Solvable Leibniz Algebras with an Abelian Nilradical

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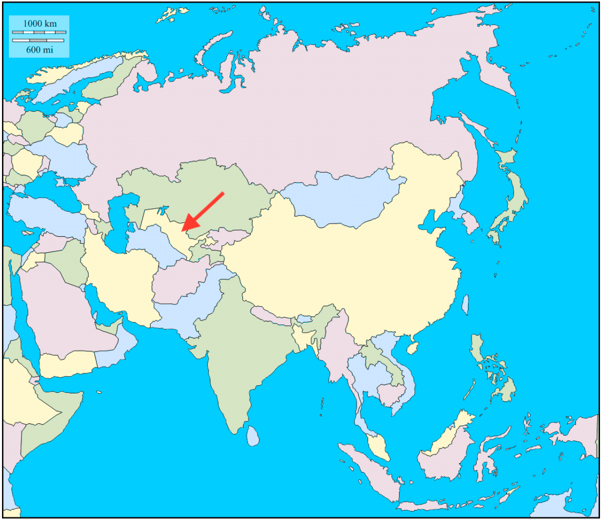
Where is Uzbekistan?



Some background on Leibniz algebras

- A Lie algebra is a vector space that has a binary bracket operation $[\cdot, \cdot]$ that abides to the following three axioms:
  (L1) The bracket operation is bilinear.
  (L2) $[xx] = 0$ for all $x \in L$.
  (L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ (where $x, y, z \in L$).

- Leibniz algebras are a "noncommutative" generalization of Lie algebras.

Some background on Leibniz algebras

- Leibniz algebras require just bilinearity and the Leibniz identity:
$$[[a,b],c] = [a,[b,c]] + [[a,c],b].$$

Preliminaries and Definitions

Definition

A linear map $d\colon L \to L$ of a Leibniz algebra $(L, [\cdot,\cdot])$ is said to be a **derivation** *if for all $x, y \in L$, the following condition holds:*

$$d([x,y]) = [d(x), y] + [x, d(y)]\,.$$

For Leibniz algebras, the right multiplication operator $R_x\colon L \to L$, defined by $R_x(y) = [y,x],\ y \in L$ is a derivation.

Definition

Let $d_1, d_2, \dots, d_n$ be derivations of a Leibniz algebra L. The derivations $d_1, d_2, \dots, d_n$ are said to be **linearly nil-independent** *if for $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ and a natural number k,*

$$(\alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_n)^k = 0 \text{ implies } \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

Preliminaries and Definitions

Let

$$L^1 = L, \; L^{k+1} = [L^k, L], \; k \geq 1,$$

and

$$L^{[1]} = L, \; L^{[s+1]} = [L^{[s]}, L^{[s]}], \; s \geq 1.$$

The following definitions will be needed:

Definition

A Leibniz algebra L is said to be **nilpotent** *(respectively,* **solvable***), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$).*

Preliminaries and Definitions

Definition

An ideal of a Leibniz algebra is called **nilpotent** *if it is nilpotent as a subalgebra.*

It is easy to see that the sum of any two nilpotent ideals is nilpotent. Therefore the maximal nilpotent ideal always exists.

Definition

The maximal nilpotent ideal of a Leibniz algebra is said to be the **nilradical** *of the algebra.*

Preliminaries and Definitions

Definition

A Leibniz algebra L is said to be **abelian** *if $L^2 = 0$. An ideal of a Leibniz algebra is called abelian if it is abelian as a subalgebra.*

Problem Setup

- Let L be a solvable Leibniz algebra. Then it can be written in the form $L = N \oplus Q$, where N is the nilradical and Q is the complementary subspace. From Casas, et al. we have:

Theorem

Let L be a solvable Leibniz algebra and N be its nilradical. Then the dimension of Q is not greater than the maximal number of nil-independent derivations of N.

- It was proven in Adashev, et al. that the maximal dimension of a solvable Leibniz algebra with a k-dimensional abelian nilradical is $2k$. Moreover, this maximal case is classified.

Problem Setup

Here, we have classified all solvable Leibniz algebras of dimension $2k-1$ with abelian nilradical of dimension k, i.e.

$$L = N \oplus Q$$

$$2k-1 = dim(L) = dim(N) + dim(Q) = (k) + (k-1)$$

Methods

Proposition

Let L be a Leibniz algebra from the class $R(A(k), k-1)$. Then there exists a basis $\{e_1, e_2, e_3, \dots, e_k, x_1, x_2, \dots, x_{k-1}\}$ of L such that the table of multiplication on this basis has the following form:

$$\begin{cases} e_i x_i = e_i + \beta_{ii} e_k & 1 \le i \le k-1 \\ e_i x_j = \beta_{ij} e_k & 1 \le i \le k, \ 1 \le j \le k-1, \ i \ne j \\ x_i e_i = \alpha_i e_i + \gamma_{i,i} e_k & 1 \le i \le k-1 \\ x_i e_j = \gamma_{i,j} e_k & 1 \le i \le k-1, \ 1 \le j \le k-1, \ i \ne j \\ x_i e_k = \sum\limits_{j=1}^{k} \nu_{i,j} e_j & 1 \le i \le k-1 \\ x_i x_j = \delta_{i,j} e_k & 1 \le i, j \le k-1 \end{cases}$$

where $\alpha_i \in \{0, \ -1\}$.

Theorem

Let L be a Leibniz algebra from the class $R(\mathbf{a_k}, k-1)$ and let $\alpha_i = 0$ for $1 \leq i \leq k-1$. Then L is isomorphic to one of the following algebras:

Theorem

$$L_1(\beta_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_i]=\beta_i e_k, & 1\le i\le k-1,\end{cases}$$

$$L_2(\beta_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_i]=\beta_i e_k, & 1\le i\le k-1,\\ [x_i,e_k]=-\beta_i e_k, & 1\le i\le k-1,\end{cases}$$

$$L_3(\beta_i):\begin{cases}[e_1,x_1]=e_1+\beta_1 e_k, & \\ [e_i,x_i]=e_i, & 2\le i\le k-1,\\ [e_1,x_i]=\beta_i e_k, & 2\le i\le k-1,\\ [e_k,x_1]=e_k, & \end{cases}$$

$$L_4(\nu_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_1]=e_k, & \\ [x_1,e_k]=-e_k, & \\ [x_i,e_k]=\nu_i e_1, & 2\le i\le k-1,\end{cases}$$

$$L_5(\delta_{i,j}):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [x_i,x_j]=\delta_{i,j}e_k, & 1\le i,j\le k-1.\end{cases}$$

Theorem

Let L be a solvable Leibniz algebra from the class $R(\mathbf{a_k}, k-1)$ and $\alpha_i = -1$ for $1 \leq i \leq k-1$. Then L is isomorphic to one of the following algebras:

Theorem

$$L_6(\beta_j):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_j]=\beta_j e_k, & 1\le j\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le k-1,\end{cases}$$

$$L_7(\beta_j):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_j]=\beta_j e_k, & 1\le j\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le k-1,\\ [x_j,e_k]=-\beta_j e_k, & 1\le j\le k-1,\end{cases}$$

$$L_8(\gamma_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_1]=e_k, & \\ [x_i,e_i]=-e_i, & 1\le i\le k-1,\\ [x_i,e_1]=\gamma_i e_k, & 1\le i\le k-1,\end{cases}$$

$$L_9:\begin{cases}[e_1,x_1]=e_1+\beta_1 e_k, & \\ [e_i,x_i]=e_i, & 2\le i\le k-1,\\ [e_1,x_i]=\beta_i e_k, & 2\le i\le k-1,\\ [e_k,x_1]=e_k, & \\ [x_1,e_1]=-e_1-\beta_1 e_k, & \\ [x_i,e_i]=-e_i, & 2\le i\le k-1,\\ [x_i,e_1]=-\beta_i e_k, & 1\le i\le k-1,\\ [x_1,e_k]=-e_k. & \end{cases}$$

$$L_{10}(\delta_{i,j}):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le k-1,\\ [x_i,x_j]=\delta_{i,j}e_k, & 1\le i,j\le k-1.\end{cases}$$

Theorem

Let L be a solvable Leibniz algebra from the class $R(\mathbf{a_k}, k-1)$ and let $\alpha_1 = \cdots = \alpha_{t-1} = -1$ and $\alpha_t = \cdots = \alpha_{k-1} = 0$. Then L is isomorphic to one of the following algebras:

Theorem

$$M_{1,t}(\beta_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_i]=\beta_i e_k, & 1\le i\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\end{cases}\quad M_{2,t}(\beta_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_i]=\beta_i e_k, & 1\le i\le k-1,\\ [x_i,e_k]=-\beta_i e_k, & 1\le i\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\end{cases}$$

$$M_{3,t}(\beta_i):\begin{cases}[e_t,x_t]=e_t+\beta_t e_k, & \\ [e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_t,x_i]=\beta_i e_k, & 1\le i\le k-1,\\ [e_k,x_t]=e_k, & \\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\end{cases}\quad M_{4,t}(\beta_i):\begin{cases}[e_1,x_1]=e_1+\beta_1 e_k, & \\ [e_i,x_i]=e_i, & 2\le i\le k-1,\\ [e_1,x_i]=\beta_i e_k, & 2\le i\le k-1,\\ [e_k,x_1]=e_k, & \\ [x_1,e_1]=-e_1-\beta_1 e_k, & \\ [x_i,e_i]=-e_i, & 2\le i\le t-1,\\ [x_i,e_1]=-\beta_i e_k, & 2\le i\le k-1,\\ [x_1,e_k]=-e_k, & \end{cases}$$

$$M_{5,t}(\gamma_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_1]=e_k, & \\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\\ [x_i,e_1]=\gamma_i e_k, & 2\le i\le k-1,\end{cases}\quad M_{6,t}(\nu_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_t]=e_k, & \\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\\ [x_t,e_k]=-e_k, & \\ [x_i,e_k]=\nu_i e_t, & 1\le i\le k-1,\end{cases}$$

$$M_{7,t}(\delta_{i,j}):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\\ [x_i,x_j]=\delta_{i,j}e_k, & 1\le i,j\le k-1.\end{cases}$$

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Thank you for your attention!