Edge-disjoint tree representation of three tree degree sequences

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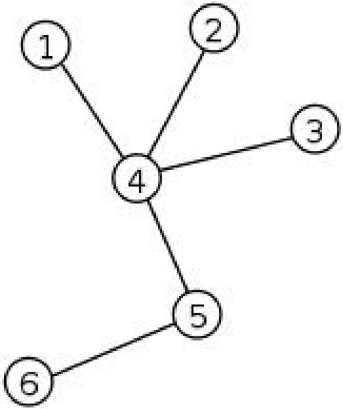
Trees

Tree



Tree

A tree is a connected graph with no cycles.
Note that

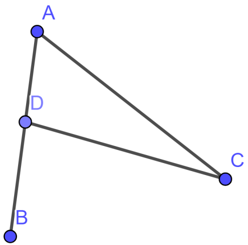
$$|E| = |V| - 1.$$

Not a tree

Degree Sequence

A sequence of degrees of vertices

Degree Sequence



$d = [2\ 1\ 2\ 3]$

Tree Degree Sequence

A degree sequence where the entries are positive integers, and the sum of the degrees is $2(n-1)$.

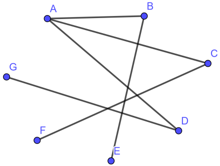
Tree Degree Matrix

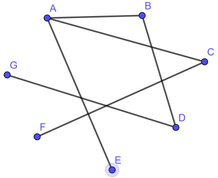
- A $c \times n$ matrix which represents the degrees of c tree degree sequences, each of which has n vertices
- In this project, we worked with $c = 3$.

Graph Realization

- The realization of the degree sequence is the graph which corresponds to the degree sequence
- Notice how the realization is not necessarily unique.
- We call a degree sequence graphical if there exists a corresponding graph realization.

$d = [3\ 2\ 2\ 2\ 1\ 1\ 1]$



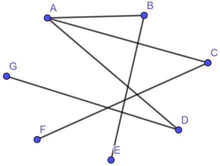


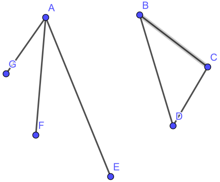
Tree Realization

A realization that is a tree

Since the sum of the vertices is twice the number of edges, the sum of the degrees should be $2(n-1)$.

$$d = [3\ 2\ 2\ 2\ 1\ 1\ 1]$$





Edge-disjoint Tree Realization

Cramming multiple trees so that two edges don't share exactly the same pair of vertices

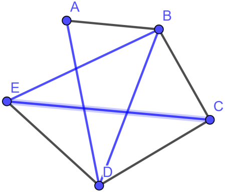


Figure: Edge-disjoint

Edge-disjoint Tree Realization

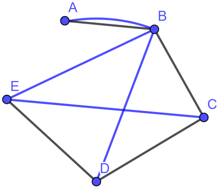


Figure: Not Edge-disjoint

Question

What is a sufficient condition for three tree degree sequences to have an edge-disjoint tree realization?

Result prior to our research

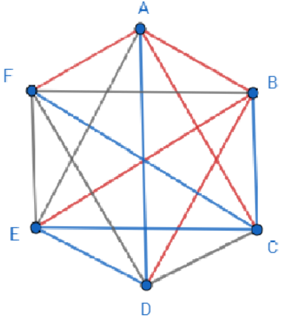
Theorem (Kundu, 1975)

For a tree degree matrix D which has a dimension $3 \times n$, if each of the columns has the sum of at least 5, and the sums of each pair of tree degree sequences and the sum of all three tree degree sequences are all graphical, then D has an edge-disjoint tree realization.

Example

Consider

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 3 & 2 & 2 & 1 \\ 3 & 3 & 1 & 1 & 1 & 1 \end{bmatrix}$$



Question

- Now that we know a sufficient condition of having an edge-disjoint tree realization for three trees, can we lower the lower bound?
- What if the minimum degree is 4, not 5?

Conjecture

Conjecture (Miklos)

For a tree degree matrix D which has a dimension $3 \times n$, if each of the columns has the sum of at least 4, and the sums of each pair of tree degree sequences and the sum of all three tree degree sequences are all graphical, then D has an edge-disjoint tree realization.

Approach

Mathematical Induction

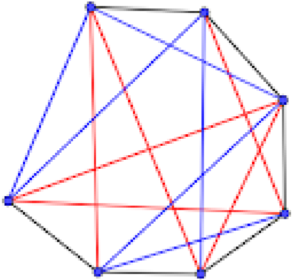
- Look at the column $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Remove it.
- Subtract 1 from a high degree vertex (degree≥ 2) in each of the first two rows (must be different vertices!).
- If we assume that the reduced tree degree matrix (with $n-1$ columns) has edge-disjoint tree realization, then the original tree degree matrix will have edge-disjoint tree realization.

Example

$$D = \begin{bmatrix} \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 1 & \mathbf{1} \\ 2 & \mathbf{2} & 2 & 2 & 2 & 1 & 2 & \mathbf{1} \\ 2 & 2 & 2 & 1 & 1 & 2 & 2 & \mathbf{2} \end{bmatrix}$$

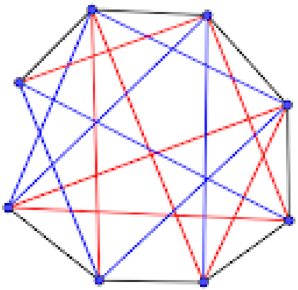
Example

$$D' = \begin{bmatrix} \mathbf{1} & 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & \mathbf{1} & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 & 2 & 2 \end{bmatrix}$$



Example

$$D = \begin{bmatrix} \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 1 & \mathbf{1} \\ 2 & \mathbf{2} & 2 & 2 & 2 & 1 & 2 & \mathbf{1} \\ 2 & 2 & 2 & 1 & 1 & 2 & 2 & \mathbf{2} \end{bmatrix}$$

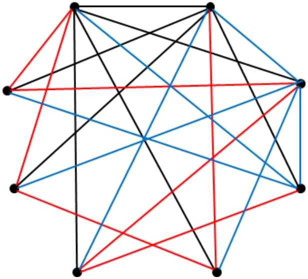


Approach

- So what are we going to do with this induction?
- Remove all the columns $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ so that we are left with vertices with degrees at least 5
- Then according to Kundu's Theorem, our conjecture is true.

$$\begin{bmatrix} 4 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 7 & 7 & 7 & 5 & 4 & 4 & 4 & 4 \end{bmatrix}$$



No!!!

Exceptional cases

- When the first column is $\begin{bmatrix} 1 \\ 1 \\ n-3 \end{bmatrix}$
- Some other cases (resolved by Yuhao Wan)

Why is this an exceptional case?

- Let j be the column that we are trying to remove: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.
- Since the sum of the first column is $n-1$, we need to subtract at least 1 from an entry in the first column.
- Since the first two entries of the first column are 1's, if we subtract 1 from one of those entries, we are left with D' with an entry 0, which is no longer a tree degree matrix.
- Oh no!!!

How do we resolve it?

- Do we have to use $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$?
- No!
- We have a lot of candidates: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.
- Our aim is to show that there are finitely many base cases, and other cases are derived by inductive steps starting from one of the base cases.

Let's do this - Case bashing

Cases

1. Three columns with the sum $n-1$
2. Two columns with the sum $n-1$
    1. No $[1;2;1]$ or $[2;1;1]$
    2. $[1;2;1]$ or $[2;1;1]$
        1. Two $n-3$
        2. One $n-3$
3. One column with the sum $n-1$
    1. $[1;2;1]$
    2. No $[1;2;1]$, but $[2;2;1]$
    3. No $[1;2;1]$ nor $[2;2;1]$ but $[3;1;1]$

Three columns with the sum $n-1$

Let D be a degree matrix with the first column $\begin{bmatrix} 1 \\ 1 \\ n-3 \end{bmatrix}$, and has a column of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Denote the i-th row j-th column entry by $d_{i,j}$, and denote the j-th column by d_j.

Case 1: D has three columns with the sum of $n-1$ each.

Proposition 1.1

If D has three columns with the sum of $n-1$ each, then D has at most 9 columns, so there are finite such cases

Three columns with the sum $n-1$

Proof:

- Minimum sum of each column is 4
- Note that the sum of all the entries in the matrix must be $6n-6$.

$$6n-6 \geq 3(n-1)+4(n-3)$$

- $n \leq 9$

Cases

1. Three columns with the sum $n-1$ ✓
2. Two columns with the sum $n-1$
   1. No $[1;2;1]$ or $[2;1;1]$
   2. $[1;2;1]$ or $[2;1;1]$
      1. Two $n-3$
      2. One $n-3$
3. One column with the sum $n-1$
   1. $[1;2;1]$
   2. No $[1;2;1]$, but $[2;2;1]$
   3. No $[1;2;1]$ nor $[2;2;1]$ but $[3;1;1]$

Two columns with the sum $n-1$

Case 2: D has exactly two columns with the sum of $n-1$ each.

We can divide this into two cases.

Case 2a: There are no columns of $\begin{bmatrix}1\\2\\1\end{bmatrix}$ nor $\begin{bmatrix}2\\1\\1\end{bmatrix}$.

Proposition 2.1

If D has exactly two columns with the sum of $n-1$ each, and there are no columns of $\begin{bmatrix}1\\2\\1\end{bmatrix}$ nor $\begin{bmatrix}2\\1\\1\end{bmatrix}$, then D has at most 8 columns.

Two columns with the sum $n-1$

Proof:

- Third tree must be $[n-3, 1, 1, \cdots, 1, 2, 2]$
- Each column with its third entry 1 must have a sum of at least 5.
- Exists constant C $(C \leq 8)$ such that $n \leq C$
- $n \leq 8$

Two columns with the sum $n - 1$

Case 2b: There exists a column of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

We have to divide this into two cases again...

Cases

1. Three columns with the sum $n-1$ ✓
2. Two columns with the sum $n-1$
   1. No $[1;2;1]$ or $[2;1;1]$ ✓
   2. $[1;2;1]$ or $[2;1;1]$
      1. Two $n-3$
      2. One $n-3$
3. One column with the sum $n-1$
   1. $[1;2;1]$
   2. No $[1;2;1]$, but $[2;2;1]$
   3. No $[1;2;1]$ nor $[2;2;1]$ but $[3;1;1]$

Two columns with the sum $n-1$

Case 2bi: There are two entries of $n-3$ in D.

Proposition 2.2

Without loss of generality, suppose $d_{3,1} = d_{2,2} = n-3$.

If $n \geq 11$, then there exists a column of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

We can use this column to reduce D to D'.

Two columns with the sum $n-1$

Proof:

- The last two rows must be $\begin{bmatrix} n-3 & 2 & 2 & 1 & 1 & \cdots & 1 \end{bmatrix}$, in some order.
- There can be at most 2 columns of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and at most 2 columns of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.
- If there is another column with the sum of 4, that column must be $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, so we're done.
- If not, each sum of other columns must be 5.

$$6n-6 \geq 2(n-1)+4\cdot 4+5(n-6)$$

- $n \leq 10$

Cases

1. Three columns with the sum $n-1$ ✓
2. Two columns with the sum $n-1$
    1. No $[1;2;1]$ or $[2;1;1]$ ✓
    2. $[1;2;1]$ or $[2;1;1]$
        1. Two $n-3$ ✓
        2. One $n-3$
3. One column with the sum $n-1$
    1. $[1;2;1]$
    2. No $[1;2;1]$, but $[2;2;1]$
    3. No $[1;2;1]$ nor $[2;2;1]$ but $[3;1;1]$

Two columns with the sum $n-1$

Case 2bii: D has at most one entry of $n-3$

Proposition 2.3

If D has at most 2 columns with the sum $n-1$, at most 1 entry of $n-3$, and there exists a column of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, then we can erase this column to reduce D to D'.

Two columns with the sum $n-1$

Proof:

- WLOG $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ exists (j-th column).
- $d_{3,j}$ connects to $d_{3,1}$.
- Suppose $d_{1,j}$ connects to $d_{1,k} \geq 2$ ($k \neq 1$).
- Consider $D' = D \setminus \{d_j\}$.
- Assume there is no edge among $D'_2 \setminus \{d_1, d_k\}$.
- Since $d_{2,1} = 1$, all the other vertices in D'_2 must connect to d_k, so $d_{2,k} = n-3$. Contradiction!

Cases

1. Three columns with the sum $n-1$ ✓
2. Two columns with the sum $n-1$ ✓
    1. No $[1;2;1]$ or $[2;1;1]$ ✓
    2. $[1;2;1]$ or $[2;1;1]$ ✓
        1. Two $n-3$ ✓
        2. One $n-3$ ✓
3. One column with the sum $n-1$
    1. $[1;2;1]$
    2. No $[1;2;1]$, but $[2;2;1]$
    3. No $[1;2;1]$ nor $[2;2;1]$ but $[3;1;1]$

One column with the sum $n-1$

Case 3: D has exactly one column with the sum $n-1$.

Proposition 3.1

In D, there exists a column of $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (up to symmetry of D_1 and D_2).

One column with the sum $n-1$

Proof:

- Consider columns with $d_{3,j} = 1$.
- Suppose each of these columns has a sum of at least 6.

$$6n - 6 \geq (n-1) + 6(n-3) + 2 \cdot 4$$

- $n \leq 5$
- Contradiction, as there are no tree degree matrices with less than 5 columns.
- There exists a column with $d_{3,j} = 1$ and the column sum 4 or 5.

One column with the sum $n - 1$

Case 3a: D has a column of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (up to symmetry).

According to Proposition 2.3, we can erase this column to reduce D to D'.

Cases

1. Three columns with the sum $n - 1$ ✓
2. Two columns with the sum $n - 1$ ✓
   1. No $[1;2;1]$ or $[2;1;1]$ ✓
   2. $[1;2;1]$ or $[2;1;1]$ ✓
      1. Two $n - 3$ ✓
      2. One $n - 3$ ✓
3. One column with the sum $n - 1$
   1. $[1;2;1]$ ✓
   2. No $[1;2;1]$, but $[2;2;1]$
   3. No $[1;2;1]$ nor $[2;2;1]$ but $[3;1;1]$

One column with the sum $n - 1$

Case 3b: D has no columns of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, but has a column of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Proposition 3.2

If D has no columns of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, but has a column of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, we can either erase this column to reduce D to D', or there are finitely many cases where such reduction is impossible.

One column with the sum $n-1$

Proof:

- d_j is the column of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.
- $d_{3,j} = 1$ connected to $d_{3,1} = n-3$
- Second row looks like $\begin{bmatrix} 1 & d_{2,2} & d_{2,3} & \cdots & d_{2,j-1} & 2 & d_{2,j+1} & \cdots & d_{2,n-1} & 1 \end{bmatrix}$.
- Suppose there are no edges among $\{d_{2,2}, d_{2,3}, \cdots, d_{2,j-1}, d_{2,j+1}, \cdots, d_{2,n-1}\}$.
- All of these connected to $d_{2,n} = 1$. Contradiction!
- There exists an edge among $\{d_{2,2}, d_{2,3}, \cdots, d_{2,j-1}, d_{2,j+1}, \cdots, d_{2,n-1}\}$.

One column with the sum $n - 1$

- Suppose this edge connects $d_{2,a}$ and $d_{2,b}$.
- Since it cannot be the case that $d_{2,a} = d_{2,b} = 1$, $d_{2,a} + d_{2,b} \geq 3$.
- Suppose there are no edges among $D_1 \setminus \{d_{1,1}, d_{1,a}, d_{1,b}, d_{1,j}\}$.
- All these vertices must be connected to $d_{1,a}$ or $d_{1,b}$.
- There exists one vertex which is connected to both.
- $d_{1,a} + d_{1,b} \geq n - 3$
- Depending on $d_{3,a}$ and $d_{3,b}$, the inequality changes, or we hit a contradiction, but for all cases that don't hit a contradiction, there exists a constant C (which turns out to be $C \leq 10$) such that $n \leq C$.
- Hence, we can reduce D to D', or there are finitely many cases which cannot be reduced.

Cases

1. Three columns with the sum $n-1$ ✓
2. Two columns with the sum $n-1$ ✓
   1. No $[1;2;1]$ or $[2;1;1]$ ✓
   2. $[1;2;1]$ or $[2;1;1]$ ✓
      1. Two $n-3$ ✓
      2. One $n-3$ ✓
3. One column with the sum $n-1$
   1. $[1;2;1]$ ✓
   2. No $[1;2;1]$, but $[2;2;1]$ ✓
   3. No $[1;2;1]$ nor $[2;2;1]$ but $[3;1;1]$

One column with the sum $n-1$

Case 3c: D does not have any $\begin{bmatrix}1\\2\\1\end{bmatrix}$ or $\begin{bmatrix}2\\2\\1\end{bmatrix}$, but has a $\begin{bmatrix}3\\1\\1\end{bmatrix}$ (up to symmetry of the first two rows).

Proposition 3.2

Suppose $n \geq 11$ and D doesn't have any column of $\begin{bmatrix}1\\2\\1\end{bmatrix}$ (equivalently $\begin{bmatrix}2\\1\\1\end{bmatrix}$) nor $\begin{bmatrix}2\\2\\1\end{bmatrix}$. Then, D must have at least two columns of $\begin{bmatrix}3\\1\\1\end{bmatrix}$ and two columns of $\begin{bmatrix}1\\3\\1\end{bmatrix}$.

One column with the sum $n-1$

Proof:

- Assume there is at most one column of $\begin{bmatrix}1\\3\\1\end{bmatrix}$.
- At most 2 columns with the sum 4
- At most 5 columns which don't have the column sum 5. (look at sum degree sequence)
- At least $n-5$ columns with the column sum 5 and the third entry 1.
- Since there are no $\begin{bmatrix}2\\2\\1\end{bmatrix}$, those columns are $\begin{bmatrix}1\\3\\1\end{bmatrix}$ or $\begin{bmatrix}3\\1\\1\end{bmatrix}$.
- At most one $\begin{bmatrix}1\\3\\1\end{bmatrix}$ and the rest are $\begin{bmatrix}3\\1\\1\end{bmatrix}$.

One column with the sum $n - 1$

$$2n - 2 \geq (n - 6) \cdot 3 + 6 \cdot 1$$

- $n \leq 10$
- Contradiction!
- Use the same method for the second row.

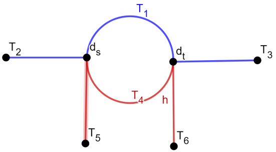
One column with the sum $n - 1$

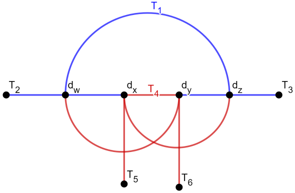
Proposition 3.3

If D has at least two columns of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and at least two columns of $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, we can reduce D to D' by changing the four columns above into two columns of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

One column with the sum $n-1$

Let d_s, d_t be the columns $\begin{bmatrix}2\\2\\1\end{bmatrix}$, d_w, d_z be the columns $\begin{bmatrix}3\\1\\1\end{bmatrix}$, and d_x, d_y be the columns $\begin{bmatrix}1\\3\\1\end{bmatrix}$.





Cases

1. Three columns with the sum $n-1$ ✓
2. Two columns with the sum $n-1$ ✓
   1. No $[1;2;1]$ or $[2;1;1]$ ✓
   2. $[1;2;1]$ or $[2;1;1]$ ✓
      1. Two $n-3$ ✓
      2. One $n-3$ ✓
3. One column with the sum $n-1$ ✓
   1. $[1;2;1]$ ✓
   2. No $[1;2;1]$, but $[2;2;1]$ ✓
   3. No $[1;2;1]$ nor $[2;2;1]$ but $[3;1;1]$ ✓

Base Cases

So how many base cases are there?

Base Cases

100000!

Computerized Proof

- We cannot check 100000 base cases by hand.
- Supervisor (Dr Miklos) wrote a code to check if there were any base cases with no edge-disjoint tree realizations.
- The computer said no.

Theorem

Theorem (Miklos, Seong, Wan; 2017)

For a tree degree matrix D which has a dimension $3 \times n$, if each of the columns has the sum of at least 4, and the sums of each pair of tree degree sequences and the sum of all three tree degree sequences are all graphical, then D has an edge-disjoint tree realization.

Next step

- Find a non-computerized proof.
- Find necessary and sufficient condition for edge-disjoint tree realizations of three tree degree sequences to exist (minimum degree 3 doesn't work, as there are counterexamples)
- Necessary and sufficient condition for edge-disjoint tree realization of n tree degree sequences to exist (the problem gets exponentially hard as n increases)

Many thanks to...

- Supervisor Dr. Istvan Miklos (in BSM)
- Yuhao Wan, my research partner
- Carleton College Math Department for giving me an opportunity to present a talk on my research
- KFC chicken which gave me an epiphany on how to resolve the exception cases

Thank you!

Questions?