

# Weighted Tilings and $q$ -Fibonacci Numbers

Eric Lindgren, Mai Anh Ngo, and Stephen Segroves

Department of Mathematics and Computer Science

Carleton College, Minnesota, USA

`lindgree@carleton.edu`, `ngo@carleton.edu`, `segroves@carleton.edu`

April 3, 2007

## Abstract

We give a combinatorial interpretation of the Fibonacci numbers, then use this interpretation as a tool for finding combinatorial proofs of  $q$ -analogs of well-known Fibonacci identities. Thought is also given to different generalizations of the Fibonacci numbers as well as to our combinatorial interpretation.

## 1 Introduction

The Fibonacci numbers have been rigorously studied and numerous identities involving them exist. We seek to obtain a more generalized version of these Fibonacci identities. To do so, we interpret the  $n^{\text{th}}$  Fibonacci number  $f_n$  combinatorially as the number of tilings of a  $1 \times n$ -board, referred to as an  $n$ -board, with squares and dominoes. We then assign a weight  $q^i$  to each domino in a tiling based on its position and multiply together all the weights of the dominoes to obtain the total weight of the tiling. By adding together all possible weighted tilings of an  $n$ -board, we obtain a polynomial in  $q$ , which we call the  $q$ -version of  $f_n$ . Under this interpretation, we obtain polynomial analogues of well-known Fibonacci identities. These are analogues because if we take the limit as  $q$  goes to 1 of one of our  $q$ -versions, we obtain the original identity. Using similar methods, we also find  $q$ -versions of some Lucas identities and examine the weighted tilings of more generalized boards with generalized tiles.

## 2 $q$ -Fibonacci Numbers

**Definition**

$$f_{n+1} = f_n + f_{n-1} \qquad f_0 = f_1 = 1. \qquad (1)$$

One way to think about  $f_n$  in terms other than the standard recurrence is to think of  $f_n$  as the number of different ways of tiling an  $n$ -board with squares and dominoes.

We can see that there is one way to tile a 0-board and one way to tile a 1-board, so this idea satisfies the initial Fibonacci conditions. Also, we can see that this will satisfy the recurrence, because an  $n + 1$  board can either end in a domino or a square. If it ends with a domino, there are  $f_{n-1}$  ways to tile the board and if it ends with a square, there are  $f_n$  ways to tile the board.

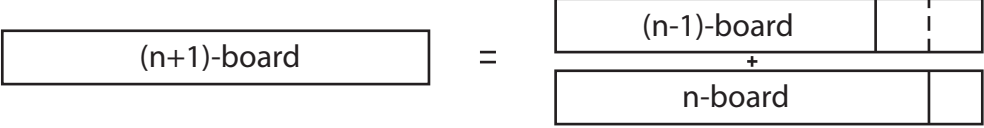


Figure 1: The Fibonacci  $n$ -board recurrence.

We are taking this approach a step further by giving each tiling a weight based on the number of dominoes and their positions in the tiling. A particular tiling's weight is the product of all  $q^i$  such that the tiling has a domino at position  $(i, i + 1)$ , or simply  $i$ .

**Example**

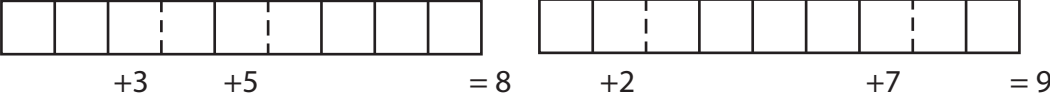


Figure 2: These tilings of a 9-board have weights  $q^8$  and  $q^9$ , respectively.

From this, for any particular  $n$ , we can construct a  $q$ -series in which the coefficient of  $q^m$  is equal to the number of tilings of an  $n$ -board with weight  $m$ . Combining this idea with the normal Fibonacci recurrence, we get the standard  $q$ -analogue:

**Definition**

$$f_{n+1}(q) = f_n(q) + q^{n-1}f_{n-1}(q) \qquad f_1(q) = f_2(q) = 1 \qquad (2)$$

where  $f_{n+1}(q)$  is the power series of the weighted tilings for an  $n$ -board. So using the recurrence, we see that

$$f_3(q) = f_2(q) + qf_1(q) = 1 + q \qquad (3)$$

$$f_4(q) = f_3(q) + q^2f_2(q) = 1 + q + q^2 \qquad (4)$$

$$f_5(q) = f_4(q) + q^3f_3(q) = 1 + q + q^2 + q^3(1 + q) = 1 + q + q^2 + q^3 + q^4 \qquad (5)$$

**Example**

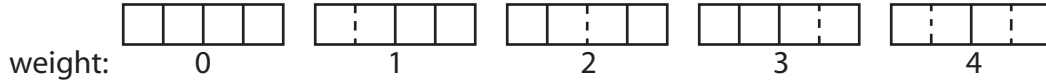


Figure 3: Each tiling contributes a  $q^t$  to  $f_{n+1}(q)$ , where  $t$  is the weight of the tiling. The sum of these powers of  $q$  gives us the corresponding  $q$ -Fibonacci series.

When proving these  $q$ -versions combinatorially, it is also helpful to define shifted  $q$ -Fibonacci numbers:

**Definition**

$$f_{n+1}^{(a)}(q) = f_n^{(a)}(q) + q^{n-1+a} f_{n-1}^{(a)}(q) \qquad f_1^{(a)}(q) = f_2^{(a)}(q) = 1 \qquad (6)$$

These are useful when dividing a board into two separate parts. We say a tiling has a *fault at position  $m$*  if there is no domino at position  $(m, m + 1)$ . So if we have an  $n$ -board with a fault at  $m$ , and thus split into two parts of length  $m$  and  $n - m$ , we can count the weighted tilings of the  $n$ -board as  $f_m(q)f_{n-m}^{(m)}(q)$ .

**Example**

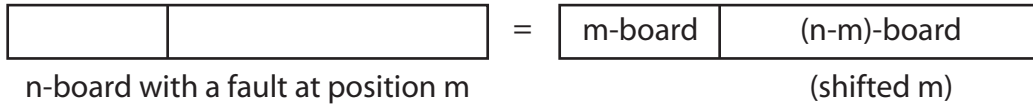


Figure 4: Here, the tilings are counted by  $f_m(q)f_{n-m}^{(m)}(q)$ .

One more important tool for proving these  $q$ -Fibonacci identities is the  $q$ -binomial coefficient:

**Definition**

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1 - q)(1 - q^2) \cdots (1 - q^n)}{(1 - q)(1 - q^2) \cdots (1 - q^k)(1 - q)(1 - q^2) \cdots (1 - q^{n-k})}. \qquad (7)$$

It is known that  $q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q$  counts the number of weighted tilings having exactly  $n$  tiles,  $k$  of which are dominoes [5]. We present a different proof, based on a bijection between tilings of a board with  $n$  tiles,  $k$  of which are dominoes and integer partitions with less than or equal to  $k$  parts, each of which is less than or equal to  $n - k$ . We say an integer partition  $\lambda$  of  $m$  is a sequence of weakly decreasing integers  $\lambda_1, \lambda_2, \dots, \lambda_r$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_r = m$ .

*Proof:*

It is shown in [1] that  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  is a polynomial in  $q$  for which the coefficient of  $q^i$  is the number of partitions  $\lambda = \lambda_1, \lambda_2, \dots, \lambda_k$  of  $i$  for which  $0 \leq \lambda_j \leq (n-k)$ . Our bijection is:

For each  $1 \leq i \leq k$ , the position of the  $i^{\text{th}}$  domino is determined by the  $\lambda_{k-i+1}$ ; there are exactly  $\lambda_{k-i+1}$  squares and  $i-1$  dominoes before the  $i^{\text{th}}$  domino, and we construct our board this way. Clearly we can do this backwards, as the value of  $\lambda_i$  will simply be the number of squares that precede the  $(k-i+1)^{\text{th}}$  domino.  $\square$

### Example

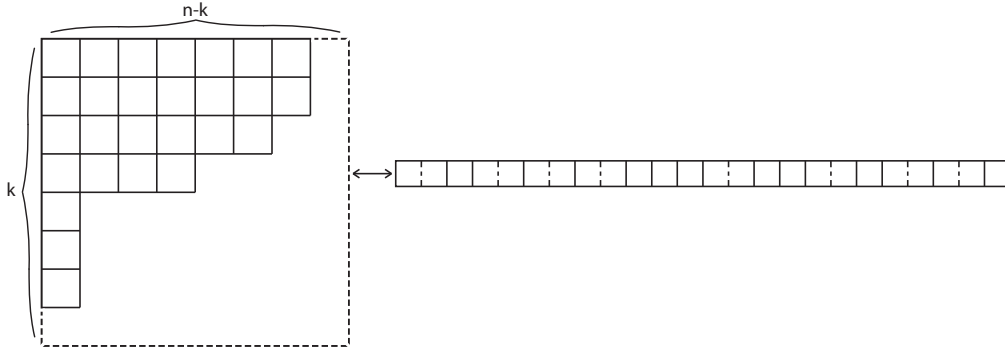


Figure 5: An example of the bijection, with  $n = 16$  and  $k = 8$ , creating a 24-board with 8 dominoes.

## 3 $q$ -Fibonacci Identities

Using the weighted tiling approach, we have been working on proving Fibonacci number identities from [2]. These  $q$ -analogs are generalized versions of the original identities because when we take the limit as  $q$  goes to 1, we obtain the original Fibonacci identities.

We now proceed to prove a number of  $q$ -analogs of Fibonacci identities.

**Fibonacci Identity:** For  $n \geq 0$ ,  $f_{2n+2} = f_n^2 + f_{n+1}^2$ . Our  $q$ -version is:

**Theorem 3.1.** For  $n \geq 0$ ,

$$f_{2n+3}(q) = q^{n+1} f_{n+1}(q) f_{n+1}^{(n+2)}(q) + f_{n+2}(q) f_{n+2}^{(n+1)}(q) \quad (8)$$

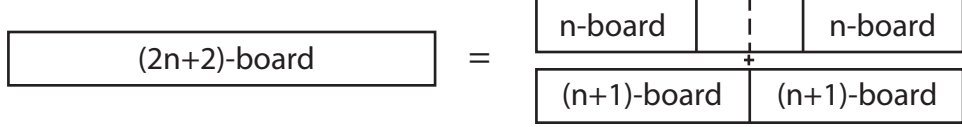


Figure 6: We can count the tilings of a  $(2n + 2)$ -board with two disjoint cases.

*Proof:*

**Left Hand Side:**  $f_{2n+3}(q)$  counts the number of weighted tilings of a  $(2n + 2)$ -board.

**Right Hand Side:** Consider position  $n + 1$  on a  $(2n + 2)$ -board. If there is a domino at  $n + 1$ , the domino contributes a  $q^{n+1}$  to the weight and we have  $f_{n+1}(q)$  weighted tilings to the left of the domino and  $f_{n+1}(q)$  to the right, but we need to shift the  $f_{n+1}(q)$  by  $n + 2$  to count them correctly. If there is not a domino covering positions  $n + 1$  and  $n + 2$ , then the  $(2n + 2)$ -board can be split into two  $(n + 1)$ -boards, counted by  $f_{n+2}(q)f_{n+2}^{(n+1)}(q)$ .  $\square$

**Fibonacci Identity:** For  $n \geq 1$ ,  $f_n^2 = f_{n+1}f_{n-1} + (-1)^n$ . Our  $q$ -version is:

**Theorem 3.2.** For  $n \geq 1$ ,

$$f_{n+1}(q)f_{n+1}^{(1)}(q) = f_{n+2}(q)f_n^{(1)}(q) + (-1)^n q^{\binom{n+1}{2}} \quad (9)$$

*Proof:*

**Left Hand Side:**  $f_{n+1}(q)f_{n+1}^{(1)}(q)$  counts the weighted tilings of two  $n$ -boards, one of which is shifted by 1.

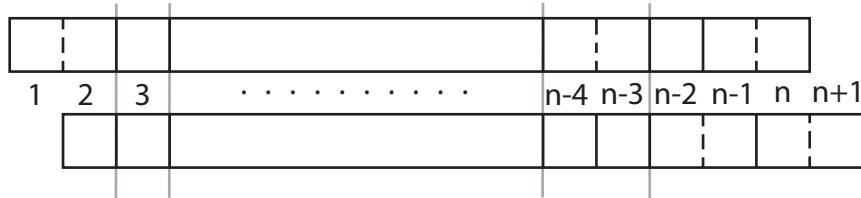


Figure 7: Two  $n$ -boards, one of which is shifted by 1.

**Right Hand Side:** Suppose we have two  $n$ -boards, one of which is shifted by 1. If we define the *tail* of a tiling to be the tiles after a fault, we can simply swap the tails of the two tilings with respect to their last common fault (if it exists).

When we do this, we now have an  $(n + 1)$ -board and an  $(n - 1)$ -board shifted by 1. This almost gives us a 1-1 correspondence. However, when  $n$  is even and the two  $n$ -boards we start with are both all dominoes (fig. 3), we can see that these tilings have no common faults, and clearly this is the only time this occurs. When  $n$  is odd, we cannot reverse the tail swapping for the case where both the  $(n + 1)$ -board and the  $(n - 1)$ -board are all dominoes (fig. 3) because again, the tilings will have no common faults. For these two cases, the combined weights of the tilings are  $q \cdot q^2 \cdot q^3 \cdots q^n = q^{1+2+\cdots+n} = q^{\binom{n+1}{2}}$ . So when

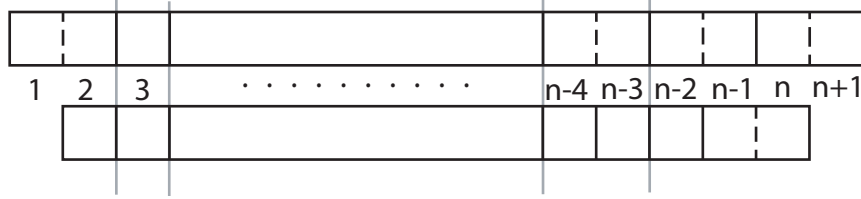


Figure 8: After the tail swap, we have two new tilings.

$n$  is even, we add  $q^{\binom{n+1}{2}}$  because it is counted by the left hand side but it is not counted by  $f_{n+2}(q)f_n^{(1)}(q)$ . Also, when  $n$  is odd, the  $q^{\binom{n+1}{2}}$  tiling *is* counted by  $f_{n+2}(q)f_n^{(1)}(q)$ , but it is not counted by the left hand side, so we subtract  $q^{\binom{n+1}{2}}$  from the right hand side.  $\square$

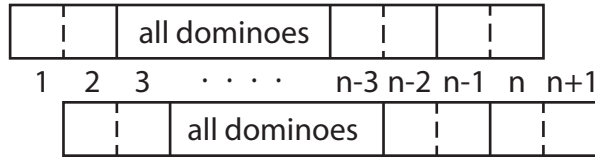


Figure 9: When  $n$  is even, this is the only case where we can not swap tails.

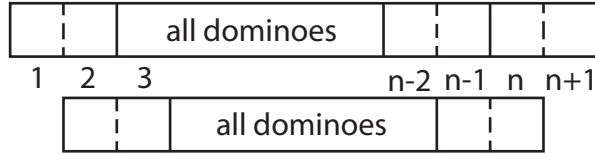


Figure 10: This is the only arrangement (when  $n$  is odd) for which we can not reverse the tail swapping.

**Fibonacci Identity:** For  $n \geq 1$ ,  $f_{n+2} + f_{n-2} = 3f_n$ . Our  $q$ -version is:

**Theorem 3.3.** For  $n \geq 1$ ,

$$f_{n+3}(q) + q^{2n-1}f_{n-1}(q) = (1 + q^n + q^{n+1})f_{n+1}(q) \quad (10)$$

*Proof:*

**Left Hand Side:**  $f_{n+3}(q) + q^{2n-1}f_{n-1}(q)$  counts the weighted tilings of an  $(n+2)$ -board and a  $2n$ -board with all squares from  $n-1$  to  $2n-2$  and a domino at  $2n-1$ .

**Right Hand Side:**

**Case 1:** An  $(n+2)$ -board is breakable at  $n$ . Then it contributes a weighted tiling of  $(1 + q^{n+1})f_{n+1}(q)$ .

**Case 2:** An  $(n + 2)$ -board is not breakable at  $n$  and contributes a weighted tiling of  $q^n f_n(q)$ . Using the  $q$ -Fibonacci recurrence,  $q^{2n-1} f_{n-1}(q)$  and the weights from case 2 equal  $q^n f_{n+1}(q)$ .

Therefore cases 1 and 2 yield  $(1 + q^n + q^{n+1})f_{n+1}(q)$ . □

**Fibonacci Identity:** For  $n \geq 1$ ,  $f_1 + f_3 + \dots + f_{2n-1} = f_{2n} - 1$ . Our  $q$ -version is:

**Theorem 3.4.** For  $n \geq 1$ ,

$$f_{2n+1}(q) - q^{n^2} = \sum_{i=1}^n q^{n^2-i^2} f_{2i}(q) \quad (11)$$

*Proof:*

**Left Hand Side:**  $f_{2n+1}(q) - q^{n^2}$  counts the weighted tilings of a  $2n$ -board that do not consist entirely of dominoes.

**Right Hand Side:** Let  $1 \leq 2i \leq 2n$  be the position of the last square that occupies an even-numbered cell. This board has dominoes at positions  $i + 2, i + 4, \dots, 2n - 1$ , which contribute an overall weight of  $n^2 - i^2$ . The remainder of the board contributes a weight of  $f_{2i}(q)$ . Therefore summing over all  $1 \leq i \leq n$ , we obtain all tilings of a  $2n$ -board except the all-domino tiling. □

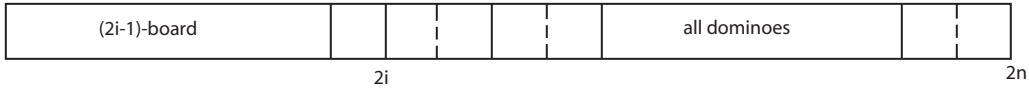


Figure 11: How we break up a  $2n$ -board in Theorem 3.4

**Fibonacci Identity:** For  $n \geq 0$ ,  $f_0 + f_3 + f_6 + \dots + f_{3n} = \frac{1}{2} f_{3n+2}$ . Our  $q$ -version is:

**Theorem 3.5.** For  $n \geq 0$ ,

$$f_{3(n+1)}(q) = \sum_{i=0}^n q^{3\binom{n+1}{2} - \binom{i+1}{2}} (1 + q^{3i+1}) f_{3i+1}(q) \quad (12)$$

*Proof:*

**Left Hand Side:**  $f_{3(n+1)}(q)$  counts the weighted tilings of a  $(3n + 2)$ -board.

**Right Hand Side:** Let  $3(i + 1) \leq 3n$  be the least number at which a  $(3n + 2)$ -board is not breakable for every  $m$  such that  $3 \mid m$  and  $3(i + 1) \leq m \leq 3n$ . Note that it must be breakable at  $3i$  by the minimality of  $3(i + 1)$ , which implies that it has a 2-board centered at  $3i + 1$  that contributes a weight of  $1 + q^{3i+1}$ . The board has dominoes at positions  $3(i + 1), 3(i + 2), \dots, 3n$ , which contribute a weight of  $q^{3\binom{n+1}{2} - \binom{i+1}{2}}$ . Finally,

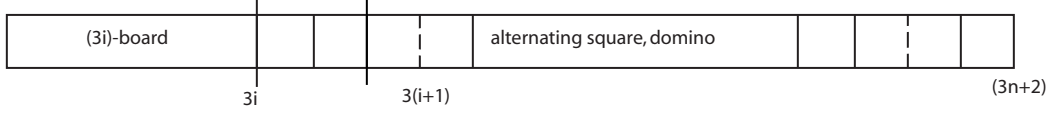


Figure 12: How we break up a  $(3n + 3)$ -board in Theorem 3.5

there is a  $3i$ -board to the left of  $3i$ . Summing over all  $0 \leq i \leq n$ , we obtain all tilings of a  $(3n + 2)$ -board.  $\square$

**Fibonacci Identity:** For  $n \geq 1$ ,  $f_1 + f_4 + f_7 + \dots + f_{3(n-1)+1} = \frac{1}{2}(f_{3n} - 1)$ . Our  $q$ -version is:

**Theorem 3.6.** For  $n \geq 1$ ,

$$f_{3n+1}(q) - q^{\frac{3n^2-n}{2}} = \sum_{i=0}^{n-1} q^{\frac{3(n^2-i^2)-5i-n-2}{2}} (1 + q^{3i+2}) f_{3i+2}(q) \quad (13)$$

*Proof:*

**Left Hand Side:**  $f_{3n+1}(q) - q^{\frac{3n^2-n}{2}}$  counts the weighted tilings of a  $3n$ -board except for the  $3n$ -board with dominoes at  $3(n-1)+1, 3(n-2)+1, \dots, 1$ .

**Right Hand Side:** Let  $3(i+1)+1 \leq 3n$  be the least number at which a  $3n$ -board is not breakable for every  $m$  such that  $3 \mid (m-1)$  and  $3(i+1)+1 \leq m \leq 3(n-1)+1$ . Note that the board must be breakable at  $3i+1$  by the minimality of  $3(i+1)+1$ , which implies that it has a 2-board centered at  $3i+2$  that contributes a weight of  $1 + q^{3i+2}$ . It also has dominoes at positions  $3(i+1)+1, 3(i+2)+1, \dots, 3(n-1)+1$ , which contribute a weight of  $q^{\frac{3(n^2-i^2)-5i-n-2}{2}}$ . Finally, there is a  $3i+1$ -board to the left of  $3i+1$ . Note that we do not count the  $3n$ -board with dominoes at every position  $k$  such that  $k \equiv 1 \pmod{3}$  when we sum over all  $0 \leq i \leq n-1$ , which means that we obtain all tilings of a  $3n$ -board except for the  $3n$ -board with dominoes at  $3(n-1)+1, 3(n-2)+1, \dots, 1$ .  $\square$

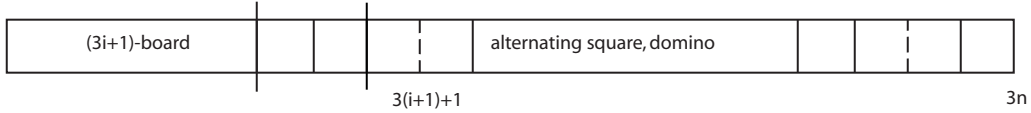


Figure 13: How we break up a  $3n$ -board in Theorem 3.6

**Fibonacci Identity:** For  $n \geq 1$ ,  $f_2 + f_5 + f_8 + \dots + f_{3n-1} = \frac{1}{2}(f_{3n+1} - 1)$ . Our  $q$ -version is:

**Theorem 3.7.** For  $n \geq 1$ ,

$$f_{3n+2}(q) - q^{\frac{3n^2+n}{2}} = \sum_{i=1}^n q^{3\binom{n+1}{2} - \binom{i+1}{2} - (n-i)} (1 + q^{3i}) f_{3i}(q) \quad (14)$$

*Proof:*

**Left Hand Side:**  $f_{3n+2}(q) - q^{\frac{3n^2+n}{2}}$  counts the weighted tilings of a  $(3n + 1)$ -board except for the  $(3n + 1)$ -board with dominoes at  $3n - 1, 3(n - 1) - 1, \dots, 2$ .

**Right Hand Side:** Let  $3(i + 1) - 1 \leq 3n$  be the least number at which a  $(3n + 1)$ -board is not breakable for every  $m$  such that  $3 \mid (m + 1)$  and  $3(i + 1) - 1 \leq m \leq 3n - 1$ . Note that the board must be breakable at  $3i - 1$  by the minimality of  $3(i + 1) - 1$ , which implies that it has a 2-board centered at  $3i$  that contributes a weight of  $1 + q^{3i}$ . It also has dominoes at positions  $3(i + 1) - 1, 3(i + 2) - 1, \dots, 3n - 1$ , which contribute a weight of  $q^{3\binom{n+1}{2} - \binom{i+1}{2} - (n-i)}$ . Finally, there is a  $(3i - 1)$ -board to the left of  $3i - 1$ . Note that we do not count the  $(3n + 1)$ -board with dominoes at every position  $k$  such that  $k \equiv -1 \pmod{3}$  when we sum over all  $1 \leq i \leq n$ , which means that we obtain all tilings of a  $(3n+1)$ -board except for the  $(3n+1)$ -board with dominoes at  $3n-1, 3(n-1)-1, \dots, 2$ .  $\square$

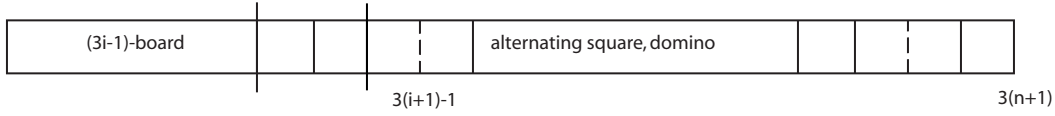


Figure 14: How we break up a  $(3n + 1)$ -board in Theorem 3.7

**Fibonacci Identity:** For  $n \geq 1$ ,  $f_n^2 - f_{n-2}^2 = f_{2n+1}$ . Our  $q$ -version is:

**Theorem 3.8.** For  $n \geq 1$ ,

$$f_{n+1}(q)f_{n+1}^{(n-1)}(q) - q^{2n-1}f_{n-1}(q)f_{n-1}^{(n+1)}(q) = f_{2n}(q) \quad (15)$$

The above identity is equivalent to:

$$f_{n+1}(q)f_{n+1}^{(n-1)}(q) = f_{2n}(q) + q^{2n-1}f_{n-1}(q)f_{n-1}^{(n+1)}(q) \quad (16)$$

*Proof:*

**Left Hand Side:** Staggering two  $n$ -boards so that the first tile of the second  $n$ -board is shifted by  $(n - 1)$ .

**Right Hand Side:** Tail-swap at the last common fault if this common fault exists.

**Case 1:** If the last tile of the top  $n$ -board and the first tile of the bottom  $n$ -board are squares, we have all possible ways of tiling a  $(2n - 1)$ -board with a square at position  $n$ .

**Case 2:** If the last tile of the top  $n$ -board is a square and the first tile of the bottom  $n$ -board is a domino, we get all possible ways of tiling a  $(2n - 1)$ -board with a domino at position  $(n, n+1)$ .

**Case 3:** If the last tile of the top  $n$ -board is a domino and the first tile of the bottom  $n$ -board is a square, we get all possible ways of tiling a  $(2n - 1)$ -board with a domino at position  $(n-1, n)$ .

These 3 cases together give all possible ways of tiling a  $(2n - 1)$ -board:  $f_{2n}(q)$ .

**Case 4:** If the last tile of the top  $n$ -board and the first tile of the bottom  $n$ -board are both dominoes, we can not do the tail-swapping. The total number of ways of tiling for this case is:  $q^{2n-1} f_{n-1}(q) f_{n-1}^{(n+1)}(q)$ .

The four cases together give us:  $f_{2n}(q) + q^{2n-1} f_{n-1}(q) f_{n-1}^{(n+1)}(q)$ . □

**Fibonacci Identity:** For  $n \geq 2$ ,  $f_n^4 = f_{n+2} f_{n+1} f_{n-1} f_{n-2} + (-1)^n$ . Our  $q$ -version is:

**Theorem 3.9.** For  $n \geq 2$ ,

$$[f_{n+1}(q)]^2 f_{n+1}^{(1)}(q) f_{n+1}^{(2)}(q) = f_{n+3}(q) f_{n+2}(q) f_n^{(1)}(q) f_{n-1}^{(2)}(q) + (-1)^n q^{\binom{n+1}{2}} (f_{n+1}(q) f_{n+1}^{(2)}(q) - q^{-1} f_{n+2}(q) f_n^{(1)}(q)) \quad (17)$$

*Proof:*

**Left Hand Side:**  $[f_{n+1}(q)]^2 f_{n+1}^{(1)}(q) f_{n+1}^{(2)}(q)$  counts the weighted tilings of four  $n$ -boards, one of which is shifted by one and one of which is shifted by two.

**Right Hand Side:** Consider four  $n$ -boards arranged as the boards on the right in Figure 15. Now we find the last common fault on both the right and left pairs, and swap the tails;  $f_{n+3}(q) f_{n+2}(q) f_n^{(1)}(q) f_{n-1}^{(2)}(q)$  counts the number of tilings after a tail swap except in a few cases.

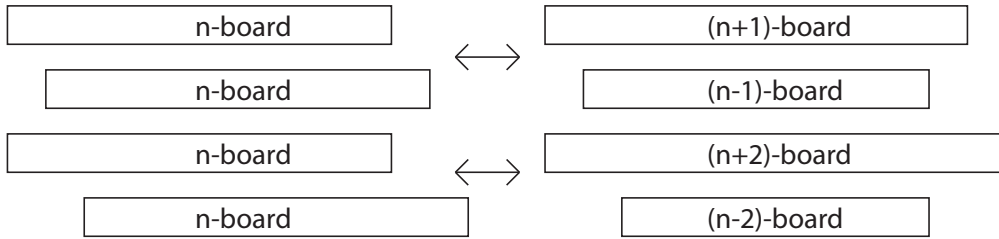


Figure 15: The four  $n$ -boards before and after tail-swapping.

**Case 1:** If  $n$  is even, we will be able to swap tails on both pairs of boards except in the case when both boards on the top left hand side are all dominoes. In this case the weight of the tilings on the left is  $q^{1+2+\dots+n} = q^{\binom{n+1}{2}}$  and the weights contributed by the  $n$ -boards on the bottom left is  $f_{n+1}(q) f_{n+1}^{(2)}(q)$ . So when  $n$  is even, we need to subtract  $q^{\binom{n+1}{2}} f_{n+1}(q) f_{n+1}^{(2)}(q)$  from the left hand side of the identity (or add it to the right hand side). Now we also need to consider the reverse direction. We will not be able

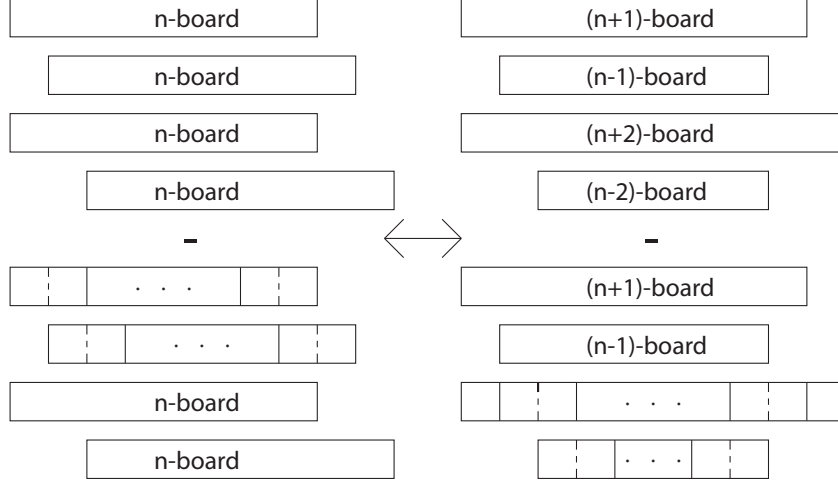


Figure 16: Tail-swapping when  $n$  is even.

to swap the tails back to four  $n$ -boards in the case when the bottom right pair of boards is arranged such that the  $(n+2)$ -board starts and ends with a square, but is otherwise entirely dominoes and the  $(n-2)$ -board is all dominoes. The weight of the two boards on the bottom right will be  $q^{2+3+\dots+n} = q^{\binom{n+1}{2}-1}$  and the weights of the pair on the top right are counted by  $f_{n+2}(q)f_n^{(1)}(q)$ . Thus we need to subtract  $q^{\binom{n+1}{2}-1}f_{n+2}(q)f_n^{(1)}(q)$  from the left hand side of our identity. So when  $n$  is even we need to add the quantity  $q^{\binom{n+1}{2}}(f_{n+1}(q)f_{n+1}^{(2)}(q) - q^{-1}f_{n+2}(q)f_n^{(1)}(q))$  to the right hand side of our identity.

**Case 2:** If  $n$  is odd, we can tail swap in all cases except when the pair of boards on the bottom left are tiled such that the  $n$ -board which is not shifted starts with a square and the  $n$ -board shifted by 2 ends with a square and the rest of the tiles in both boards are dominoes. The weights of these tilings are counted by  $q^{\binom{n+1}{2}-1}f_{n+1}(q)f_{n+1}^{(1)}(q)$ , so we need to add these weights to the left hand side of the identity when  $n$  is odd. Again, when  $n$  is odd, there is a chance that we will not be able to swap back to 4  $n$ -boards. This case is when both the  $(n+1)$ -board and the  $(n-1)$ -board on the right are tiled entirely with dominoes. The total weights contributed by these arrangements are counted by  $q^{\binom{n+1}{2}}f_{n+3}(q)f_{n-1}^{(2)}(q)$ , a quantity we need to subtract from the left hand side when  $n$  is odd. So when  $n$  is odd, we need to subtract the quantity  $q^{\binom{n+1}{2}}(f_{n+3}(q)f_{n-1}^{(2)}(q) - q^{-1}f_{n+1}(q)f_{n+1}^{(1)}(q))$ .



counted by  $f_{4(n-k)+1}^{(2k+1)}(q)$ , so in this case, the tilings are counted by  $q^{(4n+1)k} f_{4(n-k)+1}^{(2k+1)}(q)$ .

**Case 2:**  $k = j$  and  $k \neq i$  ( $i > j$ ). Then there are  $k + 1$  consecutive dominoes from the left of the tiling of our  $(4n + 1)$ -board and also dominoes from position  $4n - 2k + 1$  to the end of the board. These dominoes contribute a weight of  $1 + 3 + \dots + (2k - 1) + (2k + 1) + (4n - 2k + 2) + (4n - 2k + 4) + \dots + 4n = k(4n + 1) + (2k + 1)$ . We also know that there is a square at position  $4n - 2k$  and also a domino at position  $(2k + 1, 2k + 2)$ . The tilings we can fit in between these positions are counted by  $f_{4(n-k)-1}^{(2k+2)}(q)$ . So in this case, our tilings are counted by  $q^{k(4n+1)+(2k+1)} f_{4(n-k)-1}^{(2k+2)}(q)$ .

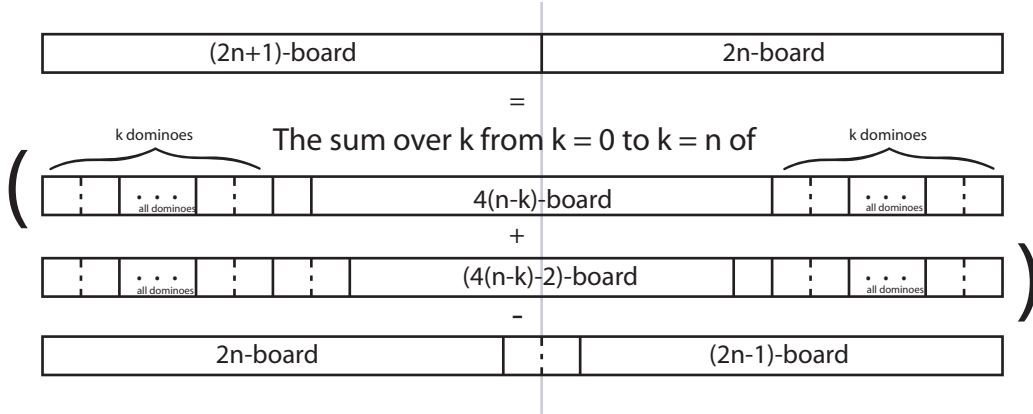


Figure 18: A synthesis of the proof of 3.10.

Since these two cases are disjoint, we can see that the number of ways to tile a  $(4n + 1)$ -board can be broken into cases based on  $k$ , but we're not looking for ways to tile a  $(4n + 1)$ -board. We're looking for the ways to tile a  $(4n + 1)$ -board *with a fault at  $2n + 1$* . Thus we need to subtract all cases where there is a domino at position  $(2n + 1, 2n + 2)$ . The  $q$ -polynomial  $q^{2n+1} f_{2n+1}(q) f_{2n}^{(2n+2)}(q)$  counts these tilings, so we just subtract this from our sum over  $k$ .

So in total, our weighted tilings are counted by  $\sum_{k=0}^n (q^{(4n+1)k} (f_{4(n-k)+1}^{(2k+1)}(q) + q^{2k+1} f_{4(n-k)-1}^{(2k+2)}(q)) - q^{2n+1} f_{2n+1} f_{2n}^{(2n+2)})$  with the convention that  $f_i(q) = 0$  if  $i \leq 0$ . Note that this is not the identity above. In order to make the identity a little nearer to the non- $q$ -version, we reindexed the sum by replacing  $k$  with  $n - k$ .  $\square$

The following three theorems are similar to Theorem 3.10 in proof and in result.

**Fibonacci Identity:** For  $n \geq 0$ ,  $f_1 + f_5 + f_9 + \dots + f_{4n-3} = f_{2n-1}^2$ . Our  $q$ -version is:

**Theorem 3.11.** For  $n \geq 1$ ,

$$f_{2n}(q)f_{2n}^{(2n-1)}(q) = \sum_{k=1}^n (q^{(4n-2)(n-k)}(f_{4k-2}^{(2(n-k)+1)}(q) + q^{2(n-k)+1}f_{4k-4}^{(2(n-k)+2)}(q)) - q^{2n-1}f_{2n-1}(q)f_{2n-1}^{(2n)}(q) + q^{(2n-1)^2}) \quad (19)$$

*Proof:*

**Left Hand Side:**  $f_{2n}(q)f_{2n}^{(2n-1)}(q)$  counts the weighted tilings of a  $(2n - 1)$ -board and a  $(2n-1)$ -board shifted by  $2n-1$ . (Or a  $(4n-2)$ -board with a fault at position  $2n-1$ .)

**Right Hand Side:** As in the proof of 3.10, we consider a  $(4n - 2)$ -board which has a fault at position  $(2n - 1)$  and starts with  $i$  dominoes and ends with  $j$  dominoes. Again, we let  $k = \min(i, j)$  and consider two cases.

**Case 1:**  $k = i$ . Then there is a square at position  $2k + 1$ , and dominoes from position  $4n - 2k - 2$  to the end of the  $(4n - 2)$ -board. The weight contributed by the dominoes is  $(4n - 2)k$ , and the tilings from position  $2k + 2$  to  $4n - 2k - 2$  are counted by  $f_{4(n-k)-2}^{(2k+1)}(q)$ , so all tilings in this case are counted by  $q^{(4n-2)k}f_{4(n-k)-2}^{(2k+1)}(q)$ .

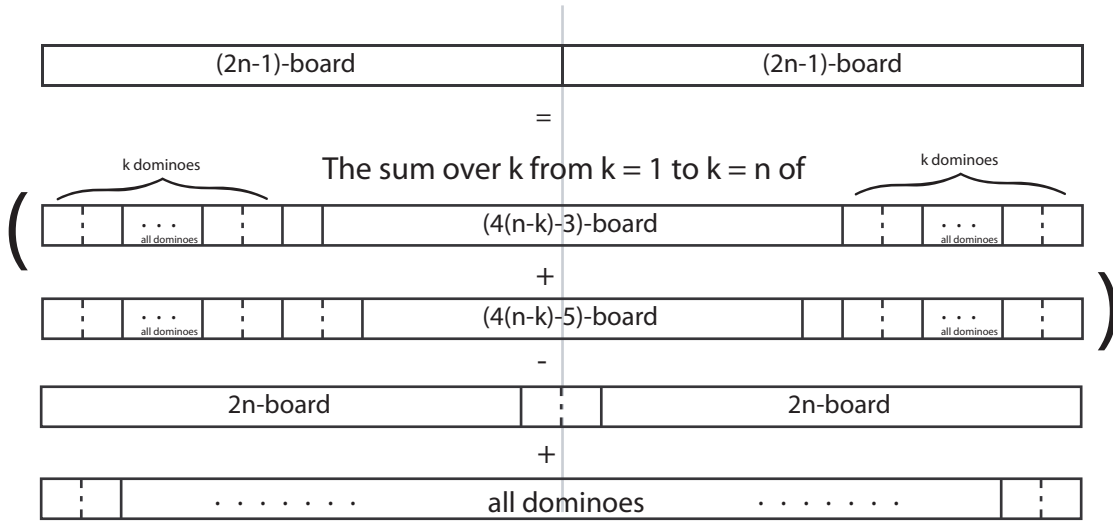


Figure 19: A synthesis of the proof of 3.11.

**Case 2:**  $k = j$  and  $k \neq i$  ( $i > j$ ). Then there are  $k + 1$  consecutive dominoes from the left of the tiling of our  $(4n - 2)$ -board, a square at position  $4n - 2k - 3$ , and also dominoes from position  $4n - 2k - 2$  to the end of the board. These dominoes contribute a weight of  $(4n - 2)k + (2k + 1)$ . The possible tilings of the rest of the board

are counted by  $f_{4k-4}^{(2k+2)}(q)$ , so in this case, all tilings are counted by  $q^{(4n-2)k+(2k+1)} f_{4k-4}^{(2k+2)}(q)$ .

We sum over  $k$  from 0 to  $n - 1$ , and we get close to our identity; however, like in the proof of 3.10, we overcounted cases where there is no fault at position  $2n - 1$ , so we need to subtract  $q^{2n-1} f_{2n-1}(q) f_{2n-1}^{(2n)}(q)$  from our sum. But when we do this, we subtract the all domino tiling of a  $(4n - 2)$ -board, which we did not count in our sum. Thus, we need to add a  $q^{(2n-1)^2}$  term back in to get our identity. Once again, as previously, we reindex the sum using  $n - k$  instead of  $k$  to get the theorem.  $\square$

**Fibonacci Identity:** For  $n \geq 0$ ,  $f_2 + f_6 + f_{10} + \cdots + f_{4n-2} = f_{2n-1} f_{2n}$ . Our  $q$ -version is:

**Theorem 3.12.** For  $n \geq 1$ ,

$$f_{2n}(q) f_{2n+1}^{(2n-1)}(q) = \sum_{k=1}^n (q^{(4n-1)(n-k)} (f_{4k-1}^{(2(n-k)+1)}(q) + q^{2(n-k)+1} f_{4k-3}^{(2(n-k)+2)}(q)) - q^{2n-1} f_{2n-1}(q) f_{2n}^{(2n)}(q)) \quad (20)$$

*Proof:*

**Left Hand Side:**  $f_{2n}(q) f_{2n+1}^{(2n-1)}(q)$  counts the weighted tilings of a  $(2n - 1)$ -board and a  $2n$ -board shifted by  $2n - 1$ . (Or a  $(4n - 1)$ -board with a fault at position  $2n - 1$ .)

**Right Hand Side:** Again consider a  $(4n - 1)$ -board which has a fault at position  $(2n - 1)$  and starts with  $i$  dominoes and ends with  $j$  dominoes. Let  $k = \min(i, j)$  and consider two cases.

**Case 1:**  $k = i$ . Then there is a square at position  $2k + 1$ , and dominoes from position  $4n - 2k - 1$  to the end of the  $(4n - 1)$ -board. The weight contributed by the dominoes is  $(4n - 1)k$ , and the tilings from position  $2k + 2$  to  $4n - 2k - 1$  are counted by  $f_{4(n-k)-1}^{(2k+1)}(q)$ , so all tilings in this case are counted by  $q^{(4n-1)k} f_{4(n-k)-1}^{(2k+1)}(q)$ .

**Case 2:**  $k = j$  and  $k \neq i$  ( $i > j$ ). Then there are  $k + 1$  consecutive dominoes from the left of the tiling of our  $(4n - 2)$ -board, a square at position  $4n - 2k - 2$ , and also dominoes from position  $4n - 2k - 1$  to the end of the board. These dominoes contribute a weight of  $(4n - 1)k + (2k + 1)$ . The possible tilings of the rest of the board are counted by  $f_{4k-3}^{(2k+2)}(q)$ , so in this case, all tilings are counted by  $q^{(4n-1)k+(2k+1)} f_{4k-3}^{(2k+2)}(q)$ .

We sum over  $k$  from 0 to  $n - 1$  and subtract all tilings of a  $(4n - 1)$ -board with a domino at  $(2n - 1, 2n)$  to get our result.  $\square$

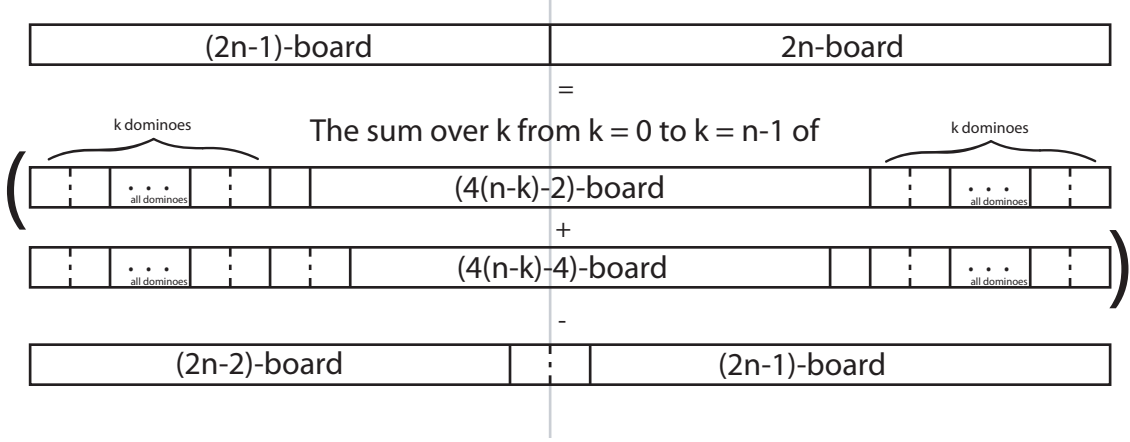


Figure 20: A synthesis of the proof of 3.12.

**Fibonacci Identity:** For  $n \geq 0$ ,  $f_3 + f_7 + f_{11} + \dots + f_{4n-1} = f_{2n-1}f_{2n+1}$ . Our  $q$ -version is:

**Theorem 3.13.** For  $n \geq 1$ ,

$$f_{2n}(q)f_{2n+2}^{(2n-1)}(q) = \sum_{k=1}^n (q^{4n(n-k)}(f_{4k}^{(2(n-k)+1)}(q) + q^{2(n-k)+1}f_{4k-2}^{(2(n-k)+2)}(q)) - q^{2n-1}f_{2n-1}(q)f_{2n+1}^{(2n)}(q) + q^{(2n)^2}) \quad (21)$$

*Proof:*

**Left Hand Side:**  $f_{2n}(q)f_{2n+2}^{(2n-1)}(q)$  counts the weighted tilings of a  $(2n - 1)$ -board and a  $(2n + 1)$ -board shifted by  $2n - 1$ . (Or a  $4n$ -board with a fault at position  $2n - 1$ .)

**Right Hand Side:** Following a familiar strategy, we consider a  $4n$ -board which has a fault at position  $(2n - 1)$  and starts with  $i$  dominoes and ends with  $j$  dominoes. Let  $k = \min(i, j)$  and consider two cases.

**Case 1:**  $k = i$ . Then there is a square at position  $2k + 1$ , and dominoes from position  $4n - 2k$  to the end of the  $4n$ -board. The weight contributed by the dominoes is  $4nk$ , and the tilings from position  $2k + 2$  to  $4n - 2k$  are counted by  $f_{4(n-k)}^{(2k+1)}(q)$ , so all tilings in this case are counted by  $q^{4nk}f_{4(n-k)}^{(2k+1)}(q)$ .

**Case 2:**  $k = j$  and  $k \neq i$  ( $i > j$ ). Then there are  $k + 1$  consecutive dominoes from the left of the tiling of our  $4n$ -board, a square at position  $4n - 2k - 1$ , and also dominoes from position  $4n - 2k$  to the end of the board. These dominoes contribute a weight of  $4nk + (2k + 1)$ . The possible tilings of the rest of the board are counted by  $f_{4k-2}^{(2k+2)}(q)$ , so in this case, all tilings are counted by  $q^{4nk+(2k+1)}f_{4k-2}^{(2k+2)}(q)$ .

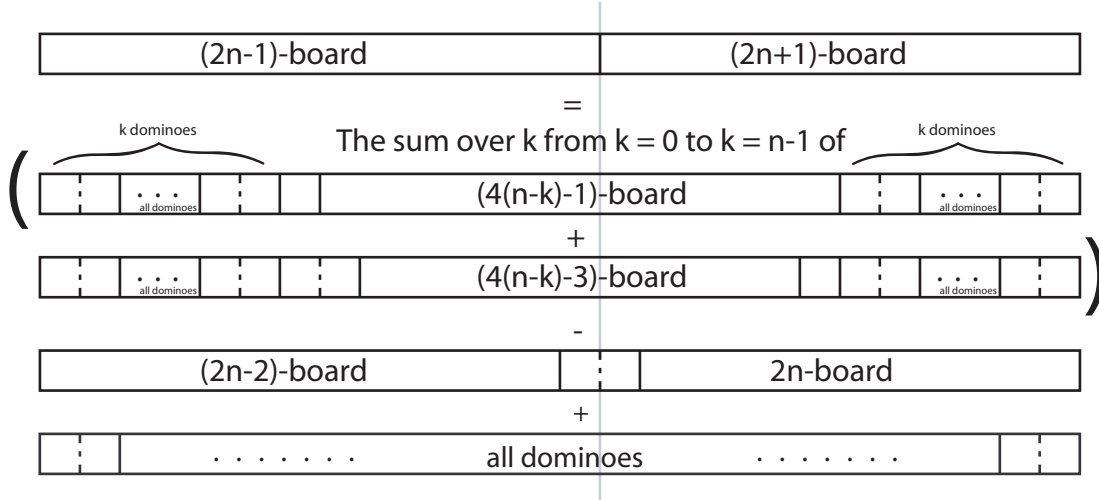


Figure 21: A synthesis of the proof of 3.13.

We sum over  $k$  from 0 to  $n - 1$ , and we get close to our identity; however, we have again overcounted cases where there is no fault at position  $2n - 1$ , so we need to subtract  $q^{2n-1} f_{2n-1}(q) f_{2n+1}^{(2n)}(q)$  from our sum. But as we do this, (like in the proof of 3.11) we subtract the all domino tiling of a  $4n$ -board, which we did not count in our sum. Thus, we need to add a  $q^{(2n)^2}$  term back in to get our identity. Once again we reindex the sum to get the theorem.  $\square$

**Fibonacci Identity:** For  $n \geq 0$ ,  $f_{2n+2} = f_{n+1}f_{n+2} - f_{n-1}f_n$ . Our  $q$ -version is:

**Theorem 3.14.** For  $n \geq 0$ ,

$$f_{2n+3}(q) = f_{n+2}(q) f_{n+3}^{(n)}(q) - q^{2n+1} f_n(q) f_{n+1}^{(n+2)}(q) \quad (22)$$

*Proof:*

**Left Hand Side:**  $f_{2n+3}(q)$  counts the weighted tilings of a  $(2n + 2)$ -board.

**Right Hand Side:** Put a  $(n + 1)$ -board on top of a  $(n + 2)$ -board such that the  $(n + 2)$ -board is shifted  $n$  units.  $f_{n+2}(q) f_{n+3}^{(n)}(q)$  counts the weighted tilings of the staggered boards.

**Case 1:** There exists a common fault between the two boards. In this case, we will swap the tails of the two boards.

*Subcase 1:* The top board ends with a square and the bottom board begins with a square. After tail-swapping, we get a  $(2n + 2)$ -board with a square at position  $(n + 1)$ .

*Subcase 2:* The top board ends with a domino and the bottom board begins with a square. After tail-swapping, we get a  $(2n + 2)$ -board with a domino at position  $(n, n + 1)$ .

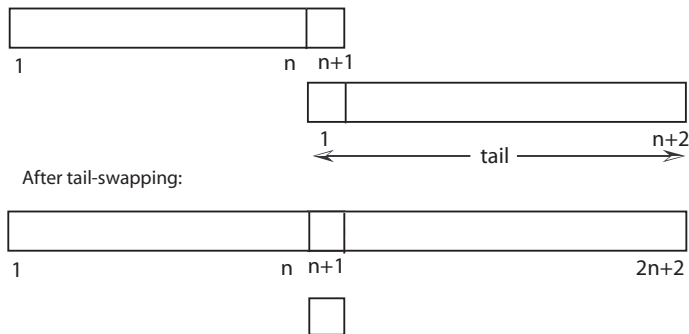


Figure 22: Tail-swapping in subcase 1.

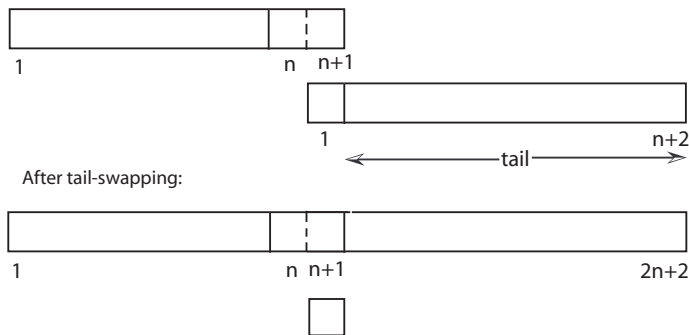


Figure 23: Tail-swapping in subcase 2.

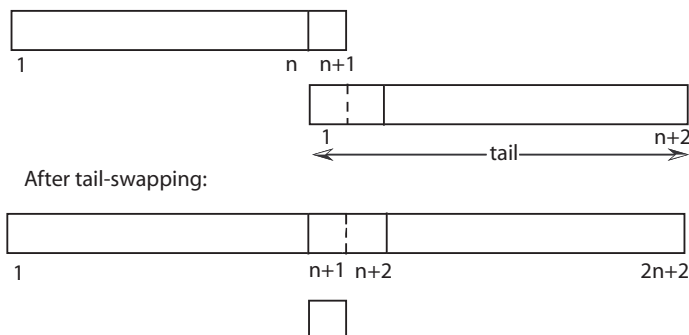


Figure 24: Tail-swapping in subcase 3.

*Subcase 3:* The top board ends with a square and the bottom board begins with a domino. After tail-swapping, we get a  $(2n + 2)$ -board with a domino at position  $(n + 1, n + 2)$ .

The three subcases together count the weighted tilings of a  $(2n + 2)$ -board since a  $(2n + 2)$ -board must either have a square at position  $(n + 1)$ , a domino at position  $(n, n + 1)$  or a domino at position  $(n + 1, n + 2)$ .

**Case 2:** There does not exist a common fault between the two boards, i.e. the top

board ends with a domino and the bottom board begins with a domino. The number of weighted tilings in this case is:  $q^{2n+1}f_n(q)f_{n+1}^{(n+2)}(q)$ .

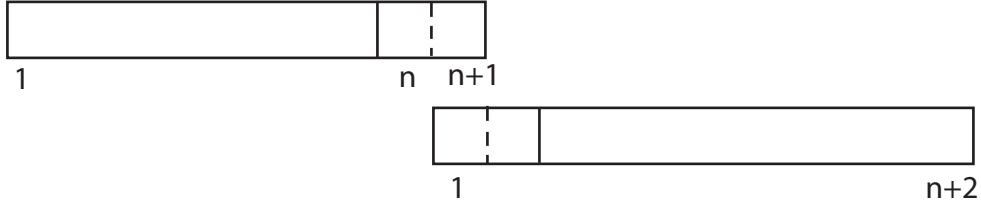


Figure 25: When the two boards don't have a common fault.

From the two cases, we have  $f_{n+2}(q)f_{n+3}^{(n)}(q) - q^{2n+1}f_n(q)f_{n+1}^{(n+2)}(q)$  counts the weighted tilings of a  $(2n + 2)$ -board.  $\square$

**Fibonacci Identity:** For  $n \geq p$ :

$$f_{n+p} = \sum_{i=0}^p \binom{p}{i} f_{n-i} \quad (23)$$

Our  $q$ -version is:

**Theorem 3.15.** For  $n \geq 0$ ,

$$f_{n+p+1}(q) = \sum_{i=0}^p q^{i^2} \begin{bmatrix} p \\ i \end{bmatrix}_q f_{n-i+1}^{(p+i)}(q) \quad (24)$$

*Proof:*

**Left Hand Side:**  $f_{n+p+1}(q)$  counts the weighted tilings of a  $(n + p)$ -board.

**Right Hand Side:** Let  $i$  be the number of dominoes in the first  $p$  tiles. Then there would be  $(p - i)$  squares in the first  $p$  tiles and the remainder is a board of length  $(n - i)$ . According to an earlier result, the number of weighted tilings that have exactly  $p$  tiles,  $i$  of which are dominoes is  $q^{i^2} \begin{bmatrix} p \\ i \end{bmatrix}_q$ . Since  $i$  is the number of dominoes in the first  $p$  tiles,

it can take any value between 0 and  $p$ . Thus,  $\sum_{i=0}^p q^{i^2} \begin{bmatrix} p \\ i \end{bmatrix}_q f_{n-i+1}^{(p+i)}(q)$  also counts the weighted tilings of a  $(n + p)$ -board.  $\square$

**Fibonacci Identity:** For  $n \geq 0$ :  $f_{n+3}^2 + f_n^2 = 2f_{n+1}^2 + 2f_{n+2}^2$ . Our  $q$ -version is:

**Theorem 3.16.** For  $n \geq 0$ ,

$$f_{n+4}(q)^2 + q^{3n+4}f_{n+1}(q)^2 = q^{n+2}(1 + q^{n+2})f_{n+2}(q)^2 + (1 + q^{n+2})f_{n+3}(q)^2 \quad (25)$$

The above is equivalent to:

$$f_{n+4}(q)^2 = q^{n+2}(1 + q^{n+2})f_{n+2}(q)^2 + (1 + q^{n+2})f_{n+3}(q)^2 - q^{3n+4}f_{n+1}(q)^2 \quad (26)$$

*Proof:*

**Left Hand Side:**  $f_{n+4}^2(q)$  counts the weighted tilings of two  $(n + 3)$ -boards; one is placed on top of the other.

**Right Hand Side:** The tilings of two  $(n + 3)$ -boards, one on top of the other, can be broken down into 3 cases:

**Case 1:** Each board has a square at the last position. The total number of weighted tilings for this case is  $f_{n+3}^2(q)$

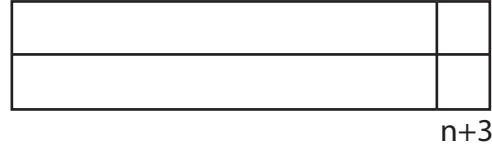


Figure 26: Tilings of two  $(n + 3)$ -boards, where each board ends with a square.

**Case 2:** Each board has a domino at the last position. The total number of weighted tilings for this case is  $q^{2n+4}f_{n+2}^2(q)$  (since each domino at the end has a weight of  $(n + 2)$ ).

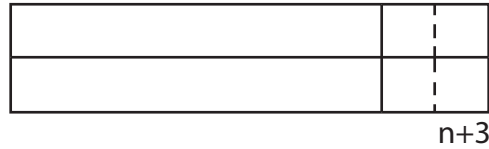


Figure 27: Tilings of two  $(n + 3)$ -boards, where each board ends with a domino.

**Case 3:** One board ends with a domino, the other ends with a square.

*Subcase 1:* The top board has a domino at the end and the bottom board has a square at the end. The domino has the weight  $q^{n+2}$ .

*Subcase 2:* The top board has a square at the end and the bottom board has a domino at the end. The domino has the weight  $q^{n+2}$ .

If we take the domino and the square at the end in each subcase out, and put an imaginary square at the end of the  $(n + 1)$ -board in each subcase, then the two subcases combined together count the number of weighted tilings of two  $(n + 2)$ -boards ( $f_{n+3}(q)^2$ ),

one on top of the other, except for the case of the two  $(n + 2)$ -boards in which each board ends with a domino  $(q^{2n+2}f_{n+1}^2(q))$ . In addition, the two subcases also count once more the tilings of two  $(n + 2)$ -boards, one on top of the other and each board ends with a square  $(f_{n+2}(q)^2)$ . So the number of weighted tilings for the two subcases is:  $q^{n+2}(f_{n+3}(q)^2 + f_{n+2}(q)^2 - q^{2n+2}f_{n+1}(q)^2)$  (since the domino taken out has the weight of  $q^{n+2}$ ).

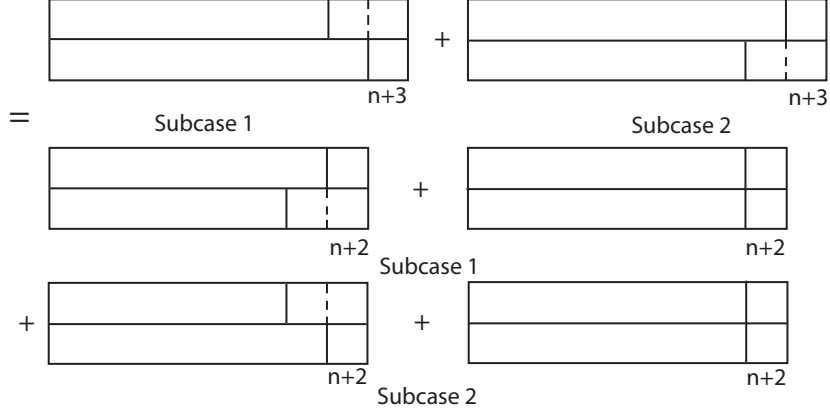


Figure 28: The tilings of subcase 1 and subcase 2.

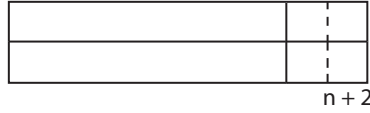


Figure 29: Tilings of two  $(n + 2)$ -boards, where each board ends with a domino.

Together, the total number of weighted tilings for the three cases is:

$$f_{n+3}(q)^2 + q^{2n+4}f_{n+2}(q)^2 + q^{n+2}(f_{n+3}(q)^2 + f_{n+2}(q)^2 - q^{2n+2}f_{n+1}(q)^2)$$

which can be simplified down to:

$$q^{n+2}(1 + q^{n+2})f_{n+2}(q)^2 + (1 + q^{n+2})f_{n+3}(q)^2 - q^{3n+4}f_{n+1}(q)^2$$

**Fibonacci Identity:** For  $n \geq 1$ :  $f_{n-1}(q)f_{n+2}(q)^2 + 2f_n(q)f_{n+1}(q)^2 = f_{n+1}(q)f_{n+2}(q) - f_{n-1}(q)f_n(q)^2$ . Our  $q$ -version is:

**Theorem 3.17.** For  $n \geq 0$ ,

$$q^{2n}(f_n(q)f_{n+3}(q))^2 + (1 + q^{n+1}f_{n+1}(q)f_{n+2}(q))^2 = (f_{n+2}(q)f_{n+3}(q))^2 - 2q^n f_n(q)f_{n+1}(q)f_{n+2}(q)f_{n+3}(q) + q^{4n+2}(f_n(q)f_{n+1}(q))^2 \quad (27)$$

**Lemma 3.18.** For  $n \geq 0$ ,

$$f_{n+3}^2(q) = q^{2n+2} f_{n+1}^2(q) + 2f_{n+2}(q)f_{n+3}(q) - f_{n+2}^2(q) \quad (28)$$

*Proof of Lemma:*

**Left Hand Side:**  $f_{n+3}^2(q)$  counts the weighted tilings of two  $(n+2)$ -boards, where one board is placed on top of the other.

**Right Hand Side:** Consider the following cases:

**Case 1:** If the two boards end with a domino, i.e. the next to last common fault of the two boards is between position  $n$  and  $(n+1)$ , the weighted tilings in this case are counted by  $q^{2n+2} f_{n+1}^2(q)$  since each domino at the end of the boards has a weight of  $q^{n+1}$ .

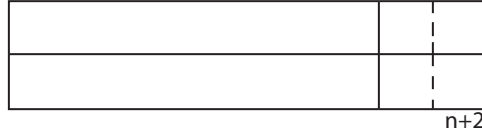


Figure 30: Tilings where the two boards end with a domino

**Case 2:** If the next to last common fault is not between position  $n$  and  $(n+1)$ , at least one of the boards has to end with a square. Consider the following 2 subcases:

*Subcase 1:* The bottom board ends with a square. The weighted tilings in this case are counted by  $f_{n+2}(q)f_{n+3}(q)$ .

*Subcase 2:* The top board ends with a square. The weighted tilings in this case are counted by  $f_{n+2}(q)f_{n+3}(q)$ .

*Subcase 3:* The two boards both end with a square. This subcase is counted by  $f_{n+2}^2(q)$ .

Subcases 1 and 2 together count the weighted tilings in case 2 but they also over-counts once the case where the two boards both end with a square (subcase 3), which is contained in both subcase 1 and subcase 2.

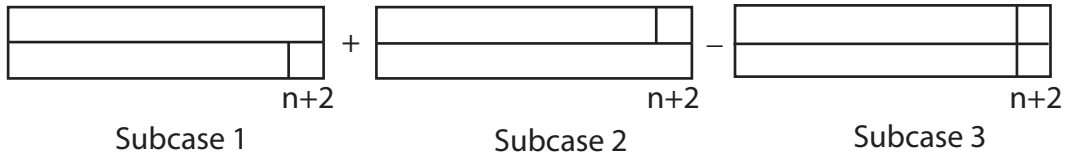


Figure 31: Counting weighted tilings in case 2.

Therefore,  $q^{2n+2}f_{n+1}^2(q) + 2f_{n+2}(q)f_{n+3}(q) - f_{n+2}^2(q)$  counts the weighted tilings of two  $(n+2)$ -boards, where one board is placed on top of the other.  $\square$

*Proof of Theorem 3.17*

We will prove

$$(f_{n+2}(q)f_{n+3}(q))^2 = q^{2n}(f_n(q)f_{n+3}(q))^2 + (1 + q^{n+1}f_{n+1}(q)f_{n+2}(q))^2 + 2q^n f_n(q)f_{n+1}(q)f_{n+2}(q)f_{n+3}(q) - q^{4n+2}(f_n(q)f_{n+1}(q))^2 \quad (29)$$

which is equivalent to the above  $q$ -version of Identity 19.

**Left Hand Side:**  $(f_{n+2}(q)f_{n+3}(q))^2$  counts the weighted tilings of 4 boards (two  $(n+2)$ -boards and two  $(n+1)$ -boards) such that the first position in each board is vertically aligned with the first position in the other boards.

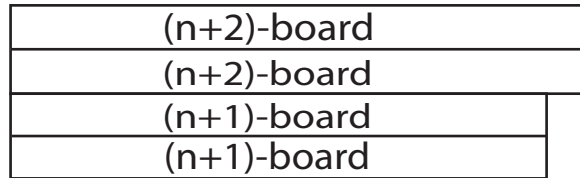


Figure 32: Arrangement of the four boards.

**Right Hand Side:** Consider the weighted tilings of the four boards described above.

**Case 1:** There exists a common fault between positions  $n$  and  $(n+1)$  of the top two  $(n+2)$ -boards. This can be divided into 4 subcases (see figure below).

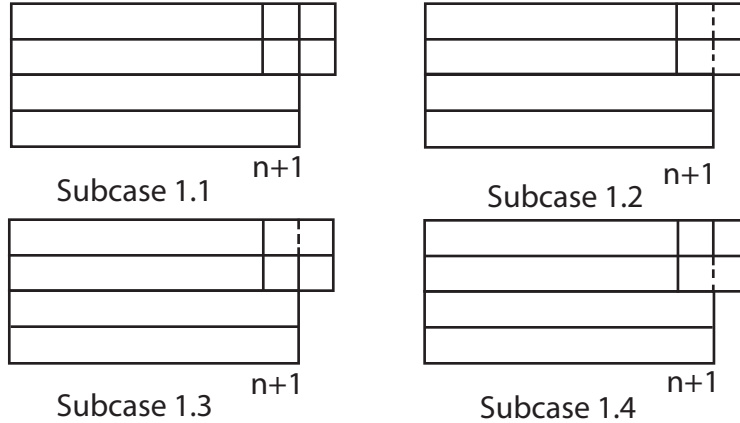


Figure 33: Tilings of case 1 broken down into 4 subcases.

The total number of weighted tilings of the 4 subcases is:

$$(1 + 2q^{n+1} + q^{2n+2})(f_{n+1}(q))^2(f_{n+2}(q))^2 = (1 + q^{n+1}f_{n+1}(q)f_{n+2}(q))^2$$

**Case 2:** There does not exist a common fault between positions  $n$  and  $(n + 1)$  of the top two  $(n + 2)$ -boards, i.e. at least one of these two boards ends with a domino at  $(n, n + 1)$  and a square at position  $(n + 2)$ .

The weighted tilings in this case can be counted by adding weighted tilings in the case that the top  $(n + 2)$ -board ends with a domino followed by a square (subcase 2.1) and the weighted tilings with the bottom  $(n + 2)$ -board ends with a domino followed by a square (subcase 2.2), and subtract the case where both  $(n + 2)$ -boards end with a domino followed by a square (subcase 2.3). The reason for the subtraction is that subcase 1 and subcase 2 both include subcase 3 and thus, adding them together we have overcounted the case in which the two  $(n + 2)$ -boards both end with a domino followed by a square.

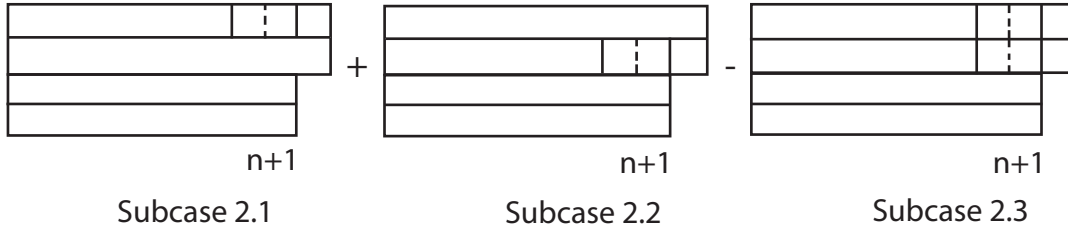


Figure 34: Tilings of case 2 equals the sum of tilings of subcases 2.1 and 2.2 minus the tilings of subcase 2.3.

The sum of weighted tilings in subcase 2.1 and subcase 2.2 can be broken down into 2 components:

a) The bottom  $(n + 1)$ -board ends with a square. The weighted tilings in this case are counted by:  $2q^n f_n(q) f_{n+1}(q) f_{n+2}(q) f_{n+3}(q)$ .

b) The bottom  $(n + 1)$ -board ends with a domino. The weighted tilings in this case are counted by:  $2q^{2n} f_n^2(q) f_{n+2}(q) f_{n+3}(q)$ .

The weighted tilings in subcase 3 are counted by:  $q^{2n} f_n^2(q) f_{n+2}^2(q)$ .

Therefore,  $2q^n f_n(q) f_{n+1}(q) f_{n+2}(q) f_{n+3}(q) + 2q^{2n} f_n^2(q) f_{n+2}(q) f_{n+3}(q) - q^{2n} f_n^2(q) f_{n+2}^2(q)$  counts the weighted tilings in case 2.

The weighted tilings in case 1 and case 2 are counted by:  $(1 + q^{n+1} f_{n+1}(q) f_{n+2}(q))^2 + 2q^n f_n(q) f_{n+1}(q) f_{n+2}(q) f_{n+3}(q) + 2q^{2n} f_n^2(q) f_{n+2}(q) f_{n+3}(q) - q^{2n} f_n^2(q) f_{n+2}^2(q)$ .

or:

$$(1 + q^{n+1} f_{n+1}(q) f_{n+2}(q))^2 + 2q^n f_n(q) f_{n+1}(q) f_{n+2}(q) f_{n+3}(q) + q^{2n} f_n^2(q) (2f_{n+2}(q) f_{n+3}(q) - f_{n+2}^2(q) + q^{2n+2} f_{n+1}^2(q)) - q^{4n+2} (f_n(q) f_{n+1}(q))^2$$

Using the lemma, we obtain the simplified version:

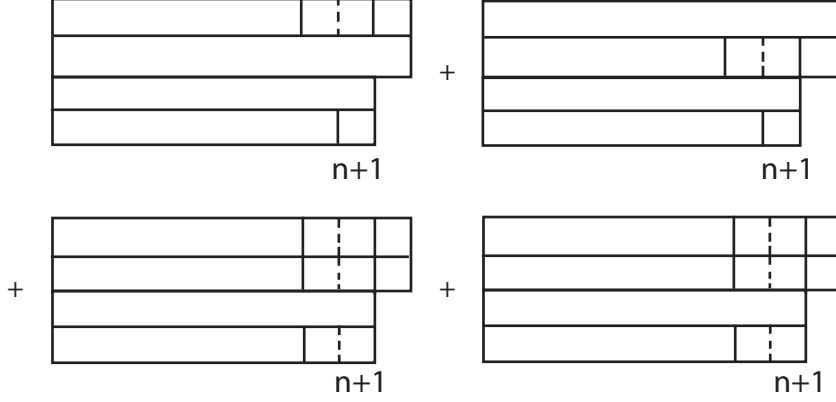


Figure 35: Counting the tilings of subcases 2.1 and 2.2.

$(1 + q^{n+1}f_{n+1}(q)f_{n+2}(q))^2 + 2q^n f_n(q)f_{n+1}(q)f_{n+2}(q)f_{n+3}(q) + q^{2n}(f_n(q)f_{n+3}(q))^2 - q^{4n+2}(f_n(q)f_{n+1}(q))^2$ , which is the right hand side of Theorem 3.17.  $\square$

## 4 $q$ -Lucas Identities

The Lucas numbers satisfy the same recurrence as the Fibonacci numbers but have different initial conditions.

### Definition

$$L_{n+1} = L_n + L_{n-1} \quad L_0 = 2, L_1 = 1. \quad (30)$$

We can interpret the  $n^{\text{th}}$  Lucas number combinatorially as the number of ways of tiling an  $n$ -bracelet with squares and dominoes. We enumerate the positions of the  $n$ -bracelet  $1, 2, \dots, n$ ; we call a particular tiling in-phase if there is no domino occupying the  $n^{\text{th}}$  position and out-of-phase if there is a domino occupying the  $n^{\text{th}}$  position.

As the empty bracelet can be either in-phase or out-of-phase and a 1-bracelet can be tiled only by a square, we see that the combinatorial interpretation satisfies the initial conditions of the Lucas numbers. The combinatorial interpretation also satisfies the Lucas recurrence: consider the last tile that does not cover the first position of the  $n$ -bracelet. If this tile is a square, we can remove it and attach the two ends of the altered  $n$ -bracelet to obtain an  $(n - 1)$ -bracelet. Likewise, if this tile is a domino, we can remove it and attach the two ends of the altered  $n$ -bracelet to obtain an  $(n - 2)$ -bracelet.

We can weight the dominoes in an  $n$ -bracelet tiling in the same way that we weight the dominoes of an  $n$ -board tiling. In this way, we can extend  $L_n$  to  $L_{n+1}(q)$ , where  $L_{n+1}(q)$  represents the generating function, in which the coefficient of  $q^i$  denotes the number of  $n$ -bracelet tilings of weight  $i$ .



Figure 36: Examples of an in-phase and out-of-phase  $n$ -bracelet.

**Lucas Identity:** For  $n \geq 1$ ,  $L_n = f_n + f_{n-2}$ . Our  $q$ -version for  $n \geq 1$  is:

**Theorem 4.1.**

$$L_{n+1}(q) = f_{n+1}(q) + q^n f_{n-1}^{(1)}(q) \quad (31)$$

*Proof:*

**Left Hand Side:**  $L_{n+1}(q)$  counts the weighted tilings of an  $n$ -bracelet.

**Right Hand Side:** Consider an  $n$ -bracelet. If it is in-phase, then it is an  $n$ -board, which is counted by  $f_{n+1}(q)$ . Otherwise, it is out-of-phase and has a domino at position  $n$ . The remainder of the bracelet is a shifted  $(n-2)$ -board, which is counted by  $f_{n-1}^{(1)}(q)$ .  $\square$

**Lucas Identity:** For  $n \geq 1$ ,  $f_{2n-1} = L_n f_{n-1}$ . Our  $q$ -version for  $n \geq 1$  is:

**Theorem 4.2.**

$$f_{2n}(q) = f_{n+1}(q) f_n^{(n)}(q) + q^n f_n(q) f_{n-1}^{(n+1)}(q) \quad (32)$$

*Proof:*

**Left Hand Side:**  $f_{2n}(q)$  counts the weighted tilings of a  $(2n-1)$ -board.

**Right Hand Side:** Suppose a  $(2n-1)$ -board is breakable at  $n$ . In this case, lay down the in-phase portion of  $L_{n+1}(q)$  on the  $(2n-1)$ -board followed by the  $(n-1)$ -board from  $f_n(q)$ . This contributes a weight of  $f_{n+1}(q) f_n^{(n)}(q)$ . Next, suppose the board is not breakable at  $n$ . Then it is breakable at  $n-1$ . In this case, lay down the  $(n-1)$ -board of  $f_n(q)$  followed by the out-of-phase portion of  $L_{n+1}(q)$ , beginning with the domino. This arrangement contributes a weight of  $q^n f_n(q) f_{n-1}^{(n+1)}(q)$  to the  $(2n-1)$ -board. Since the board is either breakable or unbreakable at  $n$ , we have that  $f_{n+1}(q) f_n^{(n)}(q) + q^n f_n(q) f_{n-1}^{(n+1)}(q)$  counts the weighted tilings of a  $(2n-1)$ -board.  $\square$

**Lucas Identity:** For  $n \geq 0$ ,  $L_n + L_{n+2} = 5f_n$ . Our  $q$ -version for  $n \geq 1$  is:

**Theorem 4.3.**

$$L_{n+1}(q) + L_{n+3}(q) = (2 + q^{n+1}) f_{n+1}(q) + q^{n+2} f_{n+1}^{(1)}(q) + q^n (f_n(q) + f_{n-1}^{(1)}(q)) \quad (33)$$

*Proof:*

**Left Hand Side:**  $L_{n+1}(q) + L_{n+3}(q)$  counts the weighted tilings of an  $n$ -bracelet and an  $n + 2$ -bracelet.

**Right Hand Side:**  $L_{n+1}(q) = f_{n+1}(q) + q^n f_n^{(1)}(q)$  by 4.1.

**Case 1:** An  $(n + 2)$ -bracelet is breakable at  $n$  and  $n + 2$ . Then it contributes a weight of  $(1 + q^{n+1})f_{n+1}(q)$ .

**Case 2:** An  $(n + 2)$ -bracelet is not breakable at  $n$ . Then it contributes a weight of  $q^n f_n(q)$  or  $q^{2n+2} f_{n-1}^{(1)}(q)$  if it is breakable or not breakable at  $n + 2$ , respectively.

**Case 3:** An  $(n + 2)$ -bracelet is breakable at  $n$  but not breakable at  $n + 2$ . Then it contributes a weight of  $q^{n+2} f_n^{(1)}(q)$ . Note that  $q^{2n+2} f_{n-1}^{(1)}(q)$ , from case 2, and  $q^{n+2} f_n^{(1)}(q)$ , from case 3, can be simplified to  $q^{n+2} f_{n+1}^{(1)}(q)$  by factoring out  $q^{n+2}$  and applying the recurrence relation for shifted  $q$ -fibonacci numbers.

Together, cases 1, 2, and 3 count the weighted tilings of an  $(n + 2)$ -bracelet and so we have that  $L_{n+1}(q) + L_{n+3}(q) = (2 + q^{n+1})f_{n+1}(q) + q^{n+2} f_{n+1}^{(1)}(q) + q^n (f_n(q) + f_{n-1}^{(1)}(q))$ .

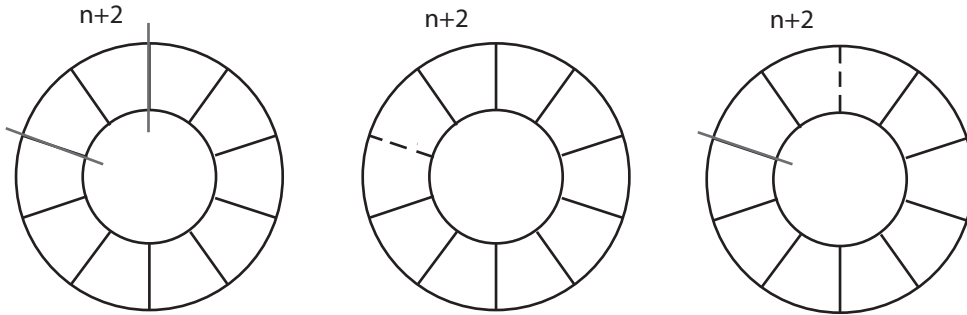


Figure 37: The boards that correspond to Cases 1, 2, and 3 above.

□

## 5 Related Topics

### 5.1 Tiling an $n$ -board with Squares, Dominoes, Tri-ominoes and More

One question that arises when thinking about these  $n$ -board weighted tilings is counting tilings with tiles other than just squares and dominoes. The first step in this is to investigate tilings with squares, dominoes, and tri-ominoes. When considering a definition for these, we use the same recursive idea as we did for the regular tilings, only weighting tri-ominoes with a  $z$  variable instead of a  $q$  in order to differentiate between dominoes and tri-ominoes. Again, to keep things similar to our  $q$ -Fibonacci polynomials, we count the tilings of an  $n$  board with  $g_{n+1}$  rather than  $g_n$ .

#### Definition

$$g_{n+1}(q, z) = g_n(q, z) + q^{n-1}g_{n-1}(q, z) + z^{n-2}g_{n-2}(q, z) \quad (34)$$

where  $g_1(q, z) = 1, g_2(q, z) = 1, g_3(q, z) = 1 + q$

This definition has an obvious combinatorial interpretation: an  $n$ -board can end with a square (this case is counted by  $g_n(q, z)$ ), a domino (counted by  $q^{n-1}g_{n-1}(q, z)$ ), or a tri-omino (counted by  $z^{n-2}g_{n-2}(q, z)$ ).

Many of our  $q$ -Fibonacci proofs can be extended easily to these  $(q, z)$ -3-bonacci tilings, but doing so also requires the definition of shifted  $(q, z)$ -3-bonacci numbers:

#### Definition

$$g_{n+1}^{(a)}(q, z) = g_n^{(a)}(q, z) + q^{n-1+a}g_{n-1}^{(a)}(q, z) + z^{n-2+a}g_{n-2}^{(a)}(q, z) \quad (35)$$

where  $g_1^{(a)}(q, z) = 1, g_2^{(a)}(q, z) = 1, g_3^{(a)}(q, z) = 1 + q^{a+1}$

These shifted  $(q, z)$ -3-bonacci numbers are defined based on the same idea of shifting that the regular  $q$ -Fibonacci numbers are.

#### $q$ -Fibonacci identity: [5]

For  $m, n \geq 0$ ,  $f_{m+n+1}(q) = f_{m+1}(q)f_{n+1}^{(m)}(q) + q^m f_m(q)f_n^{(m+1)}(q)$ .

Our  $(q, z)$ -3-bonacci version is:

**Theorem 5.1.** For  $m, n \geq 0$ ,

$$g_{m+n+1}(q, z) = g_{m+1}(q, z)g_{n+1}^{(m)}(q, z) + q^m g_m(q, z)g_n^{(m+1)}(q, z) \\ + z^m g_m(q, z)g_{n-1}^{(m+2)}(q, z) + z^{m-1} g_{m-1}(q, z)g_n^{(m+1)}(q, z) \quad (36)$$

*Proof:*

**Right Hand Side:**  $g_{m+n+1}(q, z)$  counts the weighted tilings of an  $(m+n)$ -board.

**Left Hand Side:** Consider breakability at position  $m$ . If the tiling is breakable there, it is counted by  $g_{m+1}(q, z)g_{n+1}^{(m)}(q, z)$ . If it is not, then there are three disjoint cases: there is a domino at position  $(m, m+1)$ , a tri-omino at position  $(m, m+1, m+2)$ , or a triomino at position  $(m-1, m, m+1)$ . Each of these cases can be counted by  $q^m g_m(q, z)g_n^{(m+1)}(q, z)$ ,  $z^m g_m(q, z)g_{n-1}^{(m+2)}(q, z)$ , and  $z^{m-1} g_{m-1}(q, z)g_n^{(m+1)}(q, z)$ , respectively.  $\square$

**$q$ -Fibonacci identity:** [5] For  $n \geq 0$ ,  $qf_1(q) + q^2 f_2(q) + \dots + q^{n+1} f_{n+1}(q) = f_{n+3}(q) - 1$ . Our  $(q, z)$ -3-bonacci version is:

**Theorem 5.2.** For  $n \geq 0$ ,

$$g_{n+3}(q, z) - 1 = (q+z)g_1(q, z) + (q^2+z^2)g_2(q, z) + \dots + (q^n+z^n)g_n(q, z) + q^{n+1}g_{n+1}(q, z) \quad (37)$$

*Proof:*

**Left Hand Side:**  $g_{n+3}(q, z) - 1$  counts all tilings of an  $(n+2)$ -board except the tiling of all squares.

**Right Hand Side:** Consider the position of the last weighted tile of an  $(n-2)$ -board. If it is a domino (at position  $(k, k+1)$ ), there are  $g_k(q, z)$  tilings to the left of it and all squares to the right, so this is counted by  $q^k g_k(q, z)$ . If the last weighted tile is a tri-omino (at position  $(k, k+1, k+2)$ ), then again there are  $g_k(q, z)$  tilings to the left of this tile and all squares to the left. This is counted by  $z^k g_k(q, z)$ . These two cases are possible for all  $k$  from 1 to  $n$ , but only a domino can be at position  $(n+1, n+2)$ . This case is counted by  $q^{n+1} g_{n+1}(q, z)$ . Also it is clear that the right hand side does not count the tiling of all squares.  $\square$

A natural next step would be to investigate tilings of an  $n$ -board with tiles of length 1 to  $k$ . We can further generalize to what we call  $z$ - $k$ -bonacci numbers:

### Definition

$$\begin{aligned} g_{k,n+1}(z_2, z_3, \dots, z_k) &= g_{k,n}(z_2, z_3, \dots, z_k) + z_2^{n-1} g_{k,n-1}(z_2, z_3, \dots, z_k) + \dots \\ &\quad + z_i^{n-i+1} g_{k,n-i+1}(z_2, z_3, \dots, z_k) + \dots \\ &\quad + z_k^{n-k+1} g_{k,n-k+1}(z_2, z_3, \dots, z_k) \end{aligned} \quad (38)$$

where

$$\begin{aligned} g_{k,1}(z_2, z_3, \dots, z_k) &= 1, \\ g_{k,2}(z_2, z_3, \dots, z_k) &= 1, \\ g_{k,3}(z_2, z_3, \dots, z_k) &= 1 + z_2, \\ &\dots \\ g_{k,k+1}(z_2, z_3, \dots, z_k) &= g_{k-1,k+1}(z_2, z_3, \dots, z_{k-1}) + z_k \end{aligned}$$

In this definition an  $i$ -omino is weighted with variable  $z_i$  (so dominoes are weighted by  $z_2$ ).

With this definition, it would now seem to possible to generalize both proofs and theorems for many of our regular Fibonacci identities to arbitrary  $z$ - $k$ -bonacci polynomials. Some proofs, however, become extremely difficult extremely quickly. For example, it would be very difficult to think about a tail-swapping analog to Theorem 3.2 using a  $z$ -4-bonacci polynomial because of the prohibitive number of cases where there are no common faults.

## 5.2 Tiling a $2 \times n$ -board with Squares and Dominoes

Let  $F_n$  denote the number of tilings of a  $2 \times n$ -board with squares and dominoes.  $F_n$  differs from  $f_n^2$  because now we have enough space to accommodate a vertical domino. Just as we can extend  $f_n$  to  $f_{n+1}(q)$  by weighting the dominoes, we can extend  $F_n$  to  $F_n(q)$  as follows: place a tiled  $2 \times n$ -board in the first quadrant of the  $x - y$ -plane with its bottom left corner at the origin such that its boxes are determined by the lattice points. We weight the dominoes by summing the coordinates of the appropriate lattice point (see Fig. 38). The weight of a particular tiling is then the sum of the weights of its dominoes. We define  $F_n(q)$  to be the generating function such that the coefficient of  $q^i$  gives the number of tilings of a  $2 \times n$ -board with weight  $i$ . Also, we take  $F_0(q) = 1$  by convention.

Often it is useful to consider  $2 \times n$ -boards that have either a square in the top right or bottom right corner. We denote the number of tilings of a  $2 \times n$ -board whose top right corner is occupied by a square by  $Y_{(n-1)_1}$  and the number of tilings of a  $2 \times n$ -board whose bottom right corner is occupied by a square by  $Y_{(n-1)_2}$ . When considering weighted tilings, we extend the respective  $Y$ 's to  $Y_{(n-1)_1}(q)$  and  $Y_{(n-1)_2}(q)$ . It is clear that  $Y_{1_1}(q) = 1 + 2q$  and  $Y_{1_2}(q) = 1 + q + q^2$ . Also note that for  $n \geq 2$ ,  $Y_{n_1}(q) = F_n(q) + q^n Y_{(n-1)_2}(q)$  and  $Y_{n_2}(q) = F_n(q) + q^{n+1} Y_{(n-1)_1}(q)$ .

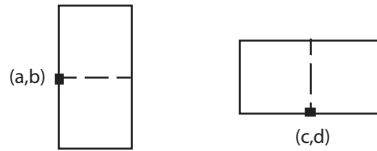


Figure 38: A vertical domino has weight equal to  $a + b$  and a horizontal domino has weight equal to  $c + d$ .

**Identity:** For  $n \geq 1$ ,  $F_{n+1} = Y_{n_1} + Y_{n_2} + F_{n-1}$ . Our  $q$ -version for  $n \geq 1$  is:

**Theorem 5.3.**

$$F_{n+1}(q) = Y_{n_1}(q) + Y_{n_2}(q) + (q^{n+1} - 1)F_n(q) + q^{2n+1}F_{n-1}(q) \quad (39)$$

*Proof:*

**Left Hand Side:**  $F_{n+1}(q)$  counts the weighted tilings of a  $2 \times (n+1)$ -board.

**Right Hand Side:** Consider the last column of a  $2 \times (n+1)$ -board. Its last column can be covered by two squares, a vertical domino, two horizontal dominoes, a square on top of a domino, or a domino on top of a square. The  $Y_{n_i}(q)$  yield two  $2 \times n$ -boards that end with two squares as well as the case when the  $2 \times (n+1)$ -board ends with a horizontal domino over/under a square. The case when it ends with a vertical domino is counted by  $q^{n+1}F_n(q)$ . The case when it ends with two horizontal dominoes is counted by  $q^{2n+1}F_{n-1}(q)$ . Thus  $Y_{n_1}(q) + Y_{n_2}(q) + (q^{n+1} - 1)F_n(q) + q^{2n+1}F_{n-1}(q)$  counts the weighted tilings of a  $2 \times (n+1)$ -board.  $\square$

**Identity:** For  $n \geq 0$ ,

$$F_n = f_n^2 + \sum_{i=1}^n f_{n-i}^2 F_{i-1}$$

Our  $q$ -version for  $n \geq 0$  is:

**Theorem 5.4.**

$$F_n(q) = f_{n+1}(q)f_{n+1}^{(1)}(q) + \sum_{i=1}^n q^i f_{n-i+1}^{(i)}(q)f_{n-i+1}^{(i+1)}(q)F_{i-1}(q) \quad (40)$$

*Proof:*

**Left Hand Side:**  $F_n(q)$  counts the weighted tilings of a  $2 \times n$ -board.

**Right Hand Side:** Consider the position of the last vertical domino in a  $2 \times n$ -board. If there is no vertical domino, then we simply have two  $n$ -boards, one of which is shifted, giving  $f_{n+1}(q)f_{n+1}^{(1)}(q)$ . Otherwise, say the last vertical domino has coordinates  $(i-1, 1)$ . To the left of this domino, is a  $2 \times (i-1)$ -board, which is counted by  $F_{i-1}$ . The domino itself gives a weight of  $q^i$ . To the right of this domino, are two  $(n-i)$ -boards; the one on the bottom is shifted by  $i$  and the one on top is shifted by  $i+1$ . Therefore, the  $2 \times n$ -board with last vertical domino at  $(i, 1)$  has a total weight of  $q^i f_{n-i+1}^{(i)}(q)f_{n-i+1}^{(i+1)}(q)$ . Summing over all  $1 \leq i \leq n$  and including the case in which there is no last domino, we count all possible weighted tilings of a  $2 \times n$ -board.  $\square$

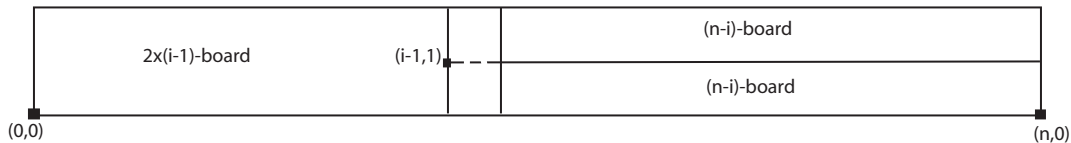


Figure 39: How we break up a  $2 \times n$ -board in 5.4

**Identity:** For  $n \geq 2$ ,  $F_{n+1} = 2F_n + 3F_{n-1} + 2F_{n-2} + 2F_{n-3} + \cdots + 2F_0$ . Our  $q$ -version for  $n \geq 2$  is:

**Theorem 5.5.**

$$F_{n+1}(q) = (1 + q^{n+1})F_n(q) + q^n(1 + q + q^{n+1})F_{n-1}(q) + \sum_{i=1}^{\lfloor \frac{n-2}{4} \rfloor} 2q^{2i(n-i+1)}F_{n-2i}(q) \\ + \sum_{i=1}^{\lfloor \frac{n-3}{2} \rfloor} q^{(2i+1)n-2i^2}(1 + q)F_{n-(2i+1)}(q) + \begin{cases} q^{\frac{n^2+2n-4}{2}}[qY_{1_1}(q) + Y_{1_2}(q)], & \text{if } n \text{ is even} \\ q^{\frac{(n-1)(n+3)}{2}}[Y_{1_1}(q) + Y_{1_2}(q)], & \text{if } n \text{ is odd.} \end{cases} \quad (41)$$

*Proof:*

**Left Hand Side:**  $F_{n+1}(q)$  counts the weighted tilings of a  $2 \times (n + 1)$ -board.

**Right Hand Side:** The proof of the right hand side is straightforward yet tedious, and so it is left to the reader.

Hint: Consider the right hand side of Theorem 5.3. □

While examining the  $F_n$  sequence, we went to [8] and discovered that  $F_n$  appeared to satisfy the recurrence  $a_{n+1} = 3a_n + a_{n-1} - a_{n-2}$ , for  $n \geq 2$  with  $a_0 = 1$  and  $a_1 = 2$ . Theorem 5.6 below proves this observation. We also note that taking the  $\lim_{q \rightarrow 1}$  (Theorem 5.5) shows that  $F_n = a_n$ .

**Identity:** For  $n \geq 2$ ,  $F_{n+1} = 3F_n + F_{n-1} - F_{n-2}$ . Our  $q$ -version for  $n \geq 2$  is:

**Theorem 5.6.**

$$F_{n+1}(q) = (1 + 2q^{n+1})F_n(q) + q^n F_{n-1}(q) - q^{3n} F_{n-2}(q) + q^{2n}(1 - q)Y_{(n-2)_1}(q) \quad (42)$$

*Proof:*

**Left Hand Side:**  $F_{n+1}(q)$  counts the weighted tilings of a  $2 \times (n + 1)$ -board.

**Right Hand Side:** Consider the tiles that occupy the last column of a  $2 \times (n + 1)$ -board. The case that the last column is covered by two squares or a domino is counted by  $(1 + q^{n+1})F_n(q)$ . The case that the last column is covered by two horizontal dominoes is counted by  $q^{2n+1}F_{n-1}(q)$ . Otherwise, the last column is covered by either a horizontal domino on top of a square or a square on top of a horizontal domino; these cases are counted by  $q^{n+1}Y_{(n-1)_1}(q)$  and  $q^n Y_{(n-1)_2}(q)$ , respectively.

Therefore, a  $Y_{(n-1)_1}$ -board equals a  $2 \times n$ -board that cannot end in a vertical domino, two horizontal dominoes, or a horizontal domino on top of a square. Then in the  $q$ -version, we have  $Y_{(n-1)_1}(q) = F_n(q) - q^n F_{n-1}(q) - q^{2n-1} F_{n-2}(q) - q^n Y_{(n-2)_1}(q)$ . Next, we can break down the  $Y_{(n-1)_2}$ -board using the standard recurrence to get  $Y_{(n-1)_2}(q) = F_{n-1}(q) + q^{2n} Y_{(n-2)_1}(q)$ .

Now we have

$q^{n+1}Y_{(n-1)_1}(q) + q^n Y_{(n-1)_2}(q) =$   
 $q^{n+1}F_n(q) + (q^n - q^{2n+1})F_{n-1}(q) - q^{3n}F_{n-2}(q) + q^{2n}(1 - q)Y_{(n-2)_1}(q)$ , which implies that  
 $(1 + 2q^{n+1})F_n(q) + q^n F_{n-1}(q) - q^{3n}F_{n-2}(q) + q^{2n}(1 - q)Y_{(n-2)_1}(q)$  counts the weighted  
 tilings of a  $2 \times (n + 1)$ -board.  $\square$

## 6 Conclusion/Remarks

We have now proved generalizations of a number of identities. That we were able to prove them without the use of much algebra demonstrates the power of the combinatorial techniques that we employed. We believe that this interpretation will lead to even more identities of  $q$ -Fibonacci numbers, many of which we have not encountered. We have seen that this interpretation also lends itself rather easily to generalizations of these polynomials (such as our  $2 \times n$ -boards), which are new and interesting in their own right. We can conceive of many harder problems, such as finding identities involving a  $3 \times n$ -board or an  $m \times n$ -board, or tail-swapping with tri-ominoes. Also, while our interpretation of the Fibonacci numbers is powerful, there are Fibonacci identities for which we have combinatorial proofs but no  $q$ -analog. For example, we do not have a satisfying  $q$ -version for:

$$f_{m+(t+1)p} = \sum_{i=0}^p \binom{p}{i} f_t^i f_{t-1}^{p-i} f_{m+i}$$

Perhaps there is another combinatorial interpretation that yields a  $q$ -version of  $f_n(q)$  that would make this problem more tractable, which we leave as an exercise for the reader.

THE END.

## References

- [1] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, Massachusetts, 1976.
- [2] A. Benjamin, J. Quinn, *Proofs that Really Count*, The Mathematical Association of America, 2003.
- [3] A. Benjamin, J. Quinn, F. E. Su, Phased Tilings and Generalized Fibonacci Identities, *The Fibonacci Quarterly* Vol. 38, No. 3 (2000) 282-288.
- [4] Carlitz, L. Fibonacci notes III, *q-Fibonacci numbers*, *Fibonacci Quarterly* Vol. 12, No. 4 (1974), 317-322.
- [5] Garrett, Kristina, *Weighted tilings and q-Fibonacci numbers*, submitted.
- [6] Garrett, Kristina, private communication.
- [7] Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, *S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl.* (1917), 302-321, reprinted in I.Schur, *Gesammelte Abhandlungen*, Vol. 2, 117-136, Springer, Berlin, 1973.
- [8] The On-Line Encyclopedia of Integer Sequences, <http://www.research.att.com/~njas/sequences/>.