Exploring Upper Bounds of Graph Proper Diameters

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Carleton College

October 2, 2018

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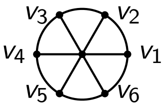
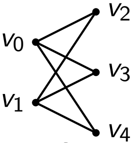


G_1



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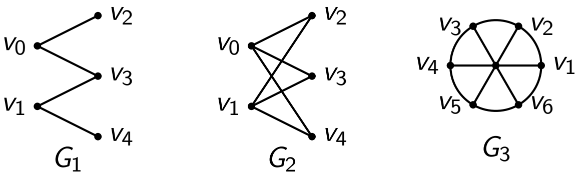

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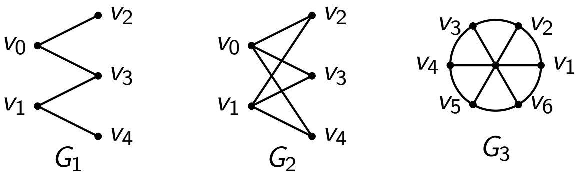


- **Bipartite Graph:** A graph containing no odd cycles

Communication Networks

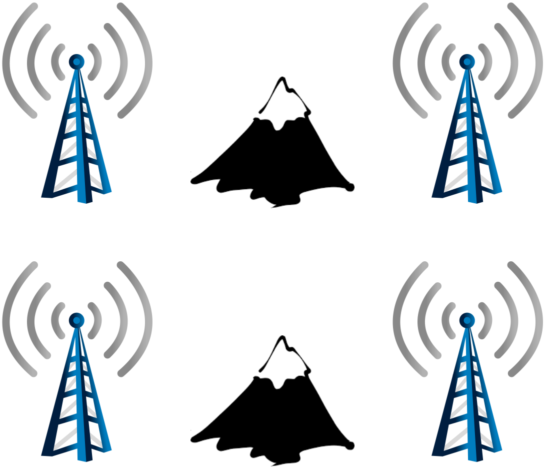


Table 1: A Communication Network

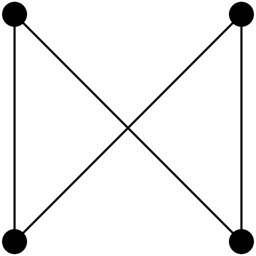
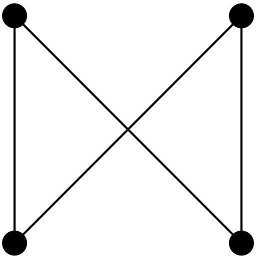


Figure 1: The Same Communication Network Represented with a Graph



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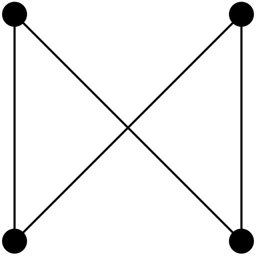

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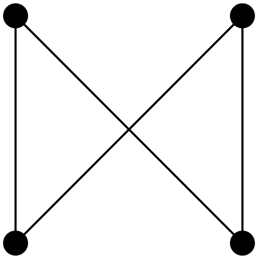

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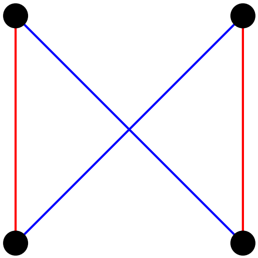

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- We can model this by giving each frequency a color

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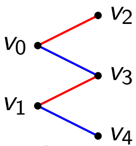

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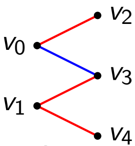

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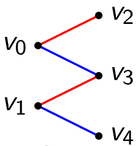

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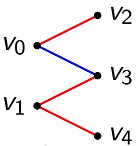

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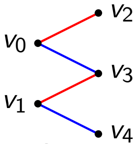

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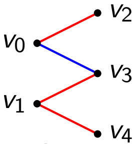

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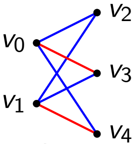

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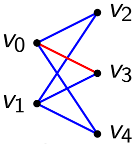

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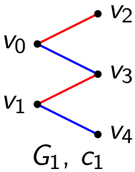

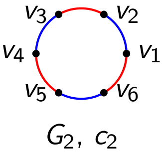

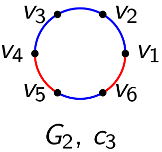

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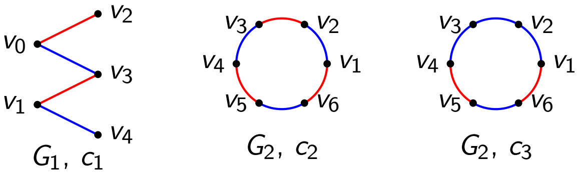
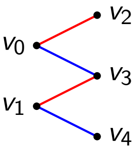
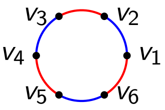


- **Observation**: $\text{diam}(G) \leq \text{pdiam}_2(G, c) \leq \#$ vertices -1

Known Properties of Proper Diameter

- The connectivity of a graph $\kappa(G)$ is the smallest number of vertices that must be removed to disconnect the graph.





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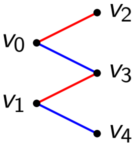

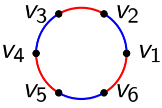

Theorem (Coll et. al. (2018))

For any properly connected 2-colored graph G of order $n \geq 2$, $\text{pdiam}_2(G) \leq n - \kappa(G) + 1$.

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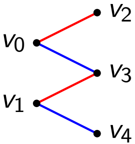

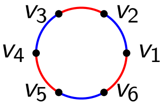

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- When $\kappa(G) = 2$, this inequality simplifies to $\text{pdiam}_2(G) \leq n - 1$. When is this upper bound attainable?

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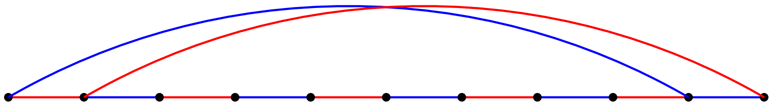

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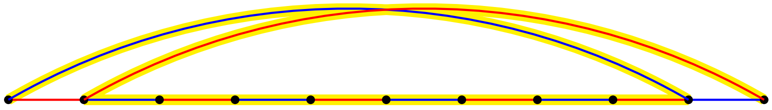

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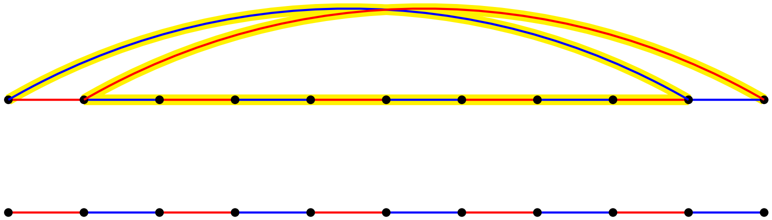

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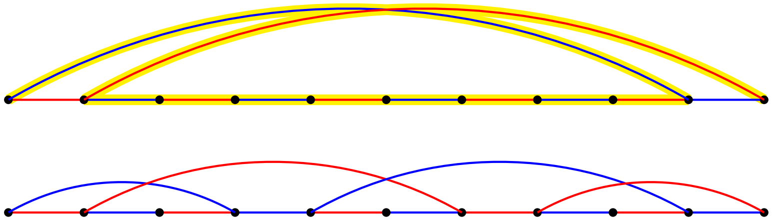

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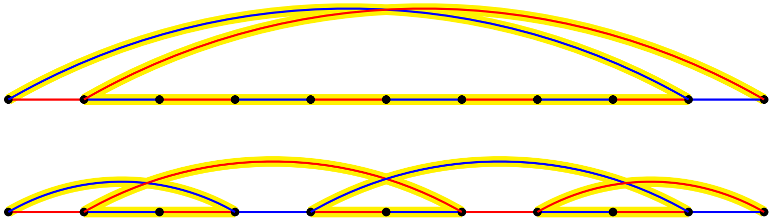

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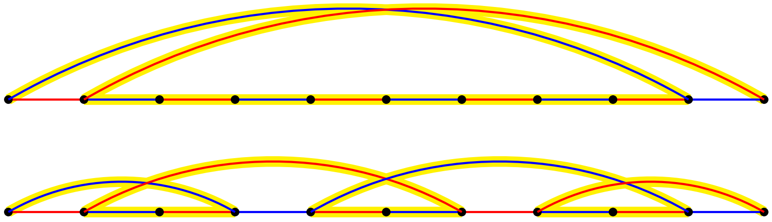

- There seem to be a variety of counterexamples.

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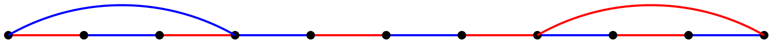

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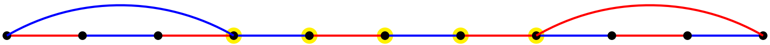

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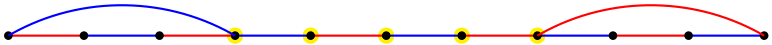

- Cut vertices still remain, so $\kappa(G)=1$. Links must cross over somewhere to increase the connectivity of G to 2.

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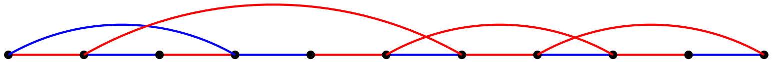

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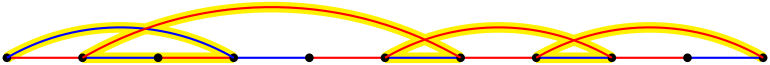

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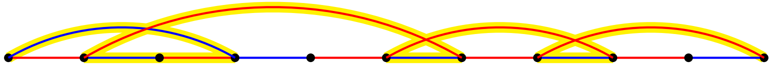

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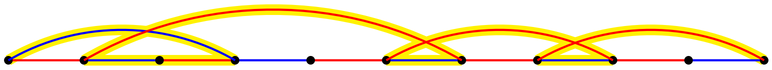

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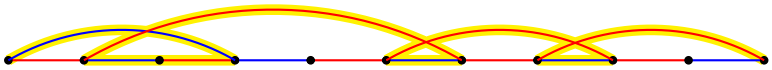

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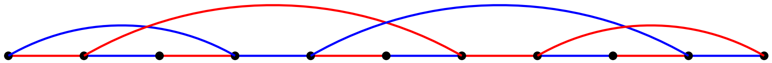

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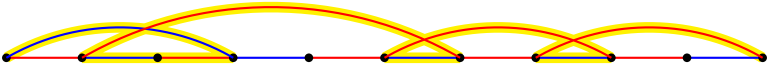

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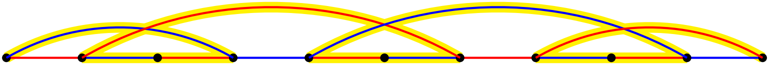

Generalizing the Counterexamples

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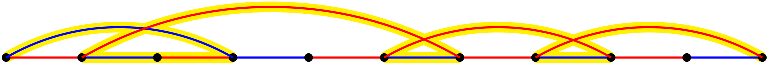

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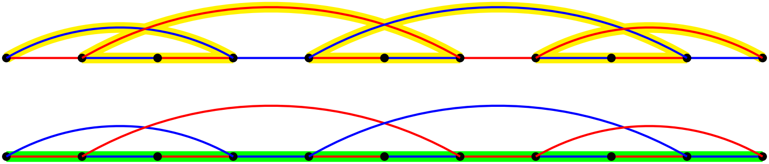

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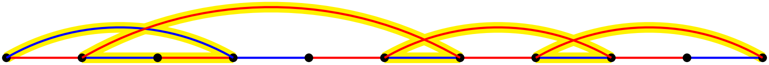

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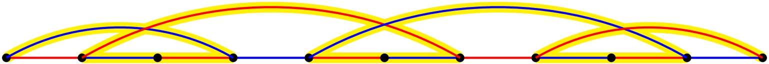

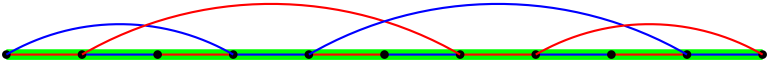

- If G is a counterexample, once it has connectivity 2, there will be two different Hamiltonian paths between the ends of the path. The yellow path, which uses links, is called a **chain**.

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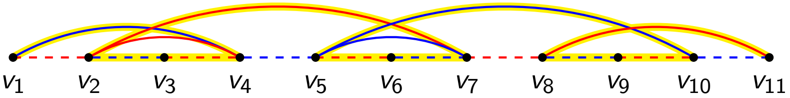

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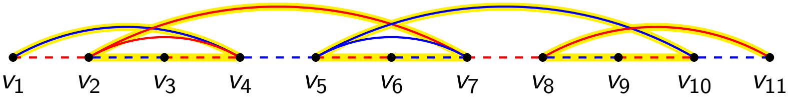

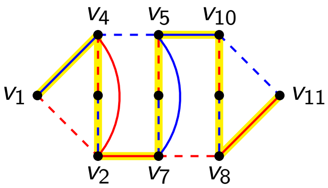

Definition of $\mathcal{T}_n$ Graphs (Version 1)

Definition: $\mathcal{T}_n$ Graph 1.0

A $\mathcal{T}_n$ **Graph** is bipartite with $\kappa(G) = 2$. The structure contains an even cycle with a pair of opposite vertices called **ears**. Add all possible **bands**, which are paths not on the cycle between pairs of vertices that have the same distance from each ear.

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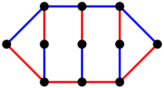

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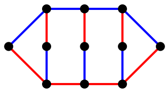

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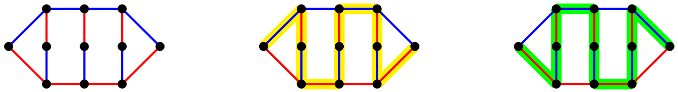

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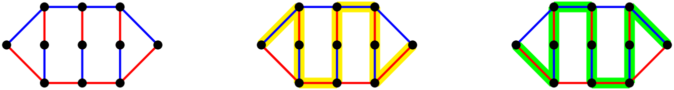

$$\text{pdiam}_2(T, c_1) = n - 1 = 10$$

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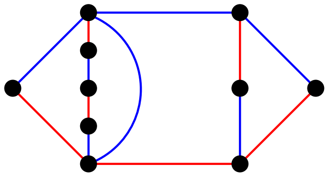

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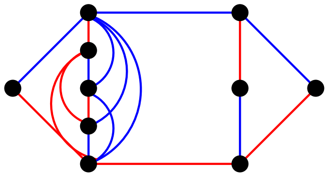

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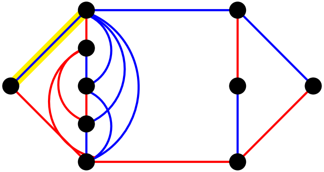

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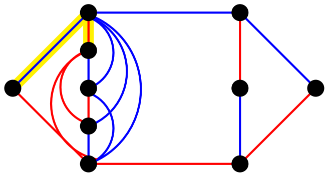

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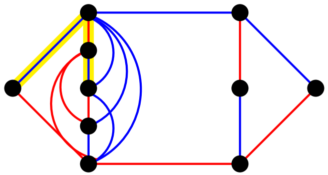

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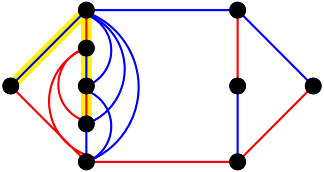

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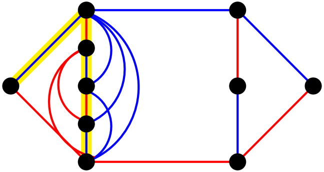

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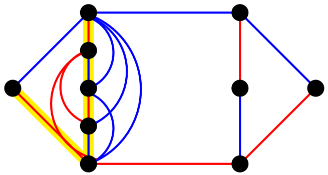

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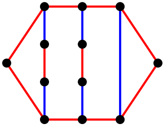

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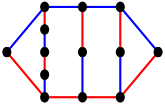

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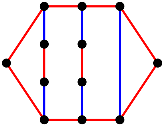

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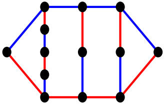


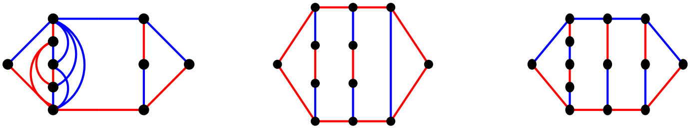
Odd $\mathcal{T}_n$

- "Mixed" $\mathcal{T}_n$ graphs would necessarily contain a shortcut

Final Formulation of $\mathcal{T}_n$ Graphs

Final Definition: $\mathcal{T}_n$ Graph

A $\mathcal{T}_n$ **Graph** contains an even cycle and all possible bands on the cycle. All bands must have lengths equivalent mod 2 and any edge may be added between vertices in a band.



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Theorem (F., G., M., R., S., S. (2018)+)

Let G be a graph on n vertices with $\kappa(G) \geq 2$. The proper diameter of G is $n - 1$ if and only if G is a $\mathcal{T}_n$ Graph.

Future Directions

- Refine the $\mathcal{T}_n$ information to submit a paper for publication
- Continue looking at proper diameter in other graph families
- Generalize $\mathcal{T}_n$ to other connectivity values
- Explore proper connectivity

References

- V. Coll, J. Hook, C. Magnant, K. McCready, and K. Ryan, The Proper Diameter of a Graph, *Discuss. Math. Graph Theory*, (2018) 1-19.
- V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, and Z. Tuza. Proper connection of graphs. *Discrete Math.*, 312(17):2550–2560, 2012.

Thank you! Questions?