac{1}{2}$ as n goes to infinity.

□

7.3 Other Findings and Conjectures

Conjecture 7.7. *The closer to the center, roots approach* $\sqrt{2}-1$*, the further to the outside, roots approach* $\frac{1}{2}$*.*

Conjecture 7.8. *We conjecture that for* $\delta < \sqrt{2}-1$ *the shadow of square a will be an interval.*

We also looked at the shadow cast by square *ac*. These calculations are very complicated because finding the tipping point for light to pass through square *ac* is not the same as finding the tipping point for light to pass through both square *ac* and the rest of the set. In many cases to find the actual tipping point, one must consider many possible interactions with square *aa*. Because of these complications we do not have any notable results from these gaps.

8 Conclusion

Prior to our research it was known that the projection of $C^{\frac{1}{3}} \times C^{\frac{1}{3}}$ with $M = 2$ and $\mu = 1$ is the interval $[0, 2]$. Our research explored what happens to the projection in terms of length and Hausdorff dimension for different values of μ, λ and M, when λ_1 and λ_2 are allowed to differ, and for different locations of the light source. We found that projections for thin sets are different than those when $\lambda \geq \frac{1}{3}$.

Some of the problems we studied remain open questions. Ideas for future research include: decoding the construction of irregular projections and finding a general length formula for the projections of thin sets for $M \geq 2$, solving the addition of two Cantor-like sets, and investigating what happens for δ between $\frac{1}{3}$ and $\sqrt{2}-1$, and for δ between $\sqrt{2}-1$ and $\frac{1}{2}$. All of our theorems, conjectures, and open questions could also be looked at in higher dimensions.

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