Linear Factorization of Hypercyclic Functions for Differential Operators

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NUMS 2018

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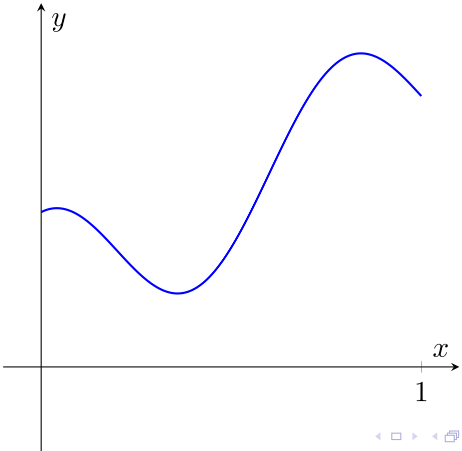

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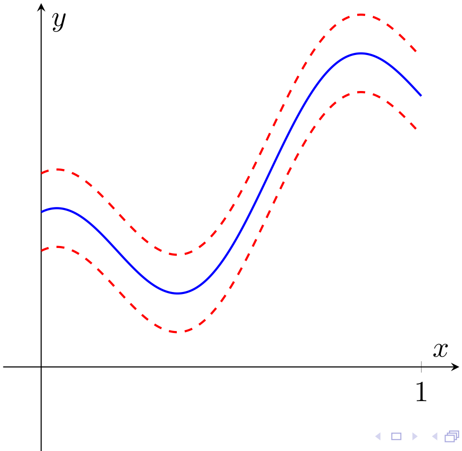

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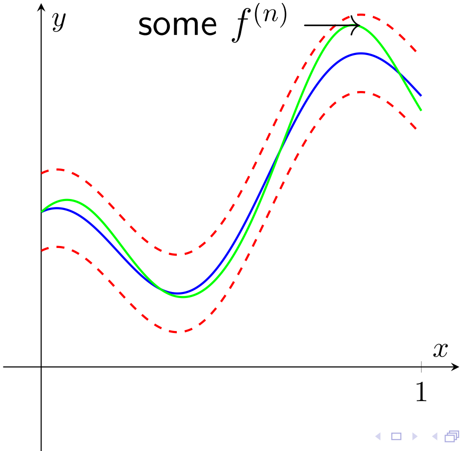

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-What about making infinite lists with a differential operator $T : \{Tf, T^2 f, T^3 f, \dots\}$

-If this list is dense, then f is a hypercyclic function for the operator T

Brief Sketch of MacLane's Theorem

Weierstrass Approximation Theorem

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-Define the integration operator A: $Af = \int\limits_0^x f(t)dt$.

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-Make f an infinite product of linear functions:

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right)$$

Finding Inverse of Operator T

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$$Tp = a_0 D^l \left(I - \frac{D}{r_1}\right)\left(I - \frac{D}{r_2}\right)\cdots\left(I - \frac{D}{r_{m-l}}\right)p$$

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-Note that

$$\left(I - \frac{D}{r}\right)\left(I + \frac{D}{r} + \frac{D^2}{r^2} + \cdots + \frac{D^m}{r^m}\right) = \left(I - \frac{D^{m+1}}{r^{m+1}}\right)$$

Finding Inverse of Operator T (continued)

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-If $l > 0$, then the integral factor takes over, and $S_n p \to 0$ as $n \to \infty$.

-If $l = 0$, $S_n p \not\to 0$.

Constructing f as a Product

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Find functions q_n such that, for any f_{k-1} and p_k as $n \to \infty$:

$$q_n \to 0 \tag{1}$$

$$T^n(f_{k-1}(1+q_n)) \to p_k \tag{2}$$

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$$T^n(f_{k-1}(q_n + 1)) = T^n(f_{k-1}q_n) + T^n(f_{k-1}) \approx p_k(z)$$

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Set q_n to equal quotient polynomial of $S_n p_k$ and f_{k-1}.

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$T = \psi(D)$ has a "factor" of $(I - D/r)$. By the chain rule, for any polynomial $g(z)$,

$$(I - D/r)(g * e^{rz}) = g * e^{rz} - \frac{1}{r} e^{rz} Dg - \frac{1}{r} r e^{rz} g = -\frac{1}{r} e^{rz} Dg$$

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$(q_n + 1)$: quotient polynomial of

$S_n p_k + (f_{k-1} - S_n p_k) \left(\sum_{i=0}^{n-m-1} \frac{(-rz)^i}{i!} \right) \left(\sum_{i=0}^{4n} \frac{(rz)^i}{i!} \right)$ and f_{k-1}

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When z is replaced with D, operator becomes the *translation* operator:

$$\psi(D) : f(z) \to \lambda f(z + a)$$

The result is in sight ... almost ...

-We have $f(z) = \prod_{j=1}^{\infty}(1 + q_j(z))$. Product is not of *linear* factors

-One may split each $(1 + q_j)$ into linear factors which each also tend to 1, *but the new product may not converge*.

-Could one *reorder* the roots of each $(1 + q_j)$ so that the product converges? Yes, but it's tricky

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-Our goal revisited: Make $\{Tf, T^2 f, T^3 f, \dots\}$ dense in the set of differentiable complex functions

End Slide

Thanks for watching! I would also like to thank the following:

-The St. Olaf CURI program for funding my research

-Dave Walmsley, my project advisor and collaborator