Constructing Generalized Gelfand-Graev Representations

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Number theory and L-functions

Number theory is one of the most ancient branches of mathematics.

Examples of L-functions:

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- Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Number theory and L-functions

Birch and Swinnerton-Dyer conjecture

- Hasse-Weil L-function

$$L(s, E) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{1-2s}}$$

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- A representation of a group G is a linear group action of G on a vector space V, called the representation space.
- A group action of a group G on a set A is a map from $G \times A$ to A such that:
  - $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all g_1, g_2 in G, a in A
  - $1 \cdot a = a$ for all a in A

Representation of S_3 on $\mathbb{R}^3$

- S_3 acts on vectors in $\mathbb{R}^3$ by permuting their coordinates.
- For example: $(123) \cdot \langle x, y, z \rangle = \langle z, x, y \rangle$ and $(23) \cdot \langle x, y, z \rangle = \langle x, z, y \rangle$

Representation of S_3 on $\mathbb{R}^3$

The action of S_3 on $\mathbb{R}^3$ can be understood by looking at specific subspaces that are stable under the group action:

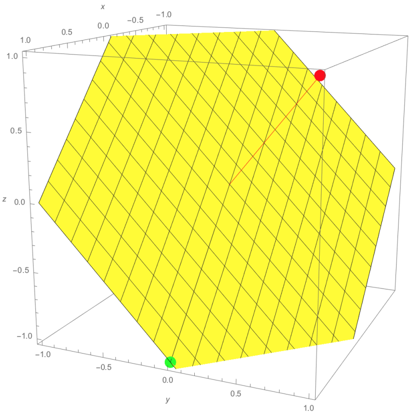
- The span of a vector $w = \langle 1, 1, 1 \rangle$
- Vectors whose coordinates add to 0, that live in the plane $x + y + z = 0$.

Representation of S_3 on $\mathbb{R}^3$

The action of S_3 on the vector $\langle 1, 2, 3 \rangle$ can be understood by expressing $\langle 1, 2, 3 \rangle$ as a linear combination of w and vectors whose coordinates add to 0.

$$\langle 1, 2, 3 \rangle = \langle -1, 0, 1 \rangle + \langle 2, 2, 2 \rangle$$

Representation of S_3 on $\mathbb{R}^3$



Generalized Gelfand-Graev Representations

- Generalized Gelfand-Graev representations (GGGRs) have originally been introduced by Kawanaka in 1985.
- They are important for integral realizations of automorphic L-functions.
- The main result of our project sheds light on the construction of GGGRs in the case of $\mathrm{GL}(n)$ defined over finite fields.

GGGRs

To make a GGGR we need the following components:

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- $Z_L = Z_L(A) = \{\ell \in L_A \mid \ell A \ell^{-1} = A\}$
- $(\eta, V) =$ a representation of $U_A Z_L$

GGGRs

Ex: $A = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix}$ $L = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$

$$U_A = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \quad Z_L = \begin{pmatrix} x & & \\ & y & \\ & & x \end{pmatrix}$$

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An induced representation is a representation of G built from a representation of a subgroup.

Nilpotent matrices

A matrix A is **nilpotent** if $A^n = 0$ for some $n > 0$.

- Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A^2 = 0$

Example of a Nilpotent Orbit

A nilpotent orbit is a set $\mathcal{O}$ of nilpotent matrices such that for any A,B in $\mathcal{O}$, $A = gBg^{-1}$ for some g in G.

Example of a Nilpotent Orbit

Orbit Representative
$\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$
Other Matrices in the Orbit
$\begin{pmatrix} 0 & 1 & 0 \\ -2 & -4 & 2 \\ -4 & -7 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 4 & 1 \end{pmatrix}^{-1}$
$\begin{pmatrix} -5 & -7 & -2 \\ 3 & 4 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$

Partitions and Nilpotent Orbits

- A partition of n is a way of expressing n as a sum of positive integers, i.e. $[\lambda_1, \ldots \lambda_r]$ such that $\lambda_1 + \ldots + \lambda_r = n$.
    - Ex: a partition of 5: $[3, 2]$
- Nilpotent orbits are in bijective correspondence with partitions of n.
- Our choice of the nilpotent orbit representative is determined by having the semisimple element in dominance order.

Partitions and Nilpotent Orbits

$$\text{Partition} = [3, 1]$$

Semisimple Element	**Nilpotent Orbit Representative**
$\begin{pmatrix} 2 & & & \\ & 0 & & \\ & & -2 & \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}$
$\begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & & \\ & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{pmatrix}$

Result

Theorem

Let $G = \mathrm{GL}(n, \mathbb{F}_q)$. *Let* Γ_λ *be the GGGR corresponding to the partition* $\lambda = [\lambda_1^{k_1}, \ldots \lambda_r^{k_r}]$. *Then* Z_L *is a subgroup of* G *consisting of* λ_i *identical* $\mathrm{GL}(k_i)$ *blocks on the diagonal.*

Theorem in action

Partition	Nilpotent Orbit Representative	Z_L
$[3,2]$	$\begin{pmatrix} 0 & & 1 & & \\ & 0 & & 1 & \\ & & 0 & & 1 \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$	$\begin{pmatrix} v & & & & \\ & u & & & \\ & & v & & \\ & & & u & \\ & & & & v \end{pmatrix}$
$[2,1^3]$	$\begin{pmatrix} 0 & & & & 1 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$	$\left(\begin{array}{c\|c\|c} x & & \\ \hline & A & \\ \hline & & x \end{array}\right)$

Justification for the Theorem

Given $[2^2]$, arrange the nilpotent orbit representative in blocks corresponding to the partition and assign variables along the diagonal for each block.

$$\left(\begin{array}{cc|cc} 0 & 1 & & \\ & 0 & & \\ \hline & & 0 & 1 \\ & & & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} x & & & \\ & x & & \\ \hline & & y & \\ & & & y \end{array}\right)$$

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The nilpotent orbit representative of the partition determines the correct positions of the variables in Z_L. Repeating patterns of n variables indicate a $\mathrm{GL}(n)$ block in Z_L.

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$$\longrightarrow Z_L = \left(\begin{array}{c|c} A & \\ \hline & A \end{array}\right)$$

Further Research

The matrix coefficients that appear in integral representations of some L-functions come from groups other than $\mathrm{GL}(n)$. With this in mind, we have the following plans for future research:

- Find a formula for Z_L for other finite groups of Lie type, e.g. $\mathrm{SO}(2n+1), \mathrm{Sp}(2n), \mathrm{SO}(2n)$.
- For the groups above, find a formula for $U_{1.5}$ (such a formula is known for $\mathrm{GL}(n)$)

Acknowledgements

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