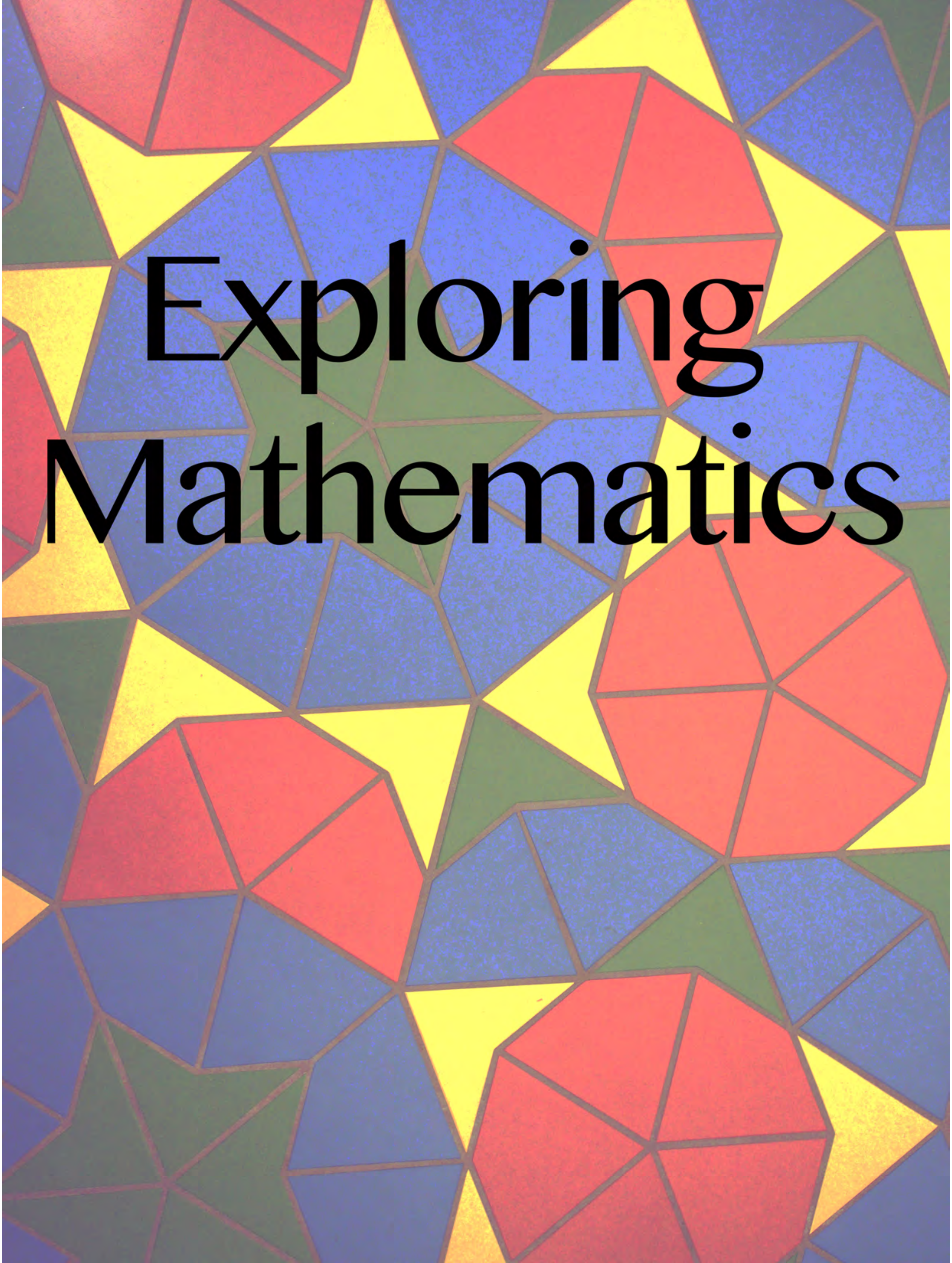
Exploring Mathematics



Exploring Mathematics

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Introduction

The intention of this book is to challenge students to explore math beyond the classroom. This collection of activities is an introductory look at multiple areas of math, including but not limited to sets, geometry and probability. Although some of these topics are more advanced than the typical high school curriculum, each activity is written with the intent of guiding the user. There is no expectation of familiarity with the material. This allows for anyone with a modest math background to be able to complete the activities and learn the enclosed material.

To get the most out of this book, it is best to choose an activity that interests you and work through it slowly, making note of questions and ideas which pique your curiosity. Each activity has a main section comprised of guided learning and questions. Once this main section is complete, look to the "Extension" or "Project Idea" sections, present in many activities, for further material on certain topics from the activity. The "Extension" and "Project Idea" sections are meant to guide your continued learning on the topic. These sections are much more self-directed than the main body of the activity. Within each topical section the book is ordered in somewhat increasing order of difficulty. It is advised to start with the first activity in each section and then work through the rest. However this is not strictly necessary because most activities do not require knowledge from previous activities.

Part I

Number Sense

Number Sense

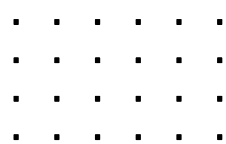
Counting Basics

You might be thinking:“But I already know how to count!" It’s one of the first things that we learn how to do. The classic method of counting goes like this: I start with the first thing I want to count, and designate it ‘one’. Then I move on, assigning the next number to the next object, and continue doing that until I’m done counting all the items. This definitely works for the most part, but it has its flaws. It can take a pretty long time. And it can be easy to lose your place, miss an object, or count an object twice! This becomes even worse if we are trying to count something that isn’t in front of us or is moving. This activity will introduce you to some of the tools that mathematicians use to count things in a more abstract way.

Principle of Multiplication

In order to build up tools to count possibilities, we will start by looking at multiplication as a way of streamlining counting.

We’ll start with something very basic. Let’s say that I want to count the number of dots in this picture:



We can certainly do it as described in the introduction, by counting each dot individually. However, we can save time by noticing that each row contains six dots, and that there are four rows. So I have $6 \times 4 = 24$ dots in this picture.

The principle of multiplication extends this intuition to situations outside of rows and columns. For example, let’s say for breakfast every day I have fruit and cereal. If the only fruits I have are apples, bananas, and pears, and the only cereals I have are Cheerios and Lucky Charms, how many different breakfasts can I make?

We could answer this with a table like this:

	Fruit	Cereal
1	apple	Lucky Charms
2	apple	Cheerios
3	banana	Lucky Charms
4	banana	Cheerios
5	pear	Lucky Charms
6	pear	Cheerios

We list every possibility, and then number them. In this case it’s pretty easy. But what if I go buy strawberries, pineapples, Cocoa Puffs, and Frosted Flakes? Well... I will have a lot more rows of my table to draw. So instead, we rely on this fundamental observation: in the initial example (two cereals and three fruits) once I've chosen a fruit, I only have to pick my cereal one of two ways. So for each fruit, I have two possible breakfasts. There are three ways to choose the fruit, for a total of $3 \times 2 = 6$ different choices.

This is known as the principle of multiplication: If I choose between some number m options (for example, 3 fruits), and then choose from between some other number n options (such 2 cereals), then I have a total of $m \times n$ options for the choices together.

1. Alex just started learning origami. Alex only knows how to fold a crane, a frog, and a star. If she buys a pack with 5 different colors of paper, how many distinguishable projects can Alex make? That is, how many unique color and shape combinations are there?
2. If I flip a coin and then roll a (standard 6-sided) die, how many results are possible? Is the answer different if I roll the die first and then flip the coin?
3. Jacob has 3 sets of sheets, 5 pillow cases, and two quilts, all with different patterns. If he uses exactly one set of sheets, one pillow case, and one quilt to make his bed, how many different ways can he do it?
4. You might recall from probability the concept of factorials. Remember that the number of permutations or arrangements of n objects is $n! = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 1$. How can we think about this in terms of the multiplication principle?
5. How many 5 digit numbers are there with no digit appearing more than once?

Overcounting

Let's take a closer look at the last question above. One solution might be: "I can pick any one of the ten digits to appear in the ones place. Then I go over to the tens place, and I have nine choices, because I can't use the one digit I chose earlier. Extending this process, I see that I have $(10)(10-1)(10-2)(10-3)(10-4) = 10 \times 9 \times 8 \times 7 \times 6 = 30240$ choices.

But something isn't quite right here: We generally don't write numbers with leading zeros. If the number I generate using that method is 02345, then it's really 2345 which is a 4-digit number. So we've overcounted by including those numbers in our count!

One way of fixing overcounting is using subtraction (you may have seen this in the Inclusion-Exclusion principle). What we need to do is count how many numbers we generate that have this leading 0, and then simply subtract this from our total count. We can see that a 5-digit number with no repeating digits that has a leading 0 is simply a 4-digit number with no repeating digits. Furthermore, none of those digits can be 0s (think about how we know this). So we have $9 \times (9-1) \times (9-2) \times (9-3) = 3024$ choices to build an improper 5-digit number (that is, a 5-digit number starting with 0). So our final answer is (total number of choices)-(number of excluded choices) $= 30240 - 3024 = 27216$ 5-digit numbers with no repeated digits.

6. Jim has two jackets, three shirts, and three ties. However, his checkered tie clashes with his pin-striped jacket, so he can't wear them together. How many combinations can he wear?

7. Four friends are driving in a car with 4 seats. Arnold refuses to drive if Betsy is sitting in the passenger seat. How many ways can they fill the seats of the car?

Another good method of handling over-counting is called the shepherd's principle, which is sometimes expressed "To count the number of sheep, count the number of legs and divide by 4". To understand what this means, let's look at an example.

I'm about to go on vacation, and I have to decide which books to bring. I've managed to narrow it down to 5, but I only have room to pack 3. I might use the multiplication principle again, and argue that since I start out being able to choose any book and lose one choice each time I pack a book, I have $5 \times 4 \times 3 = 60$ ways to grab the books. This is true: there are 60 distinct ways for me to take the books off of the shelf. The problem is that it doesn't really make any difference to me if I put *Anne of Green Gables* in my bag before *Crime and Punishment*– I still have the same books when I get there. But these choices are distinct according to my counting system. This seems even hairier than the previous example, because it's not that we just need to exclude "improper" elements, but now some selections of books are counted more than once!

If I know that I will pack books A, B, and C, how many different ways could I have selected them? (In other words, how many permutations of 3 objects are there?) There are $3! = 3 \times 2 \times 1 = 6$ of them. This means that a unique combination of packed books corresponds to exactly 6 ways of taking the books off the shelf. So if I divide my total number of ways to select a book by 6 then we get the number of distinct choices of books to pack.

Although we are really interested in the collection of books that is ultimately chosen, it's easier to count the number of ways to take the books off the shelf. Since we know that there are exactly $3!$ different ways to grab the same set of books we can just divide the total number of ways to take the book off the shelf by the number of times each collection of books is listed. This principle is useful when we know we are over-counting consistently (that is, we are overcounting each choice the same number of times).

8. Write a couple sentences explaining when you know you can use this principle. (What happens if we have a three-legged sheep?)

9. How many ways are there to rearrange the letters of the word "happy"? How about for "mammal"? Rearrangements don't have to be words, but you do need to use all the letters!

10. We know that there are $n!$ ways for n people to sit in a line. How many ways can n people be arranged at a circular table if we do not distinguish between different seats (that is, rotating the table does not change the arrangement - all that matters is who each person is next to)?

11. A certain ice cream store sells 12 flavors of ice cream. How many ways can you order a cone with three scoops? Your friend says "I have $12 \times 12 \times 12$ ways to order my cone, because I have 12 choices for the bottom scoop, 12 for the middle, and 12 for the top. For every selection of flavors, there are $3 \times 2 \times 1$ ways to arrange them in the stack. So if I divide $12 \times 12 \times 12$ by 3×2 I will have the number of ways to order 3 scoops in a dish (where order doesn't matter)." Do you agree with her? Write a couple sentences explaining your reasoning.

Extensions

- We saw some examples of the number of ways to arrange the letters in words. In general, how many ways are there to arrange the letters in a word? What factors affect that number?
- How many 3-scoop ice cream cones can I order (assuming 12 flavors still) if I don't like to have chocolate and pistachio touch?
- Four people are sitting in a row. If they decide to get up and move, so that no one returns to their original seat, how many ways can they sit? What about for five people? More?

Number Sense

Grocery Shopping

Fun Fact: The average supermarket sells 43,844 different products.[1]

Imagine this scenario: Someone goes to the grocery store, fills up their basket to the very top, perhaps with products spilling over the edges, and when they go to the checkout, they find out they didn't bring enough money. How could they have possibly known how much a entire cart of food would cost?

In this exercise we will learn tips and tricks to avoid this embarrassing situation. This involves the use of coupons, estimating prices including taxes, dealing with produce and many more estimation tricks. By the end of this, you will be an expert grocery shopper and will never have to worry at the checkout again!

How much can you buy with $20?

The following items are available to purchase at the grocery store:

Oreos: $2.00 per package

Fruit Roll Ups: $0.50 each

Potato Chips: $2.45 per bag

Pop: $0.55 per can

Gum: $0.35 per package

Bread: $2.25 per loaf

Milk: $3.35 per gallon

Cheese: $2.00 per block

Meat: $2.50 per package

1. Choose exactly $20 worth of food from the grocery store. Make sure you have at least 7 unique items in your basket. There are many different correct answers so don't worry if your answer is not the same as someone else's. If you have time try coming up with five different baskets that follow these rules, or one basket that uses all nine items.

[1]Food Marketing Institute

How much does your basket cost?

In an actual grocery store you don't have a nice list of the objects but have to remember all of their prices and do the math in your head. There can be a lot of strange numbers when working with the actual prices of the goods. One trick that can be used to make the mental math a little bit easier is rounding up the prices to the nearest $.10. This forces us to overestimate our total and ensure that we have enough money to pay for our basket. If we were to do this for an item in the list above, say potato Chips, we would round it up to $2.50 a bag. This technique allows us to more easily mentally add prices and keep track of the value of our basket.

2. Now we are going to give you a basket. Without paper, mentally estimate the cost of the basket of goods.

   2 packages of oreos

   4 fruit roll ups

   2 bag of chips

   2 cans of pop

   4 packages of gum

   2 loaves of bread

   1 gallon of milk

   3 blocks of cheese

   2 packages of meat

What about sales tax?

All of the work we did above is helpful for planning your grocery trip, but in real life there are added complications. Often the price of an item in a store doesn't fully reflect what it costs at the register. Sales tax is usually added only when you pay. In the following exercise we will practice estimating the total cost of a basket of goods including sales tax. It is mindful to note that Minnesota does not have a sales tax on food.

We are going to begin with a 10% sales tax rate.

3. Using the cart in exercise 2, practice adding 10% to the price of each item before adding its cost to the total. This takes into account the tax as you add items to your basket.
4. Repeat exercise 1, where you create a basket of goods that costs $20, but this time include tax in the cost of each item. Add the tax in as you estimate the price of each good. There is not a basket that will cost exactly $20 including tax but you should be able to get within $0.25 of $20.

There is a different way to calculate the price of a full basket with tax. If you have an idea of what your total budget including tax should be, you can use algebra to figure out what the sum of the sticker prices should be.

5. Can you figure out how much you can spend on the sticker price of goods if you want the total to be $20? (Hint: Whatever you spend on the sticker price plus 10% of the sticker price should equal $20.)
6. Now try estimating a $20 cart again. This time use a tax rate of 5%. Do it with both the add as you go technique and the total sticker price calculation technique.
7. Which technique did you prefer with a tax rate of 5%?

Project Idea

For this project you will plan out a picnic lunch for a large group. You will use the new skills you developed for estimating the cost of food in carts and taking sales tax into account to budget for this event.

- Before you start, decide how much you can spend per person.
- Make sure you have drinks, food, and dessert for each person coming to the picnic.
- For prices to calculate your total, go to your local grocery store.
- There must be at least 15 unique items on your grocery list.
- After you are done, write out a grocery list of every item you will be purchasing and the number of units you will purchase (e.g. pounds or packages).
- You also need to write out what you expect each person at the picnic to eat and what the costs for those items will be. For example, one can of pop: $0.55, two slices of bread $0.75 and so on. Also include the total that you spent on each person.

Number Sense

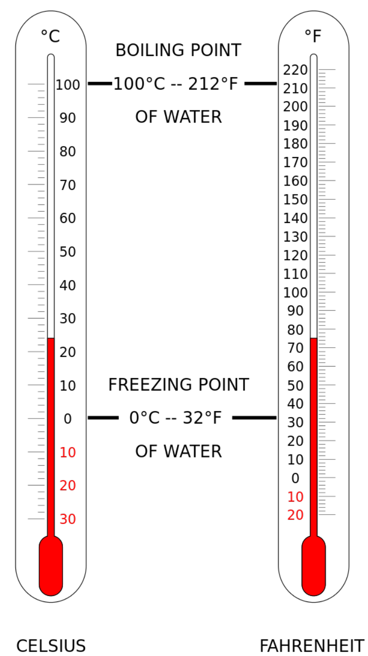
Units

In the world outside your math book, most numbers come labeled with a unit. They're often explicit ($500, 17 students, 143 pounds) but can be implied (in a store, a sign of "2 for 12" would be pretty clear). Yet sometimes units can be unclear, especially when there are different names for the same thing. If I need 4 ounces of milk, how many gallons do I need to buy? Can I get away with only buying a quart? Additionally, if the wrong implied units are assumed, things can get confusing. If I told you it was 20 out, how will you dress? What if I then told you I was actually using Celsius?

Now let's have some fun with different units!

Temperature

If you're given a temperature in °F, do you know how to convert it to Celsius? You might have learned the formula, but it's also really easy to find on your own. All you need to remember are the freezing and boiling temperatures of water in each system - much easier to keep straight than a conversion formula. The Celsius system was created based on water, so it makes a lot of sense: water freezes at $0°\mathrm{C}$ and boils at $100°\mathrm{C}$. Fahrenheit, although probably more familiar to you, is not based on water. Instead, water freezes at $32°\mathrm{F}$ and boils at $212°\mathrm{F}$. (**Fun Fact:** Mr. Fahrenheit, who invented the scale, tried to base it on human body temperature as $100°\mathrm{F}$ and how cold he could make brine (salt water) for $0°\mathrm{F}$. This isn't very precise, so it's now defined mostly by water instead.)



http://en.wikipedia.org/wiki/Thermometer#/media/File:Thermometer_CF.svg

Now let's look for a formula to convert between the two systems. Let's start by converting Celsius to Fahrenheit.

1. These systems start at two different values. $0°\mathrm{C}$ does not correspond to $0°\mathrm{F}$. What does it correspond to? Write an equation to convert $0°\mathrm{C}$ to Fahrenheit.

2. In Celsius, water must pass through 100 degrees to get from freezing to boiling, but in Fahrenheit it must pass through $212 - 32 = 180$ degrees of temperature. This means that in addition to shifting the temperatures, we need to stretch each Celsius degree to fit on the Fahrenheit scale. How much do we need to stretch the Celsius degrees?

3. To get from $100°\mathrm{C}$ to $212°\mathrm{F}$, do we have to shift by 32 or stretch first? What does that make our final equation? (Check your answer when you're done!)

4. Let's practice. Refrigerators are often kept at $4°\mathrm{C}$, while room temperature is approximately $21°\mathrm{C}$. What are these in Fahrenheit?

5. Now that we have an equation to get from Celsius to Fahrenheit, how could we change it to go the other way?

6. Let's practice more. Green tea should be $175°\mathrm{F}$, and body temperature is about $98°\mathrm{F}$. How hot are these in Celsius?

Even though we can calculate temperatures, it's also important to be able to estimate them. This is most common when talking about weather.

7. What would you wear if it's $100°\mathrm{F}$ outside? What about $50°\mathrm{F}$, $15°\mathrm{F}$, or $-10°\mathrm{F}$?

8. World travelers have to learn to convert temperatures - other countries usually report the weather in Celsius. Remember that $21°\mathrm{C}$ is room temperature (about $70°\mathrm{F}$) and $0°\mathrm{C}$ is freezing ($32°\mathrm{F}$). For each of the following temperatures, how would you dress? That is, estimate their temperature in Fahrenheit. In London, it's $7°\mathrm{C}$, in St. Petersburg it's $-10°\mathrm{C}$, and in Dubai it's $40°\mathrm{C}$.

9. Check your estimations - convert these temperatures to Fahrenheit using your equation from earlier.

10. What is unusual about $-40°\mathrm{C}$? Are there any other temperatures like that?

Smoots

Oliver Smoot was a 5'7" student at MIT in the 1950s. He became famous after using his body as a ruler to measure the Harvard Bridge, finding it to be 364.4 Smoots long. Do you know how tall you are? If not, get a friend to help you measure yourself.

For each of the next questions, estimate an answer in feet (make an educated guess), check your estimate (find the actual height or length), estimate how that compares to your height, and finally convert the measurement into units of your height, like Smoots, to check that estimate.

11. Is your desk shorter or taller than you?

12. Is the ceiling more or less than twice your height?

13. How long and wide is your classroom?

14. How much longer than you is your bed?

15. How many of you would it take to reach the top of the Eiffel Tower?

16. Find at least two more buildings, rooms, etc. to approximate with your height.

17. Go explore - find a unit you've never heard of (Wikipedia has lists of units that can be a good starting point) and use it to estimate the measure of several things around you. Be careful that you're using it for the right sort of quantity - just as I can't measure my weight in Smoots or my height in seconds, it's important to know what type of number a new unit can be used for.

Number Sense

Modular Arithmetic

Fun Fact: Modular arithmetic can be used to determine what day of the week a certain date fell on, i.e. February 11, 1978 was on Saturday. (See the extension.)

Clocks

If it was 9:00 AM and you had a math test in 26 hours, would you wait for 35:00 to roll around? Probably not – that doesn't make any sense as a time since hours are usually only counted up to twelve. That is, at noon and midnight (12 PM and 12 AM), the hour counter resets to zero.

You can probably understand this intuitively.

1. What time is 7 hours after 7 PM?
2. What time is 4 hours after 5 AM?
3. What time is 16 hours after 3 PM?

Another way of describing what you're doing uses an operation called modular arithmetic. Like addition, multiplication, division, and so forth, modular arithmetic is a rule describing an interaction between two numbers. Do you remember remainders from when you learned long division? Now the remainder is what we're looking for! With time, we are working "mod 12," meaning that we divide by 12, because clocks reset to zero at 12 (noon or midnight). For example, $16 \bmod 12$ is the same as asking for the remainder of $16/12$ which is 4.

So what time is your math test? It's at $35 \bmod 12$ o'clock, or 11:00. But what day? To know that, we have to go back to normal division to figure out how many full twelves there were before you got to the 12 you couldn't finish. There has to be at least one full twelve, because 35 >12. To figure out how much time you have (half a day? a week?) before your math test, do the other part of the division: $35/12$, ignoring any remainder or fractional portions of the answer. That is, two 12's ($2 \times 12 = 24$, which is less than 35, but $3 \times 13 = 36$, which is too large) so 24 hours or a full day exist before your math test. Pulling the two pieces together, your math test will be at 11:00 A.M. tomorrow.

A final observation on notation: Instead of "=," modular equations are written with "$\equiv$," as in $35 \equiv 11 \bmod 12$. Now let's practice!

4. What is $19 \bmod 12$?
5. What time is 11 hours after 8 A.M.?
6. What is the connection between the previous two questions?
7. What is $127 \bmod 12$?

8. How about $3 \bmod 12$?
9. What time is 45 hours after 3:00 A.M.?

Military Time

Some people, especially pilots and boaters, use 24-hour time. Instead of calculating their time mod 12, they use mod 24, so the clocks reset only once every day at midnight.

10. What is $16 \bmod 12$? In other words, what time is 16:00 in 12-hour time?
11. Using 24-hour time, what is 16 hours after 10:00?
12. What is 16 hours after 8:00 in the morning?

In mod 24, 24 and 0 are exactly the same. Midnight could be considered either 24:00 or 00:00.

Other Bases

We aren't limited to these two bases - we can use any number we want to take mods by.

13. What is $19 \bmod 7$?
14. Does mod 7 look familiar? Where is it used in everyday life?
15. What other bases are commonly used in time keeping?
16. What is $3 \bmod 2$?
17. What is $8 \bmod 2$?
18. How many different possible answers are there when modding out by 2?
19. What is another word for a number that is equivalent to zero mod 2? What about numbers equivalent to 1 mod 2?
20. What is 2 mod 6? (Remember, you can have zero sixes.)

Doomsday

In this extension you are going to learn how to figure out what day of the week any date in history fell on. This might seem like a difficult problem but by using modular arithmetic and the Doomsday Algorithm it is possible to do this with mental math. This algorithm is broken down into steps to make the calculations easier.

Step 1

The first step is treating the days of the week as numbers, starting with Sunday as 0. This would mean that Monday is 1, Tuesday is 2, Wednesday is 3, Thursday is 4, Friday is 5, and Saturday is 6.

Step 2: The anchor days

The next step is memorizing the anchor days. These are days of the week that are used as reference points in each century.

Century	Anchor Day
1700s	Sunday
1800s	Friday
1900s	Wednesday
2000s	Tuesday

If the date you are working with falls out of the range of the chart, calendars repeat themselves every 400 years so just repeat the pattern.

Step 3: The doomsday

We now need to find the doomsday of the year. The doomsday is a day of the week that a group of dates falls on. These dates are represented in the chart below. If the doomsday for a year was Tuesday, then all the dates in the chart would be on a Tuesday.

As you can see, some of these boxes have two dates in them. This is because during these months, the dates the doomsday falls on change based on if the year was a leap year. The easiest way to find out if the year is a leap year is if the year mod 4 equals 0. So if we take the year from the fun fact, 1978 mod 4, it equals 2. So the year 1978 was not a leap year and because 1978 was not a leap year, the dates we use for January and February are the 3rd and 28th respectively.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Date	3/4	28/29	0	4	9	6	11	8	5	10	7	12

Now to do the calculations to find the doomsday follow these steps. Remember the doomsday is the day of the week that all the dates in the chart above fall on. We will work through the fun fact example after this.

- How many times does the number 12 go into, as a whole, the last two digits of the year?
- Now find the difference between the last two digits of the year and 12 times answer 1. The calculation looks like this, Last two digits - ($12\times$answer 1).
- How many times does 4 go into the answer from 2?
- What is the century anchor day?
- Add up the answers from 1, 2, 3, 4 and then take mod 7. Take this answer (it should be between 0 and 6) and find the corresponding day of the week. This is the doomsday for the year.

Using 1978 as an example the calculations go as follows:

- The answer is 6. 78 / 12 =6.5. So 12 goes into 78 as a whole 6 times.
- Now finding the difference, 78 - (12 x 6) = 6
- The number 4 goes into 6 only 1 time. So the answer is 1.
- The century anchor day for the 1900s is Wednesday which corresponds to the number 3.
- Now adding up the answers from the previous four answers, 6 + 6 + 1 + 3 = 16. We know $16 \equiv 2 \mod 7$. Now that we have 2 as an answer, we go find the corresponding day to the number 2 and it is Tuesday. This means that the doomsday for the year 1978 was Tuesday. All the dates in the doomsday chart fall on a Tuesday.

Step 4: Finding the day of the week

Now that you know the doomsday for the year 1978, it is pretty simple to find the day of the week a certain date is on. Let's again use the date in the fun fact, February 11, 1978.

- Find the closest doomsday date to known date.
- Find out how many days are between the two dates.
- Take that number mod 7 and then count that number of days from the century doomsday to find the day of the week.

Working through February 11, 1978 as an example:

- The closest doomsday to February 11 is going to be February 28th. It is February 28th because 1978 was not a leap year.
- There are 17 days between the two dates. $17 \equiv 3 \mod 7$.
- This is where is gets tricky. How do we know to count forward or backward from Tuesday? Because our doomsday date, February 28th, came after the original date, February 11, we will count backward 3 days. Three days from Tuesday, counting backwards, is Saturday. SO February 11, 1978 was a Saturday.

Problems

Find what day of the week these famous events fell on.

21. The day The Declaration of Independence was signed, July 4, 1776
22. The day Christopher Columbus set sail, October 12, 1492
23. The day Pearl Harbor was attacked, December 7, 1941.
24. The first man on the moon, July 20, 1969

Project Idea

- This project idea requires a background in computer science. Take the Doomsday Algorithm you just learned about and try to create a program that will take any date in as an input and return to the user which day of the week that date was on.

Number Sense

Logarithms and Exponents

You may have seen logarithms and exponents before, but it is possible you still don't understand exactly when they will be of use. We'll first look at an example that relates to repeated doubling.

Doubling Allowances

You and your mother are deciding on your allowance. Your mother is extremely generous, and allows you to choose between two different payment plans for the month. You can choose to take $100 a day, or to receive a penny on the first day and double this amount each day. That is, 1 cent on the first day, 2 cents on the second day, 4 on the third, and so on for the entire month.

1. Which option would you prefer? Why?
2. If you choose the first option ($100 per day), how much money will you make in the whole (30 day) month?
3. Using a calculator, fill out the following table for the doubling option.

Day	Formula	Amount
1	.01	.01
2	$.01 \times 2$	.02
3	$(.01 \times 2) \times 2$	.04
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		

4. If you choose the second option, how much money will you receive on the 30th day?
5. When is the first day where someone who chose the second option will receive more money than someone who chose the first option?

6. What if the first option gave you $1,000 per day? Which is a better choice then? What if is was $10,000 per day? Can you estimate what constant daily amount would make the first option a better deal?

We think of growth as multiplication– whatever input we have going into the day, we multiply that x by two to get the next day's allowance. But, when we are using repeated multiplication, we can use exponents to write a simpler version of that same expression. Instead of writing out $2 \times 2 \times ...$ we simply write 2^x, or "2 to the power of x".

If we want to use this function in calculations, we need to find a way to reverse it. In this expression the variable (that changes depending on the day) is the exponent, and 2 is a constant base. You have probably seen exponents before. Usually, though, we put the variable in the base, not the exponent. For example when we take the square root of a number we are finding the base that, when raised to the power of two, is equal to the original number. Today we are going to do the opposite, we are going to solve for the unknown exponents when the base is constant. So to reverse this equation, we can use log_2.

In other words, $2^a = b$ means the same thing as $log_2(b) = a$.

Using exponents and the inverse operation of log_2, we can answer questions about this type of growth.

7. Without using a calculator, compute $log_2(8)$ and $log_2(64)$.

8. Using exponents, write a general formula that tells you how much allowance you get on the nth day. Remember, $1 = 2^0$ (multiplying 0 copies of 2 together)

9. Can you rewrite this formula using a logarithm?

10. One day in the month, you will receive $163.84. Can you use logarithms to figure out which day?

11. What is the first day you will receive more than $6,000?

Interest Rates

The allowance example above may seem a little bit silly to you, surely your mom would never offer you either of the above schemes. Exponential growth, however, does have some very important uses in the real world. If you've ever heard of interest rates or compounding interest then you already know about one very important use of exponents and logarithms.

We will once again use a table to think about how exponents and interest are related.

In the first example we are going to look at how a $100 investment with 10% annual interest grows over time.

Note : 10% interest means that each year you earn 10% of what you had in the account at the beginning of the year. To illustrate, the first three lines are filled in for you. Also, if you would like to simplify the expressions, think about factoring out $100.

12. Using a calculator, fill out the following table.

Year	Formula	Amount
0	\$100	\$100
1	\$100×(.10)+\$100	\$110
2	((\$100×(.10)+\$100)×.10)+(\$100×(.10)+\$100)	\$121
3		
4		
5		
6		
7		
8		

13. How many years will it be before you double your initial \$100 investment?
14. Write down a formula for your investment (A) after t years.

    Hint : Remember that any number raised to the 0 power is 1, or $x^0 = 1$ for any number x.
15. Generalize your formula for the value of any investment (A) after t years with a starting amount P and an interest rate r.

Now let's think about an investment for which the annual interest or annual growth rate is unknown. Hopefully, you found the generalized formula: $A = P \times (1 + r)^t$.

16. You invested \$750 one year ago, you now have \$806.25. What was your interest rate?
17. You invested \$1,000 **two** years ago. You went to the bank today and found that you now have \$1,166.40 in your account, what was your annually compounded interest rate?
18. Your money doubled in 8 years. What was your annual interest rate?

    Hint : If you don't know how to complete the problem without a starting amount of money try solving it with two different starting amounts.
19. You have two options for \$100. You can invest your \$100 at a 6% interest rate, or if you promise to leave it in the bank for 5 years they will give you \$145 at the end of the 5 years.

    Which scheme is will deliver you more money at the end of 5 years, and which would you choose?

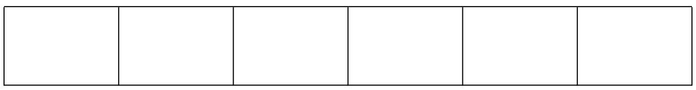
    What is the effective interest rate of the flat rate plan?

Number Sense

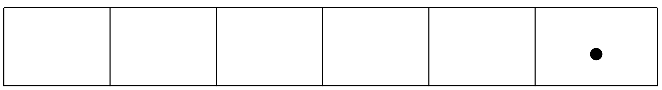
Changing Bases

1 ← 2 Machines

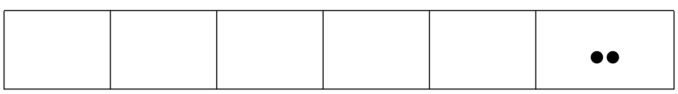
My friend invented a machine called a "two-one machine," or "1 ← 2 machine". Let's play with it and see what it can do. It doesn't look like much:

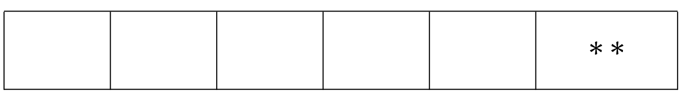


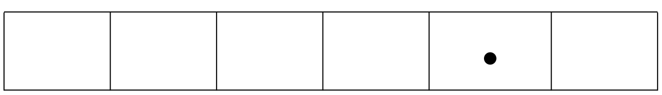
You can put dots into the box all the way to the right. One dot still doesn't look very exciting:



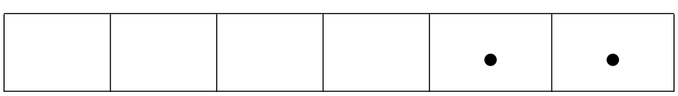
But when we try to put two dots in, they explode! This removes them from the box they're in and creates one dot in the next box to the left.

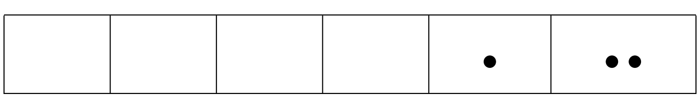


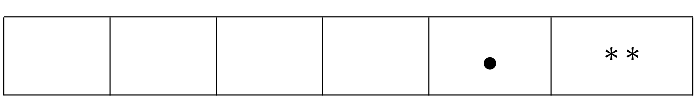




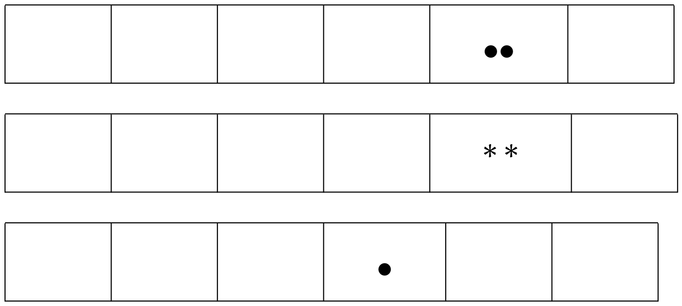
Nothing explodes when we add another dot, but start with four dots and all of a sudden:







First, the two dots explode into a dot in the second box. But then there are two dots in that box, so there's another explosion!



There can be as many boxes to the left as we want, but I only showed a few in this machine to keep it simple. We can also write the output of the machine (what it looks like after all the explosions happen) as numbers. Each place represents a square in the machine, and you write out how many dots are in it. So 1 dot in the rightmost box is just 1. 2 dots, after they explode, leave a box with 1 dot followed by a box with 0. This is written 10, squishing the boxes together. Similarly, 3 dots have one explosion leaving two boxes with one dot each so is written 11, and 4 dots is written 100. Did you notice we ignore all the empty boxes to the left? Otherwise you would have to write an infinite number of zeroes before you got to the part of the number you cared about.

1. Now keep going with the dots and fill out this table!

Dots	$1 \leftarrow 2$ Machine
1	1
2	10
3	11
4	100
5	
6	
7	
8	
9	
10	

2. How does this machine write 13?

3. What about 21?

4. One dot in the farthest right square represents exactly one dot. But one dot in the next square over actually represents two dots. A single dot in the third square signifies four dots before they exploded. Keep going to the left and determine how many dots are represented by a single dot in each square of this machine.

				8	4	2	1

5. These follow a very precise pattern - what is it? (Hint: think of the square all the way on the right as square zero)

6. A dot in the third square from the right (which we labeled as square 2) represents 4 dots, which is another way to say that 4 is the first number needing at least three squares. What is the first number needing at least 10 squares?

$1 \leftarrow 3$ Machines

Now let's take a similar, but very slightly changed, machine. Instead of exploding when there are two dots in a square, it will wait to explode until it has three dots together. So one dot is still 1, but two dots won't explode and will instead remain 2. Now the excitement happens when we have 3 dots:

					• • •

					* * *

				•	

7. Fill out the table for the $1 \leftarrow 3$ machine:

Dots	$1 \leftarrow 3$ Machine
1	1
2	2
3	10
4	
5	
6	
7	
8	
9	
10	

8. What is 13? 21?
9. How much does a dot in each square represent? Finish the table.

						3	1

10. What is the pattern between boxes?
11. How many boxes do you need to write 35?
12. What's the first number to need 10 boxes?

$1 \leftarrow 10$ Machines

One more machine! This one doesn't explode until there are 10 dots in a box together.

13. Fill out the table for the $1 \leftarrow 10$ machine:

Dots	$1 \leftarrow 10$ Machine
1	1
2	2
3	
4	
5	
6	
7	
8	
9	
10	

14. What is 13? 21? 42? 100? 9387? Don't draw all 9387 dots.
15. How much does a dot in each square represent? Finish the table.

							1

16. What is the pattern between boxes?
17. What does the $1 \leftarrow 10$ machine do? Does it look familiar?

Machines

This last machine probably looked really familiar - numbers are written the same way before and after the machine, and each number needs as many boxes as it has digits. This can also be called a "decimal" machine. The other two machines also represent systems for counting, although they may be less familiar. The first machine, the $1 \leftarrow 2$, corresponds to a language called binary that is used by computers. Each digit, called a bit, is either zero or one, because at their most basic level the computers only respond to whether a signal is on or off.

Project Ideas

- What would a 2 ← 3 machine do? A 2 ← 1 machine? How about 1 ← 1?
- Algebra extension: It is also possible to construct an 1 ← x machine. You don't know how many dots it takes before it explodes, but you can use it to write phrases such as $3x^4 + 10x^2 + 5$. You can also create anti-dots (what should they look like?) for negative numbers - an antidot will cancel a dot in the same box. What can you do with these machines?
- Negative and fractional numbers make things even more complicated when you move to other bases. Do some research into different ways to represent these sorts of numbers in binary.
- Other languages have used a variety of non-decimal bases. What are some examples? Is it known why they chose the base they used?
- Roman numerals are notable for lacking a zero. Why is this so important? When was the concept of zero invented?

Thanks to James Tanton (`www.jamestanton.com`) for sharing his exploding dot machine. If you want to see more, explore his YouTube videos and website!

Part II

Geometry

Geometry

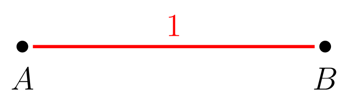
Construction I

Euclid is known as the "Father of Geometry," as his book (Elements of Geometry) is the foundation of all the geometry learned even today. However, he didn't have all the tools we have now. They had compasses, allowing them to draw circles, and straight edges to draw straight lines, but that's all! Because they didn't have a concept of zero or fractional lengths, they didn't have rulers or protractors. If you can't measure things, how do you know they're the same length? Without angle measures, how can you draw a square? Can you cut an angle in half? What about a line? Let's find out!

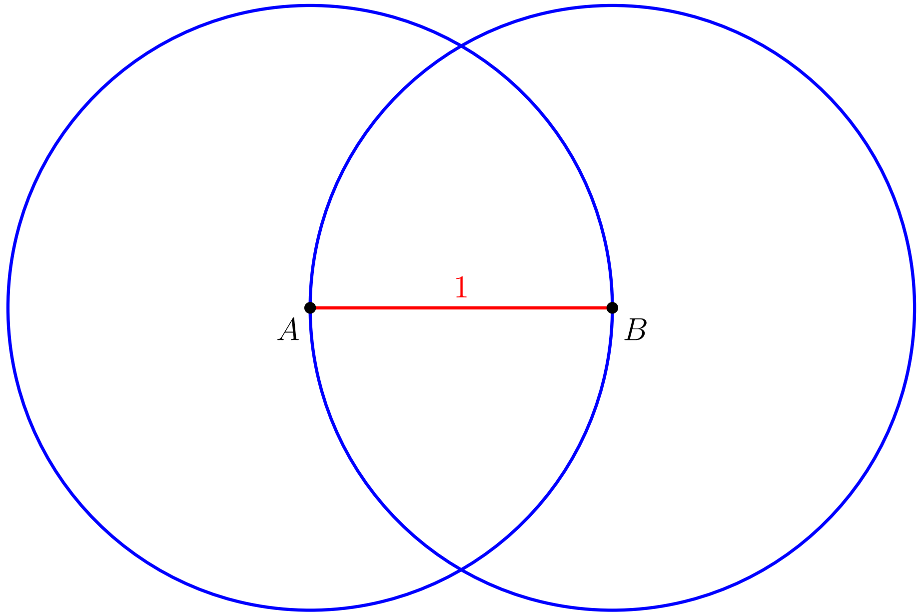
Equilateral Triangle

Follow along with the drawings in the pictures - it will prepare you to do constructions yourself. Let's start by constructing an equilateral triangle from just two points. Draw two points, A and B, as shown directly below. What can you do with them?

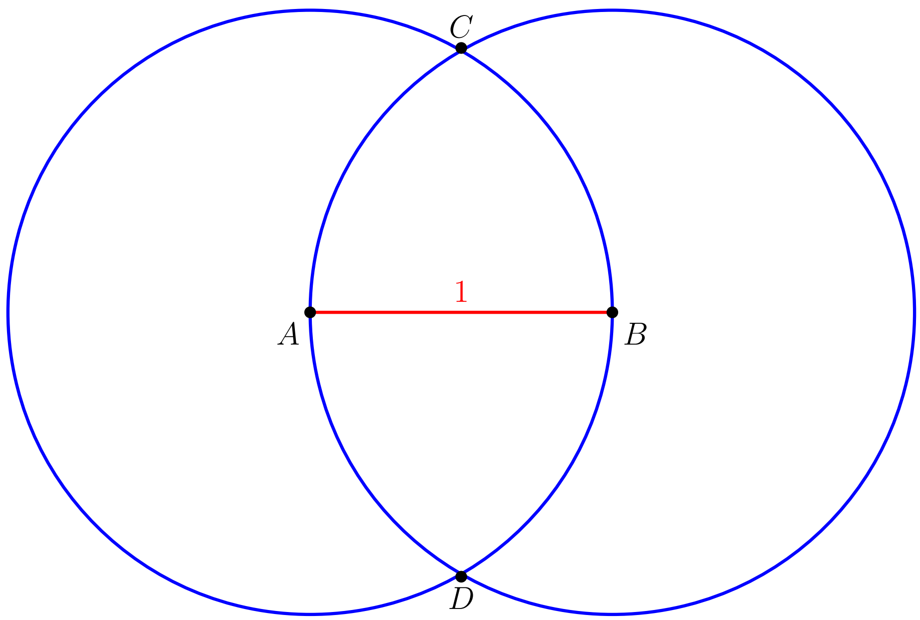
An easy first step is connecting them, yielding line segment 1.



That used the straightedge, but we also have a compass at hand. Remember that we can't measure anything with numbers; what sort of meaningful circles could we make with only the information in the picture? Any circle has two important pieces - a center and a radius. In this picture, what are the logical choices? Since we have two points and one line segment, it is probably the most obvious to use the points as our centers and the line segment as each circle's radius.



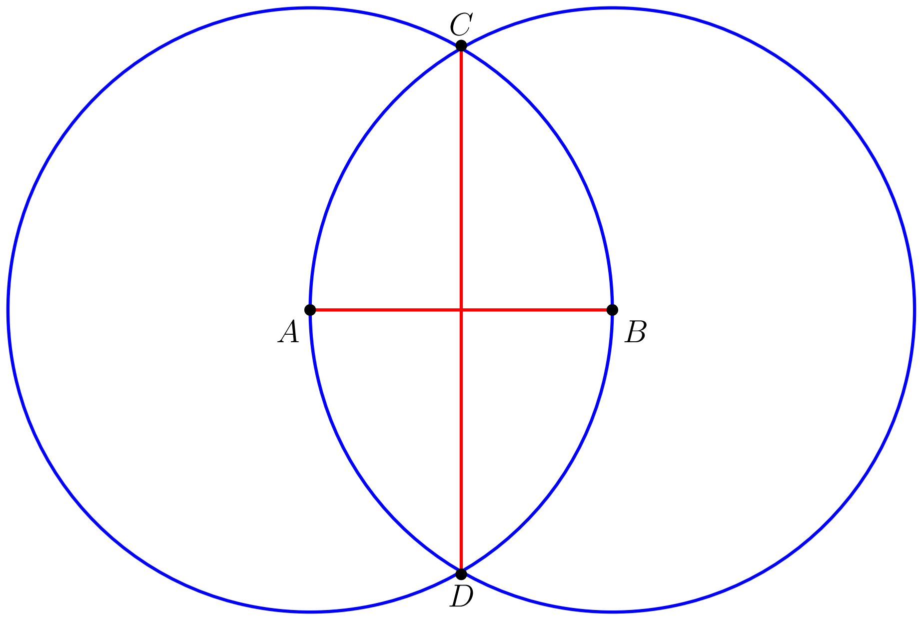
When we draw these two circles, two more points stand out as being important: the points where these circles intersect. We'll label them C and D.



Can you see how to build an equilateral triangle using the points we have labeled? Write a few sentences explaining how you know each side is equal in length. Don't measure them!

Right Angles

If we can draw a triangle, our natural next step is to try to draw rectangles. To do that, we need to figure out how to make a right angle. If you started with two points A and B, how could you draw a line perpendicular to the line AB? Let's take a look at the previous picture, but with one more line added.



Those lines sure look perpendicular! But we don't want to rely on our intuition– an 89 degree angle looks an awful lot like a right angle. Do you think they really are perpendicular? Also think about whether the line segments on either side of this perpendicular are equal – is it a perpendicular bisector? Write a few sentences explaining why or why not and how you know.

Center of a Circle

Have you ever drawn a circle and not been able to find the center? Draw a circle by tracing something round. The bottom of a bottle, cup, or lid, a large coin, a rolled-up poster, or any other round object can work. Don't use your compass for this - then you already know the center! Instead, try drawing a line which intersects the circle in two places. What can you do with this line? If you get stuck, try looking up properties of chords. Once you finish your construction, use a few sentences to explain how you know it must be the center.

Squares

Now that we know how to construct a right angle using compass and straightedge, we can try to build a square! Remember, a square is a shape with 4 equal right angles, and all four sides equal in length. What are some ways that we can be sure two line segments have equal length? How can you use the right angle construction you know to draw a line perpendicular to a specific point? Can you construct a square using the techniques we have learned so far? Be sure to write a few sentences explaining how you know it must be a square.

Project Ideas

- Look into the history of construction a bit - why didn't Euclid measure any of his line segments? When did people start assigning lengths to lines and degrees to angles?

- What other shapes can you construct using a compass and straight edge? What about hexagons? What about stars?
- There are many variations on constructions. Do some research into the Mohr-Mascheroni theorem (compass only), the Poncelet-Steiner theorem (ruler only), or Neusis constructions (rulers with marks).

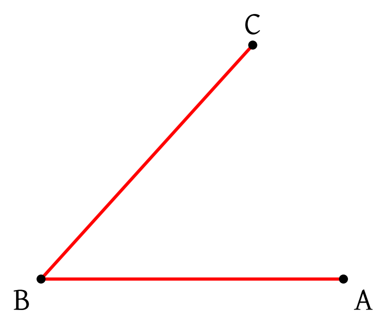
Geometry

Construction II

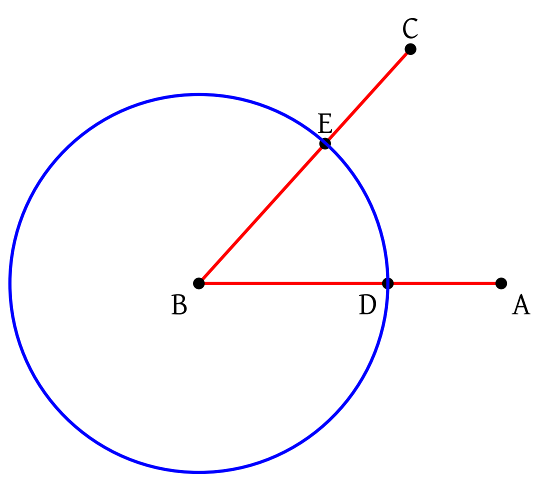
You might be wondering what these ruler and compass constructions are good for. And that's a pretty good question! They were very important to Euclid but for us these constructions are just a game– we set the rules and then follow them. But maybe you're not a huge fan of the compass and ruler. What are some other construction games we might play? We can explore this question by finding several different ways to make the same construction.

Bisecting an Angle

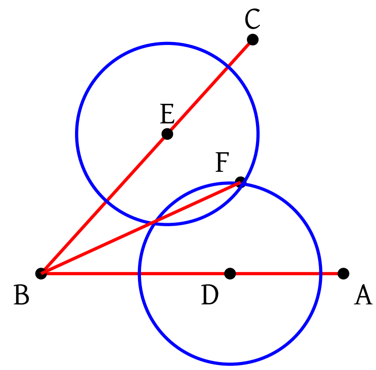
Remember that none of these constructions use protractors, so you can never know exactly how big your acute and obtuse angles are. How can you cut something in half without measuring it? Let's start with an angle ABC.



Now choose a convenient radius for your compass and draw that circle centered at B. We'll call the points where the circle intersects the angle D and E.



Now draw a circle centered at D and a circle centered at E; make sure these circles have the same radius! Even though we can't measure how long that radius is, the compass can "remember" the distance to draw another circle. These circles should intersect at a point inside the angle (call it F). Connect B and F to yield equal angles ABF and CBF.



How do you know that these angles are equal? Write a few sentences explaining why.

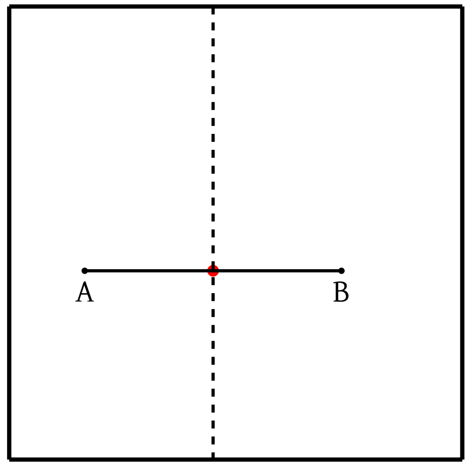
Marked Ruler

We know how Euclid would have solved the problem, but there are still other methods to explore. For instance, we have rulers with measurements on them! Could we still bisect the angle if we had no compass but added marks to our straight edge? Write a few sentences explaining how you could do this.

Paper Folding

One way that Euclid was limited is by the absence of paper, which was not available in Greece at the time. (**Fun fact:** Paper was invented on the other side of the world in China during the Han dynasty.) But we do have paper, and by using the properties of paper folding, we can construct the same things in a very different way. We can either make a fold with two points on it (this is the same as creating a line) OR we can make a fold which "matches up" two points (that is, when the paper is folded up, these two points will be pressed against each other). One thing that is very easy, then, is bisection!

Imagine that you have a line drawn on a piece of paper, and you want to find the midpoint. Now that we can "pick up" the paper, it's simply a matter of folding the paper so that the points A and B are on top of each other, and then making a crease (represented by the dashed line) along this fold, as pictured below. No more circles and circles to find the perpendicular bisector. Try it! Then write a couple sentences explaining how you know that the midpoint of the line must fall along the crease. Then, think about the angle formed by this crease and the line AB. (It may be useful to think about what would happen to a non-right angle when reflected across the crease by folding.)



Can you extend this line bisection to think of how to bisect an angle? If you are given an angle, how could you create a crease that divides the angle in half? What points should you try to match up with the fold? Give it a try!

Now, think back to the other constructions you've done with the compass and straightedge. Can you figure out how to create a square by creasing a blank piece of paper? What about an equilateral triangle? Can you think of any shapes that you can draw with a compass and straight edge that you couldn't draw using paper folding? After experimenting with the paper, write a paragraph comparing the benefits and drawbacks of each construction method.

Project Ideas

- Some constructions are actually impossible with compass and straightedge. In this activity we bisected an angle. So it doesn't seem like trisection (dividing it into three equal parts) should be that much harder. People tried and failed to complete this construction for hundreds of years before Pierre Laurent Wantzel proved it impossible in 1837. However, it is possible to trisect an angle using paper folding. Can you figure out how? If you get stuck, this website has step-by-step instructions: `https://www.math.lsu.edu/~verrill/origami/trisect/`
- Look up some other construction rules! Among other possibilities, matchstick construction simply assumes that we have a box of identical matchsticks which we can arrange into shapes. In other words, you can draw line segments that you know are exactly the same length. Once you've chosen a set of construction rules, write instructions for some constructions using those rules. Compare and contrast these with other constructions you have seen.

Geometry

Möbius Strips

In 1858, two German mathematicians (August Ferdinand Möbius and Johann Benedict Listing) discovered an oddity of mathematics. This exercise explores the Möbius strip and some of its unexpected properties.

Start by cutting out two narrow strips of paper. Take one strip and join the ends using tape. Then, make a second loop. This time, add a half-twist to the strip by turning one end upside down while keeping the other one steady. Then tape the ends together as before (see images).

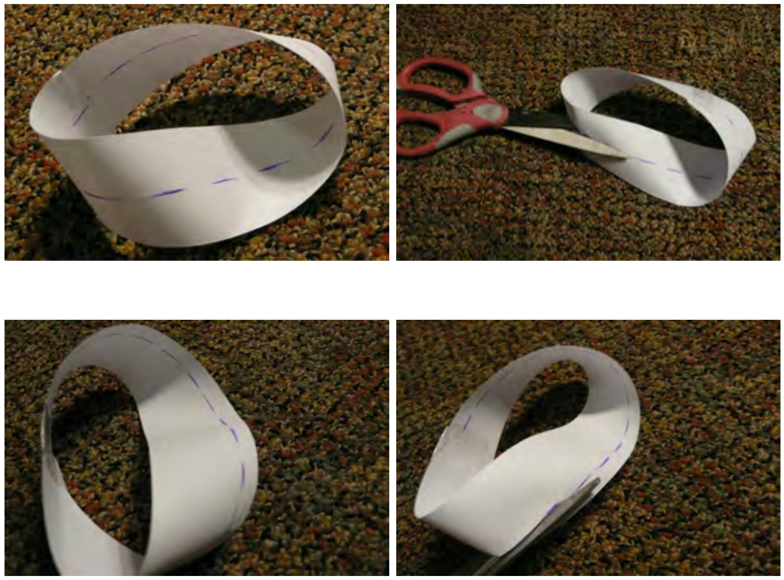
1. How many sides (i.e. flat surfaces that you could write on) does the strip with no twist have?
2. Trace along an edge of both strips with your finger. How many distinct edges does the twisted strip have? How many edges does the untwisted strip have? If you're having trouble figuring this out, try holding one edge and tracing along what might be a different one to see if your fingers intersect.
3. Draw a line around the outside of the untwisted strip. Now try to do the same thing with the strip with a twist in it. What happens?

The strip with a twist in it is called a Möbius strip. As you have seen, it only has one side, unlike the untwisted cylinder strip, which has an inside and an outside. In the strip without a twist, you can trace your finger around either the inside or the outside, but you need to lift your finger off the surface to move from the inside to the outside on the untwisted strip. With a Möbius strip, though, we only have one side - if you try to trace just one side, you will end up on the other! Although it is easy to make, this object has some very wild properties.

Cutting the Strip

Take your Möbius strip from earlier, and cut along the line that you drew. If your strip's line doesn't look like the following, near the middle of the strip, you might want to make a new one to cut.

4. What object do you get? Can you tell how many "twists" are in it? (You might have to untape the shape to figure it out!)
5. Write down your prediction of what happens if you take the new shape you've made and cut that in half. Try it!
6. Was your prediction correct? Why do you think it happened the way it did?



7. Using a new Möbius strip, try cutting in a similar way, but with your line closer to one of the edges (about 1/3 of the way across, see images below as a starting point). What do you think you will get if you cut along this line? Try it!

8. What happened? If you stopped cutting before getting back to where you began the cut, go back to the Möbius strip and finish cutting back around to where you started. What are the two shapes that resulted? How many twists do they have? Can you tell how they used to fit together to form the whole strip? Write a paragraph explaining the difference between cutting the Möbius strip at the middle versus along a line closer to the edge.

As you've seen from these first explorations of Möbius strips, the "twistiness" of the strip is very important. When we have no twist, the strip is a simple cylinder, but when we add half a twist, it is transformed into a Möbius strip. But weirdly, we can create new shapes from our Möbius strips that have a different number of twists than the strip they came from. A natural question to ask might be what happens when we twist the strip even more times before taping the ends together.

9. What happens if we put a whole twist (2 half-twists) in your paper before connecting the ends? Try making strips with 2 and 3 half-twists. You might need to use longer paper or thinner strips. What do they look like? What happens when you cut them in half using the same method as above? Can you guess what happens with 4 or 5 half-twists? Investigate using the following chart, then write a paragraph discussing your findings.

Number of half-twists	number of sides	result of cutting in half
0	2	two cylinders (zero twists)
1	1	
2		
3		
4		
5		

Extensions

- Does it matter which direction the twist is? Can you make two Möbius strips that are mirror images of each other?
- What happens if you cut a Möbius strip in half and then cut it in half again? Again? Do you see any patterns? If you have access to butcher paper (or other large sheets of paper), get a friend to help you make a really large Möbius strip for easier cutting!

Project Ideas

- If you cut a strip of paper like this:

  and tape a to d and b to c, then cut around the hole you just formed (that is, cut from a around the semicircular cutout to b on to c and the rest of the way around to d), you will have a ring with a bigger hole and a small ring. What happens if you twist one or both of the ends before taping them? That is, perhaps you twist d before attaching it to a. Does it matter which way you twist them? What other things could you do with this shape?
- What would happen if you wrote a sentence on your strip before you taped it up? Try drawing a Möbius picture or writing a Möbius story, poem, or comic!

Recommended Reading

- The Magical Möbius strip, by Julia Collins:

  `http://www.maths.ed.ac.uk/~jcollins/MobiusInstructions.pdf` has further extension activities, plus diagrams
- *The Möbius Strip: Dr. August Möbius 's Marvelous Band in Mathematics, Games, Literature, Art, Technology, and Cosmology*, by Clifford Pickover includes lots of extensions and applications of these strips
- The Möbius Puzzle Challenge: `http://plus.maths.org/content/puzzle-page-12` has further discussion of dissecting Möbius strips and more explanation of why it works the way it does

Geometry

The Pythagorean Theorem

Fun Fact: Did you know that President James Garfield, our nation's 20^{th} president, has been credited with finding a proof for the Pythagorean Theorem?

The Pythagorean Theorem is one of the most useful theorems in geometry. You've probably heard it before, but here is a reminder: the Pythagorean Theorem states that $a^2 + b^2 = c^2$ where a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse. Most students believe this theorem is true solely because a teacher or a book told them so. But how do they know? Why should you trust them? In this activity you are going to find out for yourself how we know, unmistakably, that this is true.

A Physical Proof

In this proof, you will be led step-by-step through a construction illustrating the Pythagorean Theorem.

1. Pick a corner on your piece of paper. On one edge, make a mark 3 inches from the corner and on the other edge make a mark at 4 inches. Now draw a line that connects the two marks and cut along the line. You should now have a right triangle with side lengths of 3 inches, 4 inches and 5 inches.
2. Now cut out three squares from each of the remaining three corners of the paper. For each of the squares, use one of the sides of the triangle as the length/width of the square. This should produce three squares: one with its sides equal to 3 inches, one with sides equal to 4 inches and one with sides equal to 5 inches.

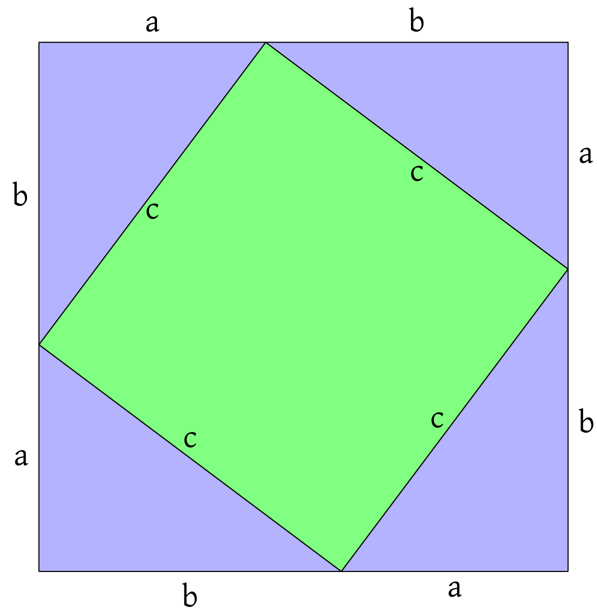
At this point it is important to note that the area of each square is equal to the length of the corresponding triangle leg squared. For example, 3 is the length of one of the legs of the triangle and the square with 3 inch sides has an area that is equal to 3^2.

To prove the theorem, we want to show that the area of the largest square is equal to the areas of the smaller squares added together. This is a visual representation of the Pythagorean theorem.

3. Work with the three squares and see if you can fit the two smaller squares perfectly into the larger square. You can cut the squares, but be careful not to make too many pieces. You do not need to make more than 5 pieces out of the two smaller squares.
4. Experiment with moving the pieces from the smaller square around and see how many different placements exist.

Now you've shown (for this triangle) that the sum of the legs squared is equal to the hypotenuse squared. This is a good way to visualize the Pythagorean theorem. However, we have not shown that this works for every right triangle, only that it works for this particular right triangle. Ultimately we would like to create a proof that can be applied to any right triangle.

An Algebraic Proof



To find a proof that can be applied to any right triangle, we will use the diagram above. This diagram was formed by placing four copies of the right triangle with side length a, b, and c as shown above. This is what creates the outer square, while the inner square is created by the hypotenuses of the four triangles. We know the inner shape is a square because all of the sides are the same length and the angles are 90°. The angles of the inner quadrilateral are 90° because the two smaller angles of the triangle are complementary, while those angles and the quadrilateral angle are supplementary. To prove the Pythagorean Theorem, $a^2 + b^2 = c^2$, we need to examine the area of this square.

5. Find the equation for the area of the whole square as usual, using the formula A = s^2.
6. There is also a different way to find the area of the whole square. This time, derive an equation for the entire square by adding the areas of the blue triangles to the area of the green square.
7. These two equations are equal to each other because they both represent the area of the entire square. Set them equal to each other. Can you simplify this expression?
8. What do you notice about this simplified equation?

You have just discovered another proof of the Pythagorean Theorem!

Extensions

In this extension, you will use the above proof using squares to work our way up to a generalized proof by putting a shape other than a square on the three sides of our given right triangle. The first shape we are going to use is a right triangle.

1. We need to make similar triangles in the same proportions as the original triangle. To do this, use each side of the original triangle to measure the hypotenuse of a new triangle. Try to make the angles on the new right triangles match each other. A good way to do this is to attempt to make the legs of the triangle the same length. Before cutting these three triangles out, check that the angles of the triangles are identical. If you attempted to make triangles with same length legs the angles should be 45°, 45°, 90°. As before, creating the 90° angle with the corner of the page is easiest.

2. Once the three similar triangles have been cut out, try to fit the two smaller triangles into the larger one like you did with the squares.

Now that you have done this with squares and right triangles, try to do the same thing with more difficult shapes like semi-circles or pentagons. Amazingly, this will work for any shape that is put on the side of the triangles and it uses the Pythagorean Theorem! Go to this Wikipedia page to find a good explanation of the generalized proof. `http://en.wikipedia.org/wiki/Pythagorean_theorem#Similar_figures_on_the_three_sides`

The amazing part is that these are not the only ways to prove this theorem. There are a ton of other proofs. We challenge you to find and research more proofs of the Pythagorean Theorem. Who knows, you might find something that is even cooler than the ones we just discovered! Use this website to get you started. http://www.cut-the-knot.org/pythagoras/.

Project Idea

In this exercise we only scratched the surface of the Pythagorean Theorem. The point of this project is to learn more about this theorem and the man who discovered the most famous proof for it, Euclid. Below we have some ideas for projects inspired by Euclid.

1. Research and learn about Euclid and how he was involved in the history of math.

2. Discover Euclid's axioms of geometry. Write at least a paragraph explaining what an axiom is and why we need Euclid's axioms for Euclidean geometry.

3. Find Euclid's proof of the Pythagorean Theorem. Study it and make sure you understand it well enough to explain it to someone else.

4. Put together a presentation about Euclid: his life, his axioms, and his continued impact on math.

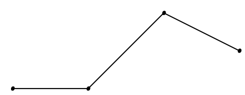
Geometry

How Far Can a Robot Reach?

An Introduction to Robot Arms

Have you ever been asked to reach an item on the top shelf, but just couldn't? Did you wish your arms were just a little bit longer? But, where would you want the extra length? In your forearm, hand or upper arm? If you change the length of one part of your arm you might be able to reach the top shelf, but surprisingly you may not be able to touch, reach, grab, access things you could before.

Have you ever thought of all of the possible spaces that your arm can reach? In this activity we are going think about the reachability region of simplified robot arms to help understand some of the geometry of our bodies.



The easiest way to represent robot arms is with each portion as a line segment (as seen above). The line segments meet at points that represent joints (like your elbow). When we draw the arms we assume that each segment can rotate 360° around its vertex (i.e. in a circle).

1. Begin by drawing two connected line segments. The first, 2 inches in length and the second, 1 inch. It is easy to find the set of points that the robot arm can reach while fully extended. The end of the shorter segment is considered the 'hand', the longer segment is attached to the body.
2. Now keep the inner line segment fixed and let the outer segment rotate around the point of connection 360°. What do you observe? What does this mean in terms of the reachability region?
3. We only need to do one more step to diagram the entire reachability region of this robot arm. How would you go about this? Do you know the area of the that you created?
   (a) To find the area of the washer, think about the space that we can't reach (the innermost circle in your diagram). Find the area of that circle. (Hint: First you need to think about what the radius of the circle is, use your diagram for help).
   (b) Now think about the reachability region and the inner circle together. What shape do they create together? Can you find the area of this space?
   (c) Just like subtraction with numbers, we can subtract shapes. In this case, we are starting with a large circle and subtracting from it the smaller, inner most circle to create our washer.

More Robot Arms

The next exercises will challenge you to take the basic geometry of robot arms and apply them in new ways.

Does Order Matter?

We will start by drawing another reachability diagram of the same robot arm. This time though, we will imagine it is anchored at the other end (the first segment is 1 inch and the second 2).

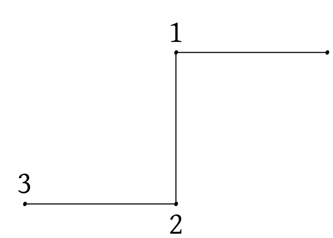
4. This time draw a 1 inch line segment first and then a 2 inch line segment.
5. Map the reachability region in the same steps as above: with both segments fixed, with only the first segment fixed and finally with both segments rotating.
6. Describe the shape of the reachability region of this robot arm. (Hint: look above for vocabulary ideas.)
7. Find the area of the reachability region.

Is it Always a Washer?

We are going to draw another reachability diagram. This time, though, we will change the length of the arms.

8. This time use two 2 inch segments to create the arm.
9. Map the reachability region in the same steps as above: with both segments fixed, with only the first segment fixed and finally with both segments rotating.
10. Describe the shape of the reachability region of this robot arm. (Hint: look above for vocabulary ideas.)
11. Find the area of the reachability region.

What about More?



We've started with relatively simple robot arms of only two segments, but they can be far more complicated. Think about a mechanical arm to package candy in a factory. It performs complex tasks and likely has many components. To further our understanding we are going to start mapping more complicated arms. In the diagram above, the arm is anchored at point 3 (and can rotate around 3), and points 1 and 2 are joints which the segments can rotate around.

12. This time draw three segments of length one inch each as your robot arm.
13. Begin with the arm fully extended. It will look like a straight line with two points that divide the line into three segments.
14. Start the diagram by rotating the final segment of the arm around point one. In other words, do not allow the arm to rotate around point 2 or 3 in the above diagram. Keep the other two segments fully extended.
15. Now allow the arm to rotate at points 3 and point 1. You will repeat the pattern you observed in step 3 as the entire arm rotates.
16. Fix the anchor point (3) with the arm fully extended again. Now allow the arm to rotate around point 2, and keep point 1 fixed.
17. Allow the arm to rotate at points 2 and 3. Repeat the pattern from step 5 as you rotate around the anchor point.
18. You should now have the entire reachability region mapped, what does it look like?
19. Can you draw an arm with only two segments that has the same reachability region as the arm with three one inch segments?
20. Find the area of the reachability region.

Extensions

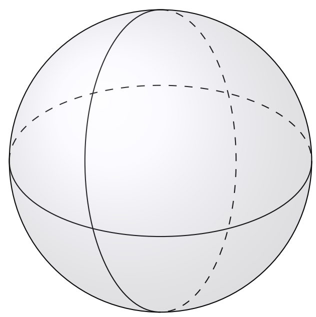
If you just can't get enough of robot arm geometry, we encourage you to attempt these variations of the above problems.

- Take the problem that involves three line segments. Instead of having three line segments of equal lengths (like above), make a robot arm with three segments of different lengths. Go through the same process above (drawing out the diagram) and calculate the reachability area of this robot arm.
- If you are looking for an even harder challenge, try to find the reachability of a robot arm that has four line segments. First try an arm with four segments of equal length, then repeat the exercise with segments of different lengths.

Geometry

An Introduction to Spherical Geometry

Over these last few exercises we have been working in Euclidean geometry. Did you know that mathematicians study other kinds of geometry as well? Today we are going to explore a type of non-Euclidean geometry, spherical geometry. Spherical geometry is geometry on the surface of a sphere. Spherical geometry has many practical applications, most importantly it was the basis for navigation and astronomy.



There are many similarities that we can see between these two types of geometry. The first similarity deals with points. Instead of plotting the points in a 2 dimensional plane, we are plotting the points on the surface of the sphere. Let's work on comparing other concepts in Euclidean geometry to spherical geometry. To do this we are going to need an object that resembles a sphere that we can draw on. A tennis ball and a dark marker will work just fine.

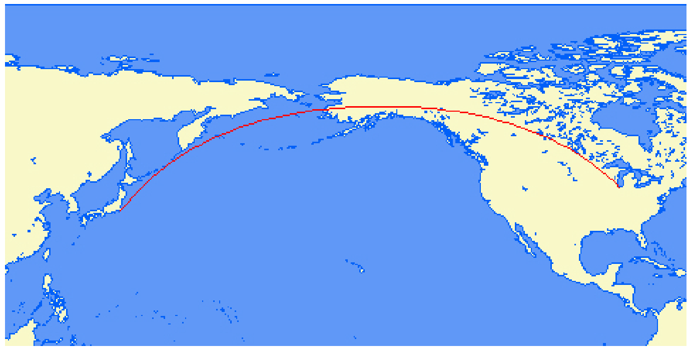
1. Take our sphere and draw two points anywhere on the sphere; the farther apart, the better.
2. Now connect the two points in a straight path.
3. Describe the shape that you just drew to connect the dots.
4. Do you think that this is the shortest distance between these two points on the sphere?

To better help us visualize the shortest distance between two points on a sphere we are going to look at the flight patterns of American Airlines, specifically from Chicago to Tokyo.

5. How do you expect the path from Chicago to Tokyo to look?

   Now look at Figure 1 to see if you were correct.

6. Does the path from Chicago to Tokyo look the way you thought it would?
7. How is it different from what you expected? (Figure 1)



Chicago (O'Hare airport) to Tokyo (Narita airport) flight path[2]

3D view of Chicago Narita flight path[3]

Now look at the 3D view of the flight path in Figure 2. This is a different way to visualize the same path as in Figure 1.

8. Now we are going to do this ourselves. Take your sphere and draw two points in the approximate locations of Chicago and Tokyo. These two points lie nearly on the same line of latitude.

9. First connect these two points, like we have in Figure 2, with a line that goes around the top of your sphere.

10. Then connect these two points using the line of latitude that they lie on.

11. Which of these distances do you think is shorter? Take a piece of string and use this to measure each distance.

We just discovered that the shortest flight path from Chicago to Tokyo involves flying over Alaska. This seems to be a little odd but it is because of how lines are drawn in spherical geometry. Just as in Euclidean geometry, a line is still the shortest distance between two points. A spherical line, the shortest distance between two points on a sphere, is a special arc called a "geodesic,"

[2]http://gc.kls2.com/cgi-bin/gcmap?PATH=ord-nrt&PATH-COLOR=red

[3]https://mattinjapan.wordpress.com/2008/07/04/flight-jl0009/

which is an arc of a great circle. Great circles are the largest circles that one can draw on a sphere; they share the same center and radius as the original sphere. The equator is an example of a great circle. The flight path from Chicago to Tokyo goes over Alaska because Alaska, Chicago, and Tokyo all lie on the same great circle. Therefore the path over Alaska is the shortest distance between Chicago and Tokyo.

Extensions

Now that we have defined what points and lines are in spherical geometry, let's investigate other shapes on a sphere as an extension.

- The first shape to investigate would be what is called a lune. Find out what is special about lunes and learn about how to find their areas. Here is a website to help you get started: `http://math.rice.edu/~pcmi/sphere/gos3.html#1`.
- Triangles and rectangles are interesting on spheres.
  - Get a protractor out. Draw two triangles of different sizes on your sphere. Remember, to draw a triangle on a sphere, you place three points, and you connect each pair with a line on the sphere. That is, you connect the points with geodesics.
    1. Fill out the table below with the angle measures of each of the triangles.

	Triangle 1	Triangle 2
Angle Measure 1		
Angle Measure 2		
Angle Measure 3		

    2. What do you notice about the sum of the angle measures for the spherical triangles?
    3. How does that compare to what you know about the angle sum of a triangle in the plane?
    4. Use the internet to find 3 more differences between regular triangles and spherical triangles.
    5. Now investigate the angle sums of spherical rectangles.
    6. How would the Pythagorean theorem be affected on a sphere? Would it remain the same?
    7. For even more information, use the website suggested above for lune research to read up on spherical triangles.

Project Idea

Below are some ideas for exploration of the relationship between maps and spherical geometry. Maps are intricately related to spherical geometry because the map maker is forced to make a decision about how to portray spherical shapes in 2-dimensions. As we have seen in this activity things can look very different on a sphere from what you would expect them to when you look at a map. The map itself can alter your expectations about what the world actually looks like. Challenge yourself to explore as many of the suggestions below as you wish.

- Investigate popular map projection types, find at least four different projection types. Suggestion: Start with the Mercator and Gall-Peters projections. Describe how the projections are created.
- Research and describe the limitations/distortions of each of the projections you discovered.
- Investigate the math behind a simple cylindrical projection. Understand the math so that given a latitude and longitude you can pinpoint the location on the projection.
- Using a globe and coordinates of major cities, create a simple cylindrical projection of the world.

Part III

Patterns

Patterns

Math in Nature

Fractals

Fractals are infinitely repeating patterns that can be created through infinite repetition of a single process[4]. If you have a true fractal, you could zoom in as much as you wanted and still see the same pattern. Although nature rarely develops true fractals, there are many natural occurrences of fractal approximations.

Branching in ferns is a common example, as the frond pattern is repeated at two or three different levels (this fern shows three levels of the fractal pattern). Lightning and other electric currents also create branching fractals, often with many more levels of repetition than ferns have. The image below, called a Lichtenberg figure, was created with high voltage electricity on a plexiglass surface. More information is at `http://www.capturedlightning.com/frames/lichtenbergs.html`.

A fern showing a fractal-like pattern[5]

"Captured lightning" in plexiglass. [6]

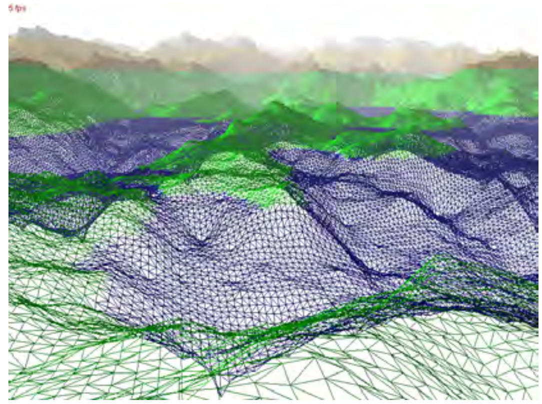
1. How many levels of branching can you see in the plexiglass?
2. What are some other examples of branching fractals in nature?
3. Can you think of any non-branching fractals, like the fern? How many levels do they have?
4. Do you think natural branching or non-branching fractals tend to be more common? Which do you think tend to have more levels? Why?

Landscapes can also be thought of as fractal-like. In fact, many movie landscapes are generated digitally using fractals. A simple version of a mountain-landscape generator is the diamond-square algorithm. In the diamond-square algorithm each of the four corners of a square are assigned random height values. The square is then divided into four new squares. Each corner

[4]There is more information on fractals in the investigating infinity activity.

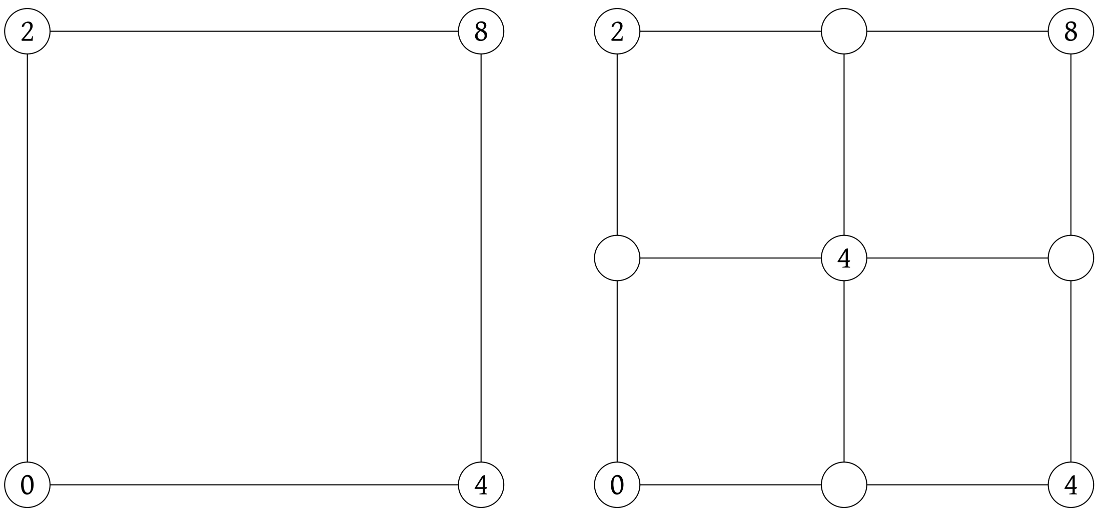
[6]`http://en.wikipedia.org/wiki/Fern#/media/File:Fern02.jpg`

[6]`http://upload.wikimedia.org/wikipedia/commons/5/55/Lichtenberg_figure_in_block_of_Plexiglas.jpg`

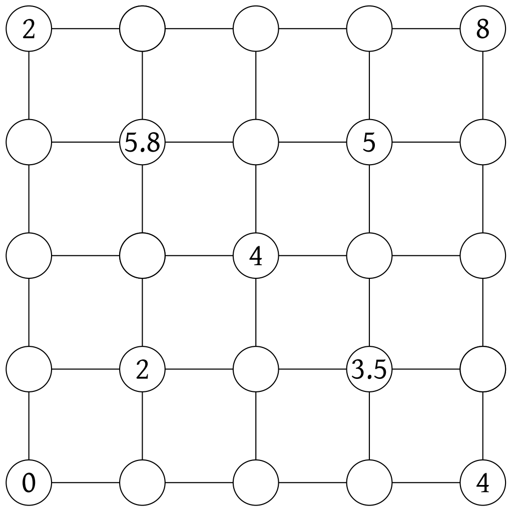


A computer generated fractal landscape.[7]

of these new squares is assigned a height determined by averaging the heights it is adjacent to. The center height is the average plus a random number from -1 to 1. This random stage assures that the terrain looks believable. Below is a square for you to practice with. All of the values requiring random numbers are provided. You should fill in the rest.



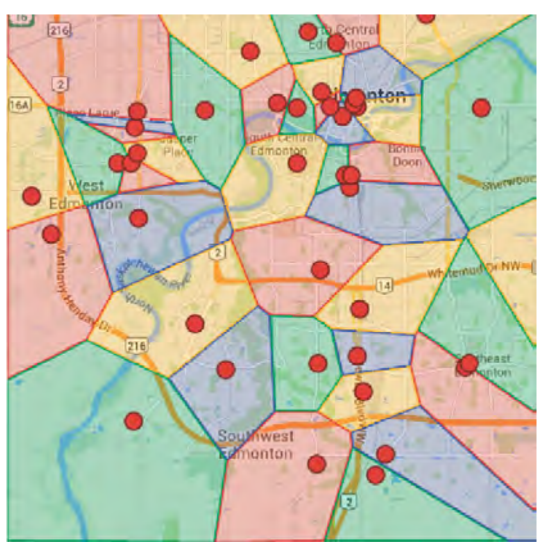
[7] `https://www.castledragmire.com/Projects/Fractal_Landscape`



Voronoi

What do soap bubbles, a giraffe's spots, John Snow's analysis of the 1854 cholera outbreak, and your GPS all have in common? They can all be described by Voronoi diagrams! In the late 1800's, Russian mathematician Georgy Voronoy defined these diagrams during his work on parallelohedra and infinite fractions. Let's say you ask your GPS to find you the nearest coffee shop. It probably looks for all the coffee shops in the same city as you. If you're in Minneapolis, that's probably a lot of coffee shops! How does it know which one is closest?

That's when the Voronoi diagram is used! If you have a bunch of points on a plane, a Voronoi diagram divides the plane into regions based on the nearest point. In this case, each of the points is a coffee shop and everywhere in that region is closer to that coffee shop than any other. For example, the figure below shows Starbucks locations in Edmonton, Canada. If you were driving along Highway 216, just about to cross the North Saskatchewan River, you would want to go north and east, as you are in the yellow region.



Voronoi Diagram of Starbucks locations in Edmonton Canada [8]

[8] http://www.mikerobe007.ca/2014/07/optimizing-your-coffee-fix.html

Now let's construct a Voronoi diagram! Start with any two points on your page, and construct the perpendicular bisector between them. If you don't remember how to do that, go back to the Construction I activity for a reminder. Now your page is divided into two regions, each of which is closest to the point it contains. When you have more points, you should have exactly as many points as you do regions - no two points should be in the same region of the page.

5. Construct a Voronoi diagram with five points.
6. Why might Voronoi patterns appear in cell growth or giraffe spots? (Hint: think about the points, sometimes also called "sources.")
7. What are some other natural places they might occur?
8. If you aren't familiar with the story, look up John Snow and the cholera outbreak of 1854. The map below is one representation of his map. The dots represent cholera cases, while the blue symbols show the pumps. How could he use a Voronoi diagram to prove a connection between cholera and the Broad Street pump (the pump in the middle of the map)?
9. The website https://www.jasondavies.com/maps/voronoi/ lets you interact with a Voronoi diagram on a sphere. What sorts of Voronoi diagrams could you create on a sphere? Why might that be more challenging than a flat piece of paper?

Map of cholera outbreak in London[9]

Project Ideas

- When you have a lot of points to work with, constructing all of the perpendicular bisectors is really slow. There are faster ways to create a Voronoi diagram, such as Fortune's algorithm. Look into Fortune's algorithm, and compare the amount of work needed to construct a diagram with 10 points. How could you describe these differing times? Research "big O notation" to see how computer scientists compare the speed of algorithms.

[9] http://www.johnsnowsociety.org/

- Can you find any examples of three-dimensional Voronoi diagrams in nature? In architecture? **Hint**: One natural example is soap bubbles.
- In a city, the closest coffee shop (by straight line distance) is not necessarily the easiest to get to. What if there are large blocks, or one way streets, or traffic? How could you adjust your Voronoi diagram to take these factors into consideration? One method uses the "Manhattan distance" rather than direct lines to compute the bisectors. Look into Manhattan distance and other variations.

Patterns

Fibonacci Numbers in Nature

Fun Fact: The Fibonacci numbers were involved in the design of the Core, at the Eden Project, near St. Austell, Cornwall, England. It is a giant greenhouse that emulates multiple different biomes.

All throughout math we find many different sequences or patterns of numbers. We can study these patterns in the classroom, but they have a surprising tendency to show up in the natural world as well. In this exercise we will look at a sequence of numbers called the Fibonacci Numbers and explore the different ways they appear outside of a textbook.

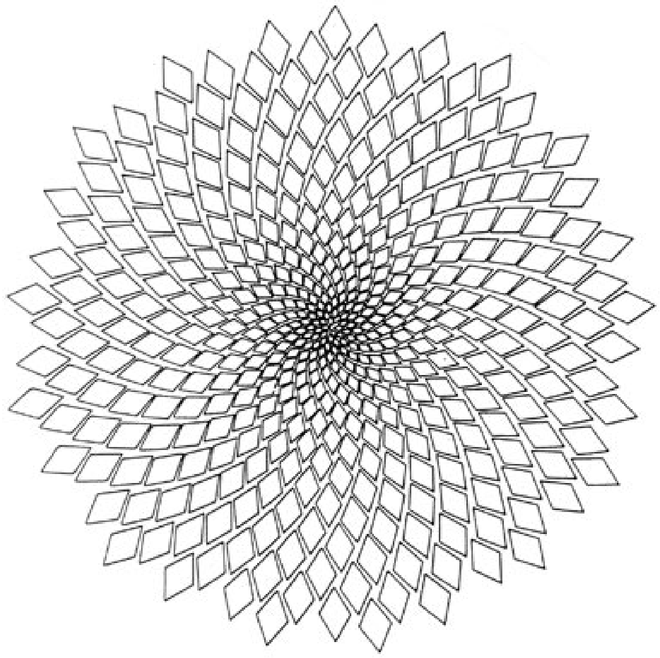
The Fibonacci numbers are the numbers that make up the Fibonacci sequence. The sequence is defined recursively: the nth element of the Fibonacci sequence is equal to the sum of the previous two numbers. The first two elements of the sequence are 1,1. To get the third number of the sequence, we sum the previous two: $1 + 1 = 2$. So the first three numbers in the sequence are 1,1,2.

1. Write down the next 10 terms of the Fibonacci sequence.

Flowers

One of the clearest examples of Fibonacci numbers in the natural world is in the seed patterns of flowers. The spiral pattern of seeds in the center of a flower contains many numbers from the Fibonacci sequence. The sunflower is one of the most common and spectacular examples of the relationship between the Fibonacci numbers and flowers. The following questions refer to an image on the next page.

2. How many 'seeds' are in the outermost ring?
   (a) Is this a Fibonacci number? If so what element of the Fibonacci sequence is it?
3. How many seeds are in each arc? (An arc begins at the center and terminates at the edge.) Is this a Fibonacci number?
4. Can you think of other plants where you have seen similar spiral patterns?
5. Why do you think plants developed to position their seeds this way?
6. Experiment with this applet `https://www.mathsisfun.com/numbers/nature-golden-ratio-fibonacci.html` to try and find the most efficient rotation for packing seeds. Remember that only the decimal portion of a rotation matters, going around the circle completely, returns you to your starting position.
7. Try finding the ratio that consecutive terms of the Fibonacci series approach. How does this relate to the efficient rotation from the applet?



A sketch of the center of a sunflower[10]

Bees

Another example of the Fibonacci numbers in nature are the ancestry trees of male bees. Bees have a very specific way of reproducing; there are two rules. The first rule of bee reproduction is that every unfertilized bee egg is born a male, or a drone bee. The second rule is that every fertilized egg produces a female bee. Therefore, every male bee has only one parent (a mother) while all female bees have two parents (a mother and a father).

To diagram this we start with one male bee. We know that the male bee has only one parent and that this parent is a female bee. If we continue up the tree, we see that the original male bee has two grandparents, a female and a male.

8. Continue diagramming this tree until you have 8 generations of bees diagrammed (21 bees across the top row).
9. Double-check your diagram by counting the number of bees in each generation. They should be equal to the Fibonacci numbers.
10. What do you think is causing the Fibonacci numbers to appear?

When Fibonnaci first started to discover this pattern he used pairs of rabbits to illustrate it in a similar fashion. Research the origin of the Fibonacci numbers and how he used rabbits to illustrate it. Draw the diagram tree for the rabbits as you did for the bees.

The Fibonacci Spiral

[10]`http://pixgood.com/sunflower-pattern-fibonacci.html`

In this final activity we will create a spiral using the Fibonacci numbers. This spiral is found in nature as the shape of the nautilus and in the shape of approaching hurricanes.

1. Begin by drawing two small squares, one next to the other. These squares should be 1×1.
2. Now below those squares, draw a square that is 2×2. This new square should be touching the bottom of the two smaller squares.
3. The next square to draw will be to the right of the ones already drawn so that the right of the 2×2 square and the 1×1 square is the left of the new square. This square will be 3×3.
4. Do the same thing now but the top of the existing picture will now be the entire bottom of the new square. It will be a 5×5 square.
5. Add one more square to the left of the picture so that the entire left of the existing picture is the right of the new square. This will be an 8×8 square.
6. Starting at the bottom right corner of the inner most 1×1 square, draw an arc that connects to the upper left corner. Continue this arc through the next 1×1 square and through the other squares. The spiral will need to grow as the squares get larger.

Patterns

Counting Patterns

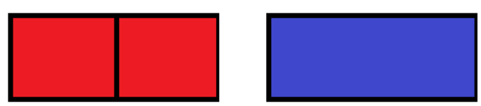
Sometimes, the work a mathematician does is not so different from the work a natural scientist might do. We might investigate a pattern, and then try to figure out why we're seeing it.

Patterns in Tilings

In this activity, we will think about the number of ways to tile an n-board: that is, a strip of n boxes. We will think about two types of tile: a square tile which covers one square on the strip, and a rectangular tile which covers up two neighboring tiles.

To begin with, note that we can only tile a 1-board one way, because there isn't room for a rectangular tile:

Things are a bit more interesting for a board of 2 squares, but only a little. There are 2 ways.



Continue with this process!

Number of Squares on Board	Number of Ways to tile
1	1
2	2
3	
4	
5	
6	

1. Draw out some strips of squares, and start filling them in. Write down some observations about the tilings. We can always tile with all square tiles. Can we always tile using only rectangle tiles?

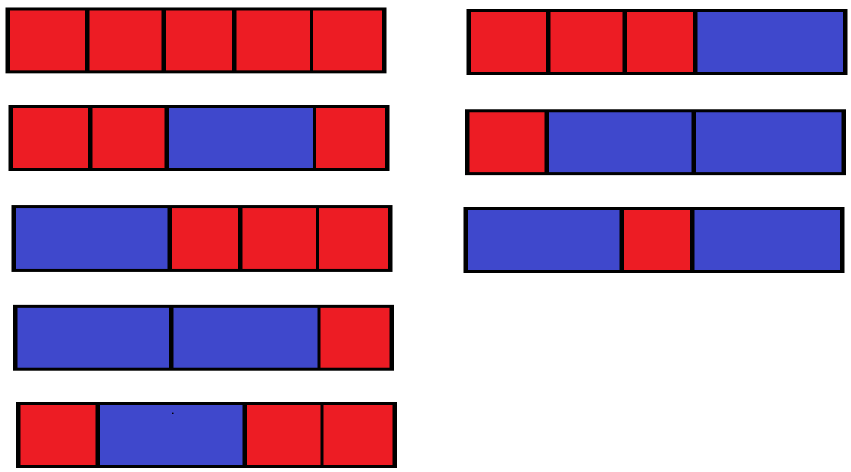
2. Fill in the above table showing the number of tilings for a strip of squares up to 6 squares long.

3. What do you notice about how fast the numbers are growing?
4. Do you recognize these numbers?

It's the Fibonacci numbers! Remember that you get the next number in the sequence of Fibonacci numbers by adding the two previous terms. (If your counts do not fit this pattern, go back and check to find which tilings you missed.)

5. Do you think it's a coincidence that these numbers match up? Is there some reason that this relationship will continue to be true for all tilings? Write a few sentences after thinking about these questions.

Consider this slightly reorganized picture of all of the 8 tilings of a 5-board:



6. What do you notice about each of the columns? How many entries are in the left column? How many are in the right column?
7. Using the drawings you created for the first table, fill out the rest of this expanded version.

Squares on Board	Number of Ways to tile	Tilings that end in a rectangle	Tilings that end in a square
1	1		
2	2		
3			
4			
5	8	3	5
6			

8. What patterns do you see in the table? Where might they come from?
9. How does this generalize to the number of tilings of any n-board?
10. Why do we see the Fibonacci numbers?

Patterns in Stair Climbing

Let's say you're climbing up a flight of stairs. You can either take them one at a time, or, if you take really big steps, take two steps at once. And you can mix these two strategies on the same flight of stairs– maybe you take double steps until you get tired and start taking them one at a time, or maybe you alternate.

11. How many different ways are there for you to go up a flight of 5 stairs? Write them out. Can you write out the number of ways to climb up a flight of 12 stairs? Why or why not?
12. Can you break your list up similarly to the way we split up the list of tilings of a strip of 5 squares? How is this question similar to the question about tilings?
13. Could you figure out the number of ways to climb up 12 stairs using this pattern?
14. Is it surprising that we can count such different objects so similarly?

Project Ideas

- Here are a couple more sequences of objects which are related to the Fibonacci numbers. See if you can figure out why these objects fit the pattern:
  - Number of ways to write n as a sum of odd numbers
  - Number of binary numbers (strings of 1s and 0s) with no consecutive 1s
- What happens if we introduce a third type of tile which takes up three squares? How does the pattern change? What is the relationship between the numbers in this sequence?
- There are other patterns of numbers that crop up when we try to count things. Research the Catalan numbers. Why do you think such a wide range of objects occur in these same patterns?

Resources

- *The (Fabulous) Fibonacci Numbers*, Alfred S. Posamentier and Ingmar Lehmann. This book contains a large number of different instances of this pattern in many different areas.
- `http://www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm` Catalan Numbers Webpage. This site provides background and applications on the sequence of Catalan Numbers.

Patterns

Tiling

Tiling patterns - a covering without gaps of a flat surface with geometric shapes - are everywhere! In bricks, on your bathroom floor, in wallpaper or mosaics, and so many other places. Mathematicians are especially interested in patterns called *tessellations* that don't have any gaps or overlaps between the shapes. Even confined to tessellations, there are endless varieties.

Vocabulary

But before we begin, let's review some vocabulary. There are two important words we'll use, both of which you have likely seen before: edges and vertices. An *edge* is a line where exactly two tiles meet and share a side. On the other hand, a *vertex* is a corner or point where at least three tiles meet.

Remember that a *polygon* is any two-dimensional, closed shape with straight edges- you could cut all the way around it on a flat piece of paper, and it would separate from the paper without any other holes. In a *regular polygon*, all the sides (and the angles between the sides) are equal. A square is the regular quadrilateral, while stop signs are regular octagons, for example.

Explore

Glue the templates for regular shapes (at the end of the exercise) to index cards, then cut them out. If you want shapes with more sides, make sure the sides are the same length as the other polygons so they fit. Take some time to play with these polygons, either tracing them or using them to cut out additional tiles.

Remember, for a pattern to be considered a tiling/tessellation it must cover a plane, such as a piece of paper, without any overlapping tiles or gaps between them. Let's explore what sort of tilings you can create.

Here are some things to think about as you create patterns by copying the provided polygons.

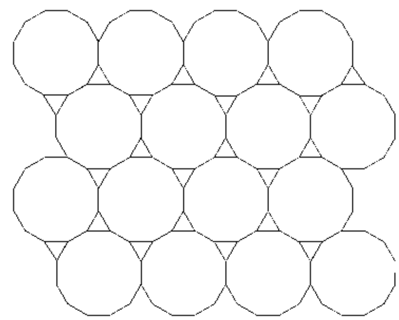
- Can you cover the entire page in just one type of regular polygon? How many different polygons can do that? Can you do it with exactly two types of polygon (for example, a tiling made only of squares and triangles)?
- Think about which polygons you can use as tiles without rotating them. Which polygons look different when rotated? When do you need to rotate the polygons differently to use them in a pattern?
- How long can you go without repeating a pattern? Using one polygon? Two of them? More?
- Write down a description for each of these tilings.

Tilings are usually described by defining what happens at the vertices. In the simplest types of tilings, called "regular" tilings, every vertex and every edge are exactly the same.

1. Is it possible to have a regular tiling with multiple types of polygons present? Why or why not?
2. How many different regular tilings can you find?
3. How many polygons meet at each vertex for each of these tilings?

On the other hand, semiregular (also called Archimedean or uniform) tilings have slightly less strict rules. The edges can be different, but every vertex must be the same.

Regular and Semiregular Tilings



4. Is the tiling shown to the right regular or semiregular? How would you describe it?
5. Can you find other semiregular tilings with multiple types of polygons present? Can you find some that use more than two types of polygons? Is it possible to have a semiregular tiling (that isn't also regular) with only one type of polygon?
6. How many semiregular tilings can you find? Make sure to think about rotating your tiles! (Note: to find all eight of them, you will need a dodecahedron - that's twelve sides!)
7. Look at these tilings in a mirror: are they all exactly the same when reflected?

Symmetry

Tessellations are defined by polygons and the behavior at vertices and edges. Now let's think about patterns defined by symmetry. How many types of symmetry can you think of?

For the so-called "wallpaper patterns," two-dimensional symmetry patterns (they can exist on a sheet of paper or, perhaps, a wall), mathematicians define three types of symmetry.

- Reflection is probably familiar to you - flipping an image with reflectional symmetry over itself, such as by turning over the piece of paper (or viewing it in a mirror) results in the same image.
- Rotation is probably familiar as well; rotating an item with rotational symmetry about its center by some amount less than a full turn (the exact rotation depends on the item) will yield the same image.

- Translation is less commonly associated with symmetry. In a repeating pattern with translational symmetry, sliding the whole thing in one direction will yield the exact same pattern.

These three types of symmetry, plus a "glide reflection" that combines translation and reflection, can combine to describe all the possible symmetries on a plane.

8. Can you think of things (shapes or examples of non-mathematical items) with each type of symmetry?
9. What sort of items have multiple types of symmetry?
10. Do any of the tessellations you found above show symmetry? Which types?

Wallpaper Patterns

Now let's design some wallpaper patterns!

11. Design a pattern with translational symmetry.
12. Can you design patterns with only translational symmetry plus one axis of reflection? Then, can you find another of these patterns that has its reflectional symmetry along a different axis from your first pattern?

    Can you make a pattern with translation plus a different axis of reflection? That is, if your first pattern reflected horizontally, try using a diagonal axis of reflection.

13. What about patterns with translational symmetry plus one rotation of $180°$?
14. The pattern to the right is an Egyptian king's floor mat. How would you describe its symmetry using the vocabulary defined here?
15. Depending on the portion of the mat you call the repeating base, you can describe the pattern differently. Can you find another way to describe its pattern?
16. What other patterns can you draw? What sort of symmetries do they have?
17. Can you make another pattern with those symmetries?
18. Find at least three different wallpaper patterns - in your home, in books, online - and try to describe the symmetries involved. Write a short paragraph about what you find.

Extensions

- Go find (in real life or in photographs and art) more examples of tessellations. Are they regular, semiregular, or something else entirely? How would you describe them to a classmate without pictures?
- What can be done if you aren't limited to regular polygons? Try looking up Hirschhorn if you don't know where to start.
- Penrose tilings are frequently used in art and architecture (including the entryway of Carleton College's Center for Math and Computing). What are they? What are some examples?
- How can you think about tessellation in higher dimensions?

Resources

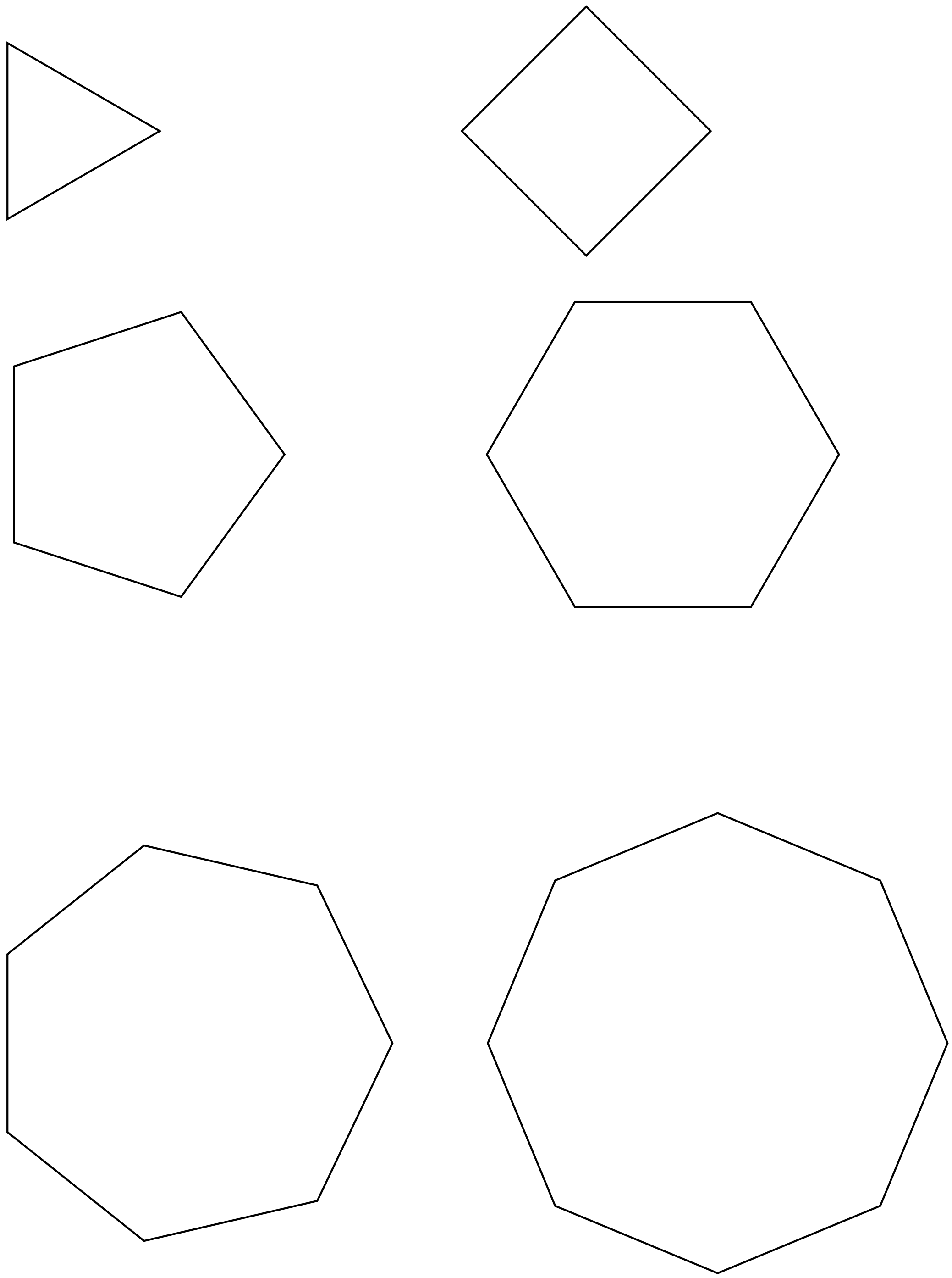
Make tessellations online:

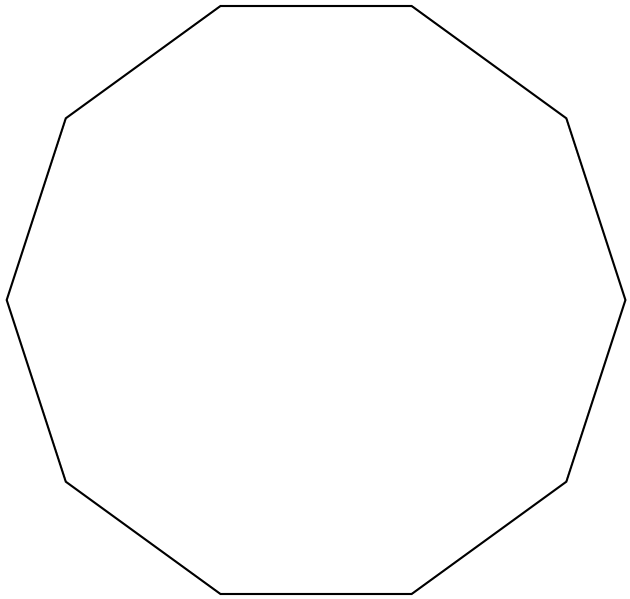
- http://www.shodor.org/interactivate/activities/Tessellate/
- http://gwydir.demon.co.uk/jo/tess/sqtile.htm

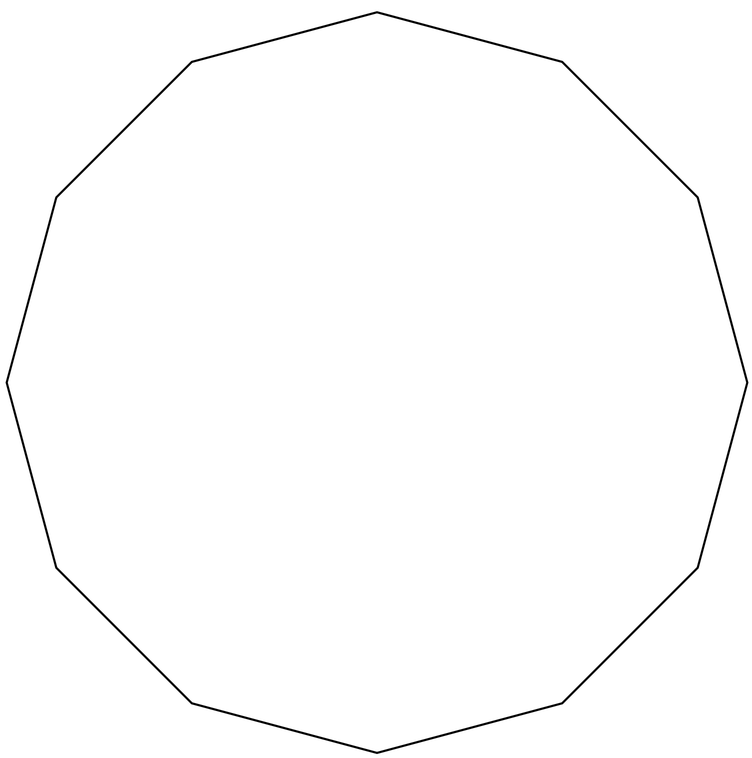
Nonregular polygons:

- http://www.mathpuzzle.com/tilepent.html

Regular Polygon Templates







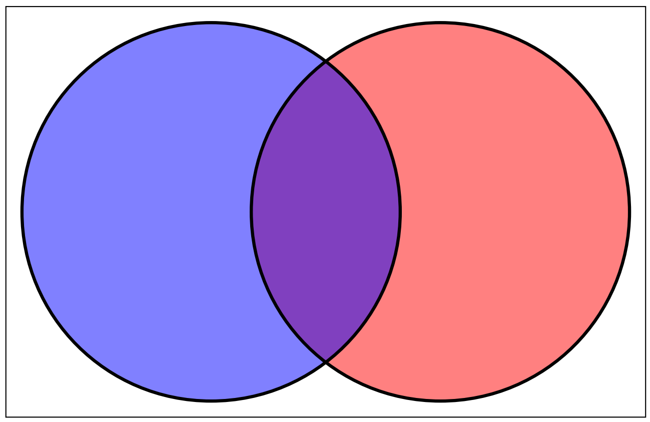
Part IV

Sets

De Morgan's Laws

Getting Started With Set Vocabulary

A set can be conceptualized as a collection of items. They are a versatile and useful concept in mathematics. Before we can get started using sets, though, we need to learn some terminology, so that we can talk about their properties and behaviors. To do that, we'll start with a concrete example.



- The *intersection* of the red and blue areas, which we write $\text{Red} \cap \text{Blue}$, contains only the area that is in both the red circle and the blue circle. In our diagram, this would correspond to the purple area.
- The *union* of the red and blue areas, which we write $\text{Red} \cup \text{Blue}$, contains everything that is in the red circle or the blue circle (or both!). This corresponds to all the colored area in this diagram.
- The *complement* of the red area, which we write Red^c, consists of all the area which is not in the circle. Red^c would be the purely blue area, as well as all the uncolored space. What is Blue^c?

As usual, parentheses tell us that what goes inside them needs to be evaluated first. Just as we would solve $(2+2)^3$ by adding $2+2=4$ before computing 4^3, we would evaluate $(\text{Red}\cap\text{Blue})^c$ by finding the area that is in both the Red circle and the Blue circle before taking the complement of that area.

When there are no parentheses, you should take a complement before finding a union or intersection.

These definitions need not be applied only to colored circles! If Alice and Bob both have collections of coins, the intersection of their collections would be all the coins that they both have in their collections. What would the union be? If we label Alice's collection as A and Bob's collection as B, how would you understand $(A \cap B)$? What about $B \cup A^c$?

Discovering the Laws

Start by drawing two overlapping circles, one red and one blue; color each circle in, and put a box around the whole diagram.

1. Using the above vocabulary, what would you call the purple area?
2. What do you call the combination of the blue, red, and purple areas?
3. How would you describe the white area?
4. Using parentheses to show what happens first, how would you use symbols to name the white area with $\cap$, $\cup$, and/or complement?

Now draw another diagram, outlining a red circle and a blue one, in a box. Now shade with orange (the complementary color of blue) everything outside the blue circle but inside the box.

5. How would you describe what you just shaded?

Similarly, use green to shade everything outside the red circle but inside the box.

6. Is the brown area (orange and green) a union or an intersection?
7. How would you write this with symbols?
8. How does the brown area compare to the first diagram?
9. Now write this as an equation!

Good job, you just found the first of De Morgan's Laws, which allows you to write a union in terms of an intersection!

There are actually two such rules combining intersection, union, and complement. Set up a new picture that is the same as before; a box with a red and a blue circle. How could you represent (Red $\cap$ Blue)^c in this picture?

Can you think of another way to represent this expression by using a union instead?

Congratulations, you have just discovered the second of De Morgan's laws, which allows you to write an intersection in terms of a union. These two laws allow us to name the same sections of the diagrams in different ways.

10. How would you write each of these laws with three colors? $(A \cap B \cap C)^c$ = ? and $(A \cup B \cup C)^c$ = ? (Hint: Draw a three-way Venn Diagram!)

11. Just as we can use our set vocabulary to talk about things besides colored circles, De Morgan's law have applications far beyond the diagrams we have used to derive them. In our example about coin collections, how would you understand $(A \cup B)^c$? Write a few sentences explaining a specific example. Can you think of a way to express that same idea with a $\cap$ using De Morgan's laws?

Building Sets

Sets as Collections

Do you collect anything? Rocks, dolls, Pokemon cards, penguin statues, coins? Any of these collections is a set, and each of these sets can be drawn in a Venn Diagram! Let's say Penelope collects stuffed animals and Quinn collects Harry Potter merchandise.

1. What sorts of items are in Penelope's collection? How about Quinn's?
2. What about items that would be in both (the intersection)?
3. What would the Venn Diagram look like?

A collection of books is the same if you rearrange their order on the shelf, but some orders make more sense or look nicer. If my collection of stuffed animals is a bear and a cat, it might have less value than a collection that has a cat and three identical bears - what if I want a bear at home, one at school, and one in my car? When we talk about collections outside of math, order and repetition might matter, but in math we only care about the unique items in a collection.

Mathematicians are also collectors. In life, items can be displayed with fancy shelving or display cases, but in math collections are always separated by commas and displayed between two curly brackets $\{\}$. Quinn's collection, for example, could be written {Hermione's wand, Ravenclaw scarf, Sorting Hat poster, Hogwarts shirt}.

Additionally, rather than collecting items such as stuffed bears or Pokemon cards, mathematicians collect numbers, letters, and other ideas or concepts. Finally, mathematicians tend to call their collections "sets." One example of a mathematical set is $\{1, 13, 592, 703, 9823\}$. Each item in a set is called an "element," so 703 is one of the elements of this set.

Our rules about order and repetition still apply when we write sets this way: $\{p, q\}$ is exactly the same set as $\{q, p\}$ which is exactly the same as $\{p, p, p, q, q, p, p, q, p\}$. We would say that this set has exactly two elements: p and q.

4. How many elements are there in the set $\{4, 5, 1, 3, 4, 1, 4\}$?
5. Using set notation, write $\{\text{trees}, \text{flowers}, \text{grasses}\} \cap \{\text{flowers}, \text{chocolates}, \text{cards}\}$ and $\{A, C, D\} \cup \{B, D, E\}$

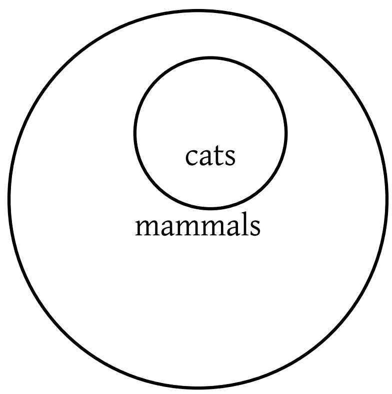
Building Set Vocabulary

So far, we have described sets by listing elements. Another way to describe a set is to define a rule or pattern that decides whether or not an element is allowed in a set rather than listing each individual item. We use this kind of language all the time when we talk about the collection of

Shakespeare's plays, or the group of Democrats in Minnesota. We avoid the impractical business of listing every play or every Democrat by referring to a common property that each item in the list has. In math, we might talk about the set of all real numbers less than 3, because we cannot list out all of the infinitely many numbers that are less than 3.

6. What are some sets you are a part of?
7. What is a set you do not belong to?
8. Think of a group of people. Can you think of two different rules that define them as a set?

Another example of a set is the set of all people. Any smaller set consisting of a specific group of people (e.g. high school students, Americans, writers, and so forth) is called a "subset" of that larger set. Considering the set of all of Shakespeare's plays, all the tragedies form a subset, as each tragic play (Macbeth, Hamlet, Othello, etc.) is also one of Shakespeare's play. Furthermore, the set of Shakespeare's plays is itself a subset of everything Shakespeare wrote as well as all plays written in Elizabethan England.



9. In the above picture, which set is the subset?
10. Add circles representing house cats, dogs, and animals kept as pets (are there pet dogs? pet cats?) to the diagram.
11. What are some subsets of students in your school? Are you in any of them? Are you in all of them? Draw Venn Diagrams like the above for some of the subsets in your school.
12. Tessa's rock collection is a subset of Grace's rock collection, say Grace has a moon rock but Tessa doesn't. Whose rock collection can't be the smaller one?

Big Sets

You can also have a set of words, such as {mouse, infinitive, hope}. One rule could be "one letter words," which defines the set {a, I}. That's a small, manageable set; it's easy to write down every element. Spend no more than five minutes on each of the next four questions.

13. What is the set of words using only the letters a, b, and t?
14. What is the set of words that rhyme with "math"?
15. What is the set of words you can make with the letters of "Minnesota"?
16. What is the set of three-letter words? How big is it?

You might have noticed that the last set is really, really large (hopefully you didn't actually try to write all of the elements). But how large is it? Does it go on forever? How do you know? Write a few sentences thinking about the size of this set.

Assuming we limit our language to English (for the sake of simplicity), this set cannot go on forever - every one of these three-letter words is contained in the dictionary. The dictionary is really large, but it clearly has an end. In other words, the set of all three-letter words is a subset of all the words in the dictionary. As you discovered above, a subset must be smaller than its set, so there are fewer three-letter words than words in the dictionary. (Does this make sense? Could it possibly be the other way around?) That means that, even though there are more three-letter words than you probably want to write down, you would be capable of writing them all if you had enough time.

Let's play with a few more sets of words. In particular, write down either a number or a sentence describing how big it is.

17. The set of words starting with the letters "dw."
18. The set of words starting with "a."
19. The set of city names.
20. The set of all words that rhyme with "silver."
21. The set of words that rhyme with "orange."

Those last two were a little bit different. Nothing rhymes with silver or orange! So how can we have sets with those rules? The "empty set" is what mathematicians call a set with nothing in it. It's written like this: $\{\}$ or $\emptyset$ and has size zero.

We've seen that to express really big sets, we can use a rule to define its elements. But just how big can a set get? Think about the set that contains all the counting numbers: $\{1, 2, 3, ...\}$. If you give me some number, I can always add one to it and get a bigger number. So no matter how many elements you list out of this set, there will always be some other elements which aren't listed. The list of Democratic Minnesotans is really long, but with enough time we could write down every element - there are *finitely* many. The list of all the Democrats is unwieldy, but the list of all the numbers is impossible to write, because it is an *infinite set*.

Write some thoughts about these questions, then discuss with a group if possible:

- How would you express the size of the set of counting numbers? Do you think all infinite sets are the same size? What would it mean for one infinite set to be larger than another?

- Do you think the set of colors is infinite? What about the set of possible grammatical English sentences?
- Think about the set of counting numbers bigger than 9: $\{10, 11, 12, ...\}$. Is this set infinite? Does it have fewer elements than the set $\{1, 2, 3, ...\}$? What about the set of even numbers: $\{2, 4, 6, ...\}$? Is this set infinite?
- Can you think of an infinite set without numbers in it? Can you find two infinite sets of numbers with no elements in common?
- We said that the set of counting numbers was infinite because numbers keep getting bigger: there's no largest element. Can you find an infinite set of numbers that does have a largest element?

Subsets

Imagine that a new film club just formed. They want to form some committees - maybe one for snacks, one to decide which movies they're watching, and one for outreach. Each member of a committee must be a member of the club, of course, but not every club member is on every committee. Some committees are more popular than others, and the committees are of different sizes. How can we tell how many possible different committees there are? Let's investigate!

For a given committee, we can imagine the sign-up sheet getting passed around between all of the members of the club. Each of the members have two choices for each committee: they can say they want to be included; or they don't want to be included. Some sheets might be signed by everyone, and others might not be signed by anyone.

Some people don't want to serve on too many different committees, or committees with certain groups of people; it can be interesting to count how many possible different committees there are. If there was only one person in a club, there could only be one unique committee, the one with that member - any committee without that individual would be empty. However, if there were seven club members, the sign-up sheet could have any of several different groupings. There could be committees with only one member, or some with two (think of all the different possible pairs!), or any number up to the committee that is the entire club!

1. Suppose that an art club has exactly two members, Allison and Brendan. Write down all the possible ways the sign-up sheet could look after going around the room. How many possible unique committees are there? (Ask your teacher to check your answer after this question to make sure you're on the right track!)
2. If one more member, Charlene, joins the club, how will the sign-up sheet change? How do these possibilities compare to the sign-up sheet with two possible members? What happens when you add a fourth person?
3. In general, how does adding one more member to the club affect the total number of possible committees?
4. Think of a way to describe the number of possible committees for any given number of group members.

Subsets and Power Sets

Committees are one application of the idea of subsets: we create a new collection by choosing elements such as people out of some collection we already have. A subset B of some set A is a set where each element of B can be found in A. Another way of expressing this is to say that A contains B. Even if A and B are the same size (for example, a committee that includes every member of the club), we can still say that B is a subset of A.

The empty set (the set with no elements, written$\{\ \}$) is a subset of every set! It's comparable to an empty sign-up sheet with nobody's name on it. In the same way, every set is a subset of itself, since every element of a set is clearly in itself.

Sets can even have other sets as elements! Do you remember our film club from the top of this page? Let's decide that it has a refreshments committee. Later, the members of that committee might realize they have far too many members to be organized, and form two subcommittees - say, a food committee and a drinks committee. However, both of these committees are also still part (subsets) of the refreshments committee, whose members are all in the film club.

Another example of this is the "power set" of A. The elements of the power set of A are all of the subsets of A. For example, if we have a set of ice cream flavors, A = $\{$chocolate, vanilla, peach$\}$, the power set would tell us all the different possible combinations of ice cream we could have. $P(A)$ = $\{$ $\{$ $\}$, $\{$chocolate$\}$, $\{$vanilla$\}$, $\{$peach$\}$, $\{$chocolate, vanilla$\}$, $\{$chocolate, peach$\}$, $\{$vanilla, peach$\}$, $\{$chocolate, vanilla, peach$\}$ $\}$. Now I know all the different cones I could get at this ice cream counter! There are eight possible subsets of the set of ice cream flavors. Did you notice the empty set? You can always choose not to order any ice cream at all, so you have zero flavors.

5. How many subsets are there of the set $\{2, 4, 7, 15\}$? Write them all down. This is the power set for this set!

6. In general, what is the size of the power set of a set with n elements? (Hint: what is the power set of the set of members of the film society described in the first section?)

7. If B is a subset of a finite set A, what can we say about the relative sizes of A and B?

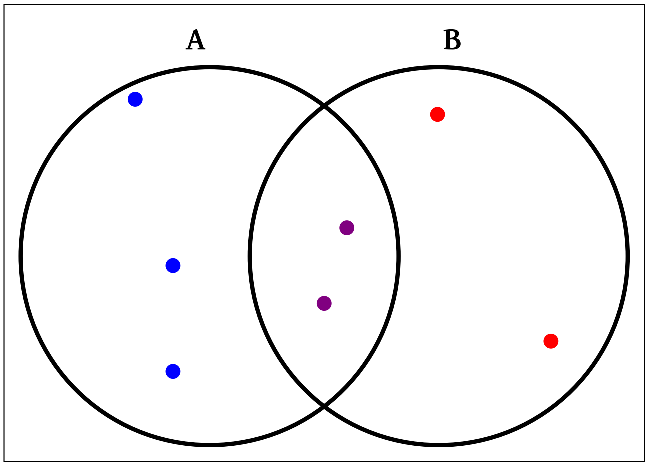
8. What does it mean if B is a subset of A and A is a subset of B?

9. If A and B are subsets of C, is $A \cup B$ a subset of C? Is $A \cap B$ a subset of C? Remember to draw out an example or two if you get stuck.

Inclusion-Exclusion Principle

The Committee For Delicious Snacks wants to have a joint meeting with the Committee For Outreach. How many chairs do they need for the meeting? Our first thought is of course that we need one chair for each member of the snack committee, and one chair for each member of the recruitment committee. But what if there is some club member who is on both committees? In that case, we would end up having two chairs for that person! We have "overcounted" this person (in this case, by counting them exactly twice). For each person serving on both committees, we need only one chair. This is the idea behind the inclusion-exclusion principle.

In terms of Venn Diagrams, the inclusion-exclusion principle says that the total number of objects in the two sets is equal to the number of objects in the first set plus the number of objects in the second set, minus the number of items that are in the overlap of the two sets.



(number of dots in circle A)+(number of dots in circle B)-(number of dots in both circles)

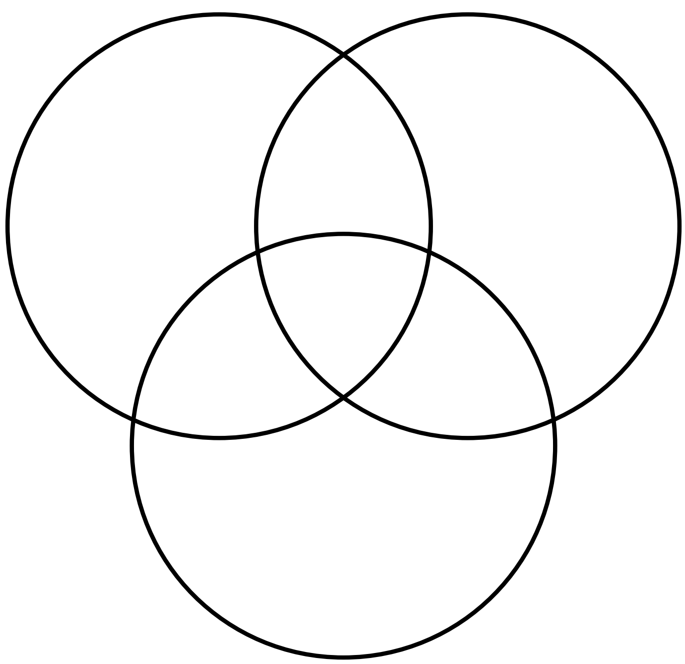
5+ 4 - 2 =7

10. Can you write this principle using set vocabulary and symbols? Remember the De Morgan's Laws exercise.

11. How many counting numbers (1, 2, 3, and so forth) less than or equal to 100 are divisible by 5? How many are divisible by 3? How many are divisible by 3 or 5? (Note that numbers divisible by 3 or 5 include 9, 10, and 15. You'll need to use the inclusion-exclusion principle.)

12. In a particularly animal-loving neighborhood, every family owns at least one pet. On the block of 30 houses, 16 families own cats and 20 own dogs. How many families own cats and dogs?

13. The inclusion-exclusion principle can be extended to unions of more than two sets. Can you determine the statement for unions of three sets? What happens if an element is in two of the three sets? What about in all three? Play around using the Venn Diagram below.



Sets

Investigating Infinity

Have you ever thought about what 1+1+1+...=? We know that the ... means there are infinitely many ones added up, but what is their sum? You would be correct in thinking it is infinity but did you know that there are different types of infinity?

Countable Infinity

1. Start by listing the first 20 natural numbers $\{1, 2, 3, ..., 20\}$.
2. Could you go on? At what point would you give up?

You would probably stop at the biggest number you could think of but what happens when you add one to it? You create a bigger number and if you add one to that, an even bigger number.

3. Now list the first 20 numbers that are divisible by three $\{3, 6, 9, ..., 60\}$.
4. Could you go on? How many numbers divisible by three are there?

We want to map the natural numbers (1,2,3, ...) to the numbers divisible by three (3,6,9,12, ...). A mapping takes one set of numbers (in this case the natural numbers) and pairs them with another set of numbers (those divisible by three) using a mathematical operation, called a function. In this case that mathematical operation is $x \times 3$. The original set of numbers is called the domain of the function and the numbers they are paired with are called the range of the function. In this activity we are always going to use the natural numbers as the domain of the function.

5. Can you think of a function with the domain of the natural numbers and range of the (natural) numbers that are odd?
   (a) Does this function match each number in the domain with one number in the range? For example, if I put three into the function do I know exactly what will come out of the function?
   (b) Does the function match each number in the range with one number in the domain? For example, if I get 11 out of the function do I know exactly what I put into the function?

When a function matches each number in the range with exactly one number in the domain and each number in the domain with exactly one number in the range it is called a one-to-one correspondence.

6. Can you think of two other one-to-one correspondences that use the natural numbers as their domain?

When a function maps the natural numbers to a set in a one-to-one correspondence, that set is countably infinite.

Extensions

As you may have noticed the type of infinity described above had a specific name, countable infinity. This leads us to the question, "Are there other types of infinity?" Yes! For a mesmerizing and mind-blowing look at other sizes of infinity, watch the incomparable Vi Hart in her video:`https://www.youtube.com/watch?v=lA6hE7NFIK0`.

Project Ideas

This project asks you to think about infinite patterns. There are infinitely many patterns of numbers, like the sets we looked at above. Also, there are infinitely many geometric patterns. Fractals are common and fascinating forms of infinite geometry. They can be relatively simple or quite complicated.

- This is the first part of your project. Do some research[11] and explore what a fractal is, then come up with your own definition of a fractal, write it down and explain how you came up with it.

  Now that you have discovered fractals, it is time to figure out what they mean mathematically.

- Sketch an approximation of the Sierpinski triangle.

  To do this, sketch an equilateral triangle; this may take a few attempts. Now connect the midpoints of each side of the triangle.

  You should have four new equilateral triangles inside the original triangle.

  Remove the middle triangle by coloring it in.

  With the remaining three triangles repeat the pattern (connect the midpoints, remove the middle by coloring it in, repeat again.)

  Keep repeating for as long as you can while still keeping the drawing neat.

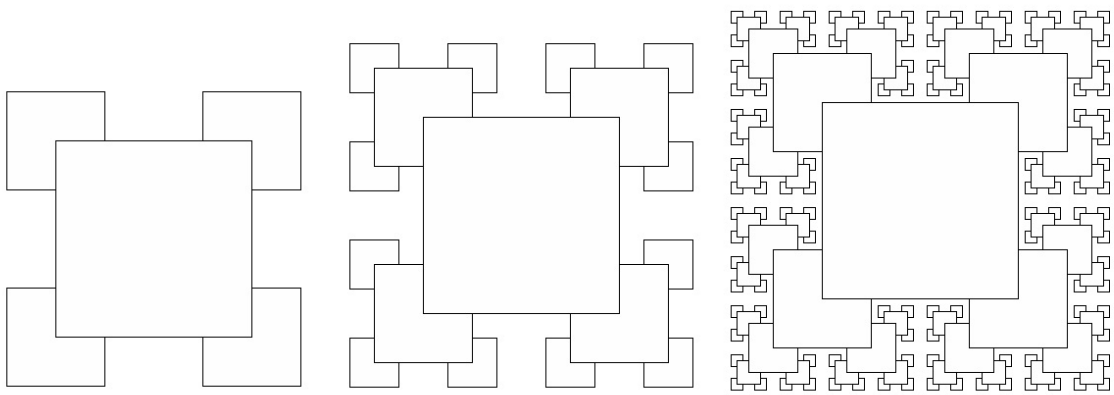
- How many iterations (repetitions of the pattern) did you complete?

  How many white triangles were in your drawing at each iteration? (Suggestion: make a table to keep track of the number of triangles while you draw.)

- Can you come up with a general rule for the number of triangles in the nth stage of the drawing?

- Now it's time to really get creative. It's your turn to create a piece of art using the fractal of your choice. We suggest using a very simple fractal (like the Sierpinski triangle) as your inspiration as they get complicated with few iterations. We have attached an image of another simple fractal to help you get started. Use your imagination to create a beautiful representation of infinity.

[11] http://fractalfoundation.org/fractivities/WhatIsaFractal-1pager.pdf



Part V

Probability

Probability vs. Reality

Some events are random: we don't know how they will turn out before we do them. In spite of this, we can still make predictions about what will happen. Understanding these predictions is the study of probability. But how much can these statistics and predictions tell us about any particular instance? How does probability compare to reality?

Coin Tosses

There's a reason that people often use coin flips to decide events– they're pretty random. In the long run, it's just as likely that a coin will land heads as it will tails. In spite of this, when we actually flip a coin many times in a row we tend to see a lot of runs where we get the same result many times in a row. Go flip a coin ten times, recording the result as you go. Then write a few sentences about what happened and whether that was what you expected.

1. If you flip a coin 20 times, what is the longest run of consecutive heads you would expect to see?
2. Give it a try! Flip a coin 20 times and record your results. How close was your prediction to the actual value?
3. Repeat this experiment four more times. Which trial had the longest run of heads? How long was it?
4. What is the mean of the lengths of the longest run of heads for your five trials? How does this compare to your initial prediction?

Expectation

Expected value is a mathematical extension of a very simple concept: we want to figure out whether something is worth our time before we can actually see how it turns out, so we have to make some kind of decision based on the likeliness of outcomes of different qualities (some outcomes are usually better than others - think about gambling!). To calculate the expected value using probability, we make a list of all possible outcomes of our model, and then multiply by the value of each of those outcomes by the probability of it actually occurring. As an example, imagine you're given the choice between two boxes. Box A has a 50% chance to contain 30 dollars, and a 50% chance of being empty. Box B, on the other hand, has a 75% chance of holding 20 dollars and a 25% chance of holding 10 dollars.

So the expected value of box A would be

$$E[A] = .50 \times \$30 + .50 \times \$0 = \$15$$

and the value of box B would be

$$E[B] = .75 \times \$20 + .25 \times \$10 = \$15 + \$2.5 = \$17.50$$

5. Which of these boxes would you choose?
6. Now suppose box A has a one percent chance to contain $1,000,000 and a 99 percent chance to hold nothing, while box B has a 95 percent chance of holding $100 and a 5% chance of being empty. Which box has the higher expected value? Which box would you choose?

In this second scenario, most people would choose a very good chance of winning a $100 over a much riskier chance at winning a much bigger amount of money. So why doesn't the expected value reflect this kind of thinking?

The expected value is not necessarily the most likely outcome. It might be helpful to think of expected value as being an "average" result if we played the game many times, we would expect to see each outcome in proportion to its probability of occurring, so if you just took the mean of all of the results you should get approximately the expected value. However, it cannot tell you what is likely to happen one particular time.

In other words, if we got to choose a thousand times it would be a pretty sure shot that one of those times Box A would contain the million dollars, which would make up for all the times it was empty. On the other hand, if we only get to choose once, it's a pretty good chance that Box A will be empty and not worth choosing in the short run.

7. A lottery ticket has a 50 percent chance of winning no prize, 25 percent chance of winning a 5 dollar prize, 20 percent chance of winning a 10 dollar prize, and 5 percent chance of winning a 100 dollar prize. What is the expected value of this ticket? What would you be willing to pay for it? Why?
8. We can calculate expected value for any numerical measurement determined by the outcome of a random event. If 60% of the shows on TV are an hour long, and 40% are half an hour long, what is the expected length of a random show?
9. If you flip a coin 3 times, what are all of the possible outcomes? What is the expected number of heads?

Adding Expectations

If X and Y are two numerical measurements for the same set of outcomes, we can find the expected value of $X + Y$ just like we found the expected value of one measurement by itself: for each possible outcome, multiply the probability of that outcome by the value of $X + Y$ in that outcome. The value of $X + Y$ is just the value of X in that situation plus the value of Y in that situation. Let's look at an example:

Suppose you have a coin, which, instead of heads and tails, has a 2 on one side and a 7 on the other. Let X be the number you get on the first flip, and Y be the total number of 7s. The possible outcomes are $(2, 2)$, $(2, 7)$, $(7, 2)$, and $(7, 7)$

$$E[X + Y] = \frac{1}{4}(2 + 0) + \frac{1}{4}(2 + 1) + \frac{1}{4}(7 + 1) + \frac{1}{4}(7 + 2)$$

By the distributive property, this is equivalent to

$$\frac{1}{4}(2)+\frac{1}{4}(0)+\frac{1}{4}(2)+\frac{1}{4}(1)+\frac{1}{4}(7)+\frac{1}{4}(1)+\frac{1}{4}(7)+\frac{1}{4}(2)$$

and we can rearrange these terms to

$$[\frac{1}{4}(2)+\frac{1}{4}(2)+\frac{1}{4}(1)+\frac{1}{4}(1)]+[\frac{1}{4}(0)+\frac{1}{4}(1)+\frac{1}{4}(1)+\frac{1}{4}(2)]$$

This is just $E[X]+E[Y]$!

In fact, this trick works in general– we can always distribute the probability of an outcome to the X and Y values, and then divide up the terms, so $E[X+Y]=E[X]+E[Y]$

10. If you flip a coin ten times, what is the expected number of heads in the first flip? What is the expected number of heads in the second flip?
11. Use the addition property of expected value to write down an equation for the expected number of heads in a series of n coin flips. (Hint: the total number of heads is equal to the number of heads you got on the first flip plus the number of heads you got on the second flip ... plus the number of heads you got on the nth flip)
12. What if the coin isn't fair, so that it only lands heads-up with probability .40? What happens to the expected number of heads? What does the equation look like?

M&M Probability

Now that we've learned how to calculate and understand expected value, we can try comparing it to some real life examples. We've seen that expected value is the average outcome in the long term. But just how long is the long term? In this section, you will compute the expected number of candies of each color in different sizes of sample, and then compare these to your actual counts for samples of those sizes. You'll need at least one standard package of plain M&Ms.

First, fill out the table of expected value assuming that each color is equally likely. (Hint: what is the expected number of blues in a sample of 1 M&M?)

Color	Sample of 10 candies	25 candies	50 candies
Blue			
Brown			
Green			
Orange			
Red			
Yellow			

Now, fill out the table of expected values using the following probabilities, taken from the manufacturer's website: .24 cyan blue, .20 orange, .16 green, .14 bright yellow, .13 red, .13 brown.

Color	Sample of 10 candies	25 candies	50 candies
Blue			
Brown			
Green			
Orange			
Red			
Yellow			

Now it's time to generate some actual data. Sort and record the number of candies of each color for each size of sample of M&Ms.

Color	Sample of 10 candies	25 candies	50 candies
Blue			
Brown			
Green			
Orange			
Red			
Yellow			

13. How do your expected values compare to your actual results for each size of sample?
14. How would you expect your table to continue if you tested 100 M&Ms?
15. Did the statistics match your table of values closely? Write a paragraph explaining whether you think the statistics are accurate.

Project Ideas

- Read about the Chi-squared test of fit. Do the manufactorer's statistics fit your data? How well? Collect a larger data sample and compare results.
- Using the internet, research the payouts for lottery tickets in real life. Calculate their expected value.
- Design your own game of chance. Decide the rules of play. How do you win? What are the prizes? List all possible outcomes along with their payouts. How much would your game cost to play? What is the expected value of the game?

Probability

Game Theory

Fun Fact: Game theory is present in a lot of popular movies. Some of these movies include *The Dark Knight, 21, The Hunger Games* and many more.

Have you ever thought about how other people make decisions? When you are making a decision does it ever depend on other people and the decisions they might make? If you don't know if your mom is making dinner, do you make dinner anyway or do you wait with growing hunger?

This is just one real world application of game theory. However, game theory is often far simpler than real life. In the example above you may have communicated with your mom about making dinner, your dad may have left you instructions for macaroni and cheese, or you may have a routine where you make dinner on Tuesdays and Fridays. In game theory (as described by Edward Packel[12]) there are at least two players, each acts in his/her own best interest, and they each have a set of choices to make. The interests of the players are often conflicting, this means that one player achieves his/her goal (in our example not having to make dinner) and the other fails (must cook dinner). In this way game theory may not be exactly applicable to daily life, but there are significant lessons we can learn from this application of mathematics.

In the following exercise you will learn some simple game theory and use your new-found knowledge to answer some questions.

Prisoner's Dilemma

The Prisoner's Dilemma is one of the most famous illustrations of game theory. It is defined as a "symmetrical game" in the language of game theory because if the positions of Prisoner A and Prisoner B were to be swapped, the payoffs would remain the same.

The game works as follows. Two people were arrested while stealing from the local game shop and they are both brought in for interrogation. They are kept separated during this time so they can't talk to each other or know what the other person is going to decide. Both players are given two options: they can either betray the other player and testify against them, or they can remain silent and cooperate with the other player. The game is represented by the chart below. It shows each player's payoffs based on the decisions of the other. To make this first game easier to understand, we will write out the payoffs in bullet points.

- If A and B both choose to betray one another, they both serve 2 years in prison.
- If A betrays B and B remains silent, A is set free and B will serve 3 years in prison.
- If B betrays A and A remains silent, B is set free and A will serve 3 years in prison.

[12] *The Mathematics of Games and Gambling* by Edward Packel

		Prisoner A Choices	
		Stay silent	Confess and Betray
Prisoner B Choices	Stay Silent	Each serves 1 year	Prisoner A goes free Prisoner B serves 3 years
	Confess and Betray	Prisoner A serves 3 years Prisoner B goes free	Each serves 2 years

- If A and B both remain silent, each of them serves only 1 year.

Using the table above, answer the following questions.

16. If Prisoner A chooses to stay silent and Prisoner B finds out, what should Prisoner B do?
17. If Prisoner A chooses to betray and Prisoner B finds out, what should Prisoner B do?

The decision for each player is made easy if they know what the other player is going to do. In this game though, remember that each prisoner has no way of knowing what the other person has chosen. So they are essentially picking blind.

18. Now without knowing what the other prisoner chose, what will each prisoner do and why?

Holiday Shopping

This is another example of a basic game in game theory but it differs from the Prisoner's Dilemma from above.

Two newlyweds have decided to forgo buying each other holiday presents in hopes of saving a few dollars this year but both, husband and wife, love giving (and receiving) gifts. They benefit from receiving a gift and from giving a gift, but if only one gives a gift they are both disappointed. Their love of presents builds doubt in the back of their minds about their decision not to buy gifts. The decision to make now is, do they buy a gift or not buy a gift? Both newlyweds have no way of knowing what the other spouse decided and must make their decisions independently.

		Wife's Choices	
		Do Not Buy Gift	Buy Gift
Husband's Choices	Do Not Buy Gift	Each receives $10 worth of happiness	Each receives $5 worth of happiness
	Buy Gift	Each receives $10 worth of happiness	Each receives $20 worth of happiness

Using the table above, answer the following questions.

19. If the the husband chooses not to buy a gift and the wife finds out, what should the wife do?
20. If the husband chooses to buy a gift and the wife finds out, what should the wife do?

This decision is again made easy if each player knows what the other one is going to do, but remember, similar to the Prisoner's Dilemma, the players do not know what the other is going to do and therefore are making their decisions blind.

21. Now without knowing what the other person chose to do, what will the husband and wife do and why?

Nash Equilibrium

If you were stumped on the final question of holiday shopping, you are not alone! There is actually not one box that the newlyweds could end up in, but two. If one of the spouses buys a gift and the other does not then the spouse that did not buy a gift has motivation to go out and buy a gift. Alternatively, the spouse that purchased something is motivated to go return the gift. This means that the top right and bottom left corners of the table are not stable solutions. If one of the "players" (newlyweds) changes his/her decision, they will both be better off.

A stable situation is referred to as a Nash equilibrium. This occurs if each player has no motivation to change their strategy when he/she assumes that the other player's strategy will not change.

22. Can you identify the Nash equilibrium in the Prisoner's Dilemma?
23. Can you identify the Nash equilibria in the Holiday Shopping game?

Mixed Strategies

In the previous question we discovered that for each newlywed it is ideal to mimic the actions of his/her spouse. We discussed running out to buy a present, or returning purchased gifts. Sometimes though, that's not an option. So what are the newlyweds to do? In real life you might flip a coin, heads to buy a present, tails to stick with no gift. In game theory we can do something similar. We can design a strategy with a random component. This is called a mixed strategy. In the Holiday Shopping example we can design a strategy for the wife where 85% of the time she buys her husband a gift while 15% of the time she does not. The following questions propose different mixed strategies for the couple and ask you to think about how that would change the chances of them receiving (or not receiving) gifts.

Each spouse begins with a 50%/50% mixed strategy; that is, they each randomly choose to buy (or not to buy) a gift.

24. What is the probability that they will end up with no gifts?
25. What is the probability they will both buy gifts?
26. What is the probability just the husband will buy a gift?
27. What is the probability just the wife will buy a gift?

Now let's change their strategies. The wife now has a mixed strategy in which 80% of the time she does not buy a gift and 20% of the times she does buy a gift. The husband now has a strategy in which 80% of the time he does buy a gift and 20% of the time he does not buy a gift.

28. What is the probability that they will end up with no gifts?
29. What is the probability they will both buy gifts?
30. What is the probability just the husband will buy a gift?
31. What is the probability just the wife will buy a gift?

Expected Value

You may be wondering what the best way to choose a mixed strategy is. Well, we mentioned before that each player in the game is trying to maximize his/her own payout. The best choice is the mixed strategy with the highest payout. However, since there are many possible outcomes there is not just one payout. Therefore, we must think about how likely each payout is (using the previous section's answers) as well as how much the payout is. This information is used to calculate the expected value of each mixed strategy.

We have already completed the first step for finding the expected value of a strategy, that is figuring out the probability of each outcome using that strategy. For each outcome (both buying, neither buying, husband buying, wife buying) we will multiply the probability of the outcome occurring by the value of that outcome. We will then sum the products to find the expected value of the strategy.

32. Why do you think we use multiplication when we are taking into account the outcome and the probability of its occurrence?
33. What is the wife's expected payout if they both follow the 50%/50% mixed strategy?
34. What is the husband's expected payout if they both follow the 50%/50% mixed strategy?
35. What is the wife's expected payout if they are following the second mixed strategy described above (80%/20%)?

Extension

One of everyone's favorite games as a kid was rock, paper, scissors but little did they know at such a young age that it is a perfect example of a game theory game. In this extension, create the table for this game and try and find the Nash equilibrium if it exists. This table will look different than the ones that we previously went over because now each player has three strategies instead of two. The payouts for this game are "1" if they win, "0" if the tie, "-1" if they lose.

Project Ideas

It was stated in the fun fact that there are a number of movies that involve game theory in them. The goal of this project will be to research these movies and determine if the players involved made the right choices.

- You will first have to find movies that involve game theory. There is a short list at the beginning of the exercise but this is not all the movies.
- Next would be to create the game tables that represent them.
- Then spend time finding the Nash equilibrium in each game and compare this result to what actually happened in the movie.
- Also explain why you think their decision was/wasn't different from the equilibria that you found.

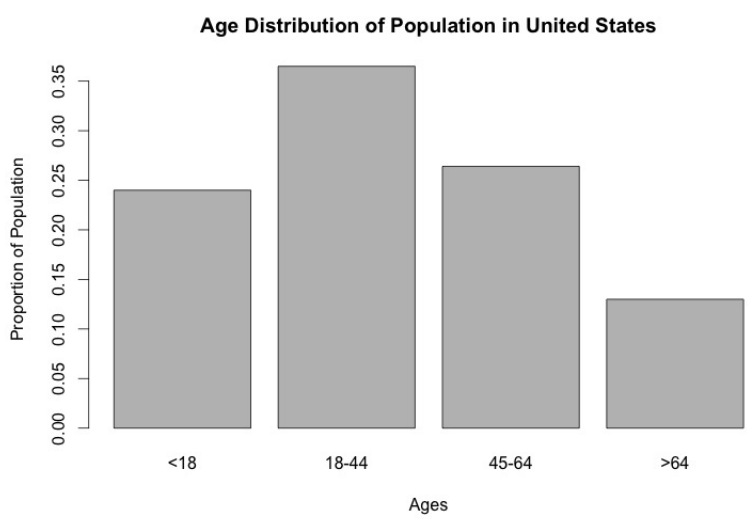
Continuous Probability

So far we've thought about probability as a fixed number of possible outcomes: heads or tails for a coin flip, numbers 1-6 for a roll of a die. But what happens when the differences between outcomes start to get fuzzier? We can start to think about continuous probability, which pertains to outcomes which are continuous, meaning that there is an unbroken range of possible outcomes.

Probability and Ages

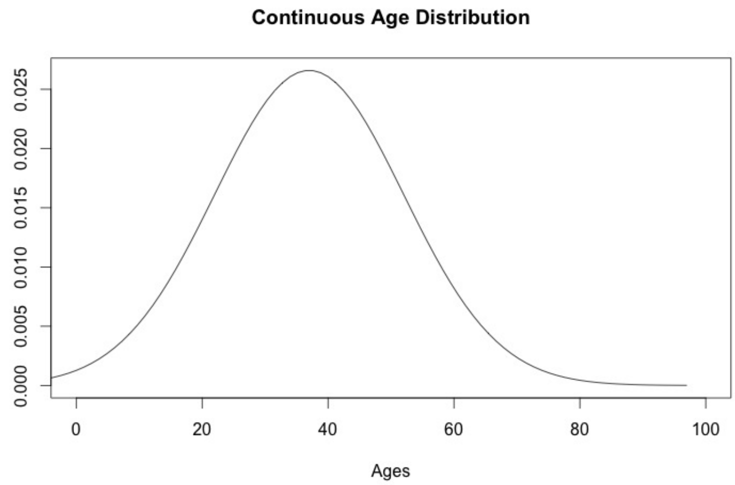
What is the probability that a person chosen at random is the same age as you?

This question depends on how strictly you mean "the same age". For instance, you could split up the population into age brackets: under 18 years old, 18-44 years old, 45-64 years old, and over 64 years old. If you counted the number of people falling into each bracket, and divided by the total population, this would give you the probability that the age of a person chosen at random would fall into that interval.



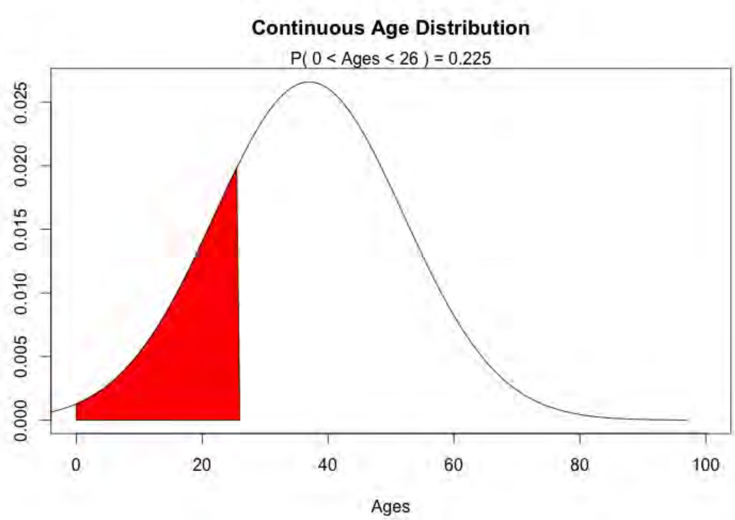
But what if you wanted to be more precise than that? You could split up the brackets into individual years. Then, the bars in the graph corresponding to each age would get smaller, and there would be more of them.

You could get even more precise! Imagine what the graph would look like if you split by the month of birth, then the week, then the day, then the hour. Eventually, the boxes will start to resemble a smooth curve:



When we get down to this small of a scale, there are an infinite number of moments of birth. Thus, the probability that a person is born at any particular one of those moments is pretty close to zero. We no longer have "chunks" that more than one age fall into, so it's no longer meaningful to talk about the probability of a single one of those moments. Instead, we can only speak about the relative probability.

We can approximate the probability of a given range of outcomes by finding the area under the probability graph for the corresponding range on the axis. To find the probability that someone is under a certain age, you would just find the area under the probability graph which is to the left of that point. For instance, this graph would show the probability that a person is under the age of 26.

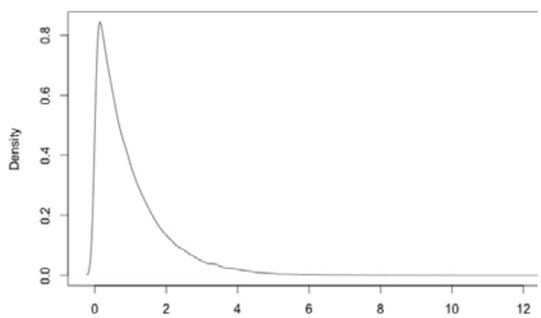


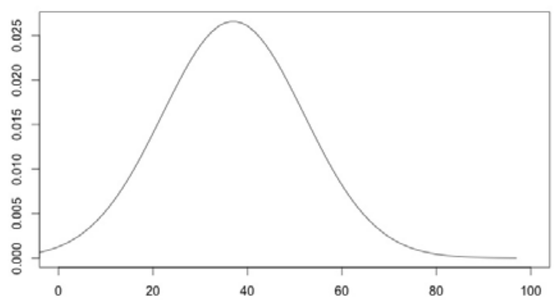
1. How would you explain the difference between continuous and discrete probability to a friend?
2. Can you think of some times when continuous probability would be a better model than discrete categories?

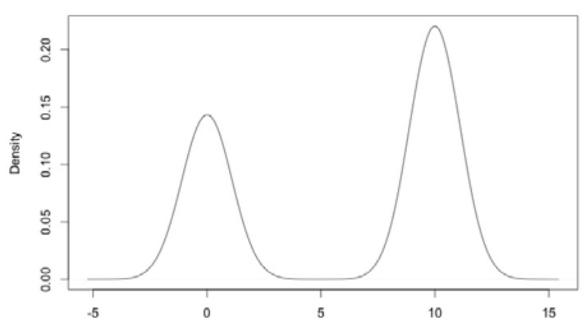
3. If the area to the left of a point represents the probability of a result less than that value, what can we say about the total area under any probability graph?

Other Continuous Distributions

In general, we use continuous probability when we want to examine probabilities where outcomes are not distinct. For example, things like time, height, weight, distance, and so on are all measured continuously, which means there is an unbroken spectrum of possible results which depend only on how precisely we measure them. With this in mind, consider these three graphs of continuous probability. What kind of data might they represent? For each of the graphs below, write a few sentences describing what kind of probabilities they represent, and what situations they could be modeling.







Extensions

- Sketch the probability density graph for a number chosen at random from between 0 and 1.
- Research different continuous probability distributions. How are they used? How do mathematicians find the area under these graphs?

Probability

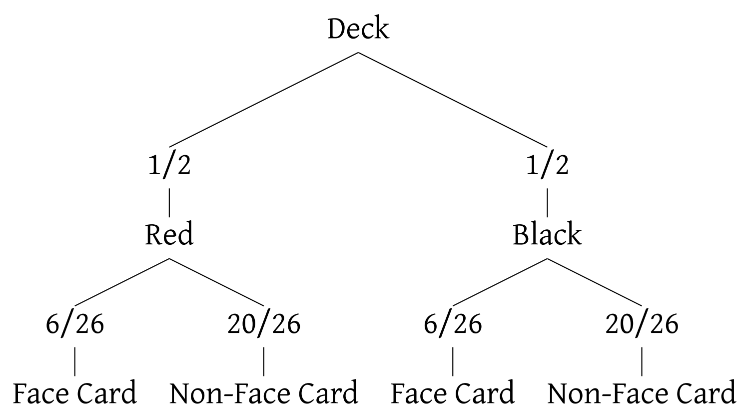
Bayes' Theorem

Fun Fact: Computer scientists are teaching computers how to make their own decisions based on Bayes' Theorem. [13]

An introduction to tree diagrams

Probability problems can be easily visualized by using a "tree diagram." The basic premise is that you break down each scenario into stages which correspond to tree branches. If we were to examine the tree below, it depicts the probability of drawing cards classified by both their color and their value. We are looking at black and red cards separately, as well as face and non-face cards separately. Face cards are jack, queen, and king only.

If we start with a standard deck of cards and simply draw a card, it is equally likely to be red or black, this means our first "branches" each have probabilities of $\frac{1}{2}$. These are the first branches of the tree. Now in this example, we split the next category into face cards and non-face cards. These are the two branches that come out of the "Red Card" and "Black Card." The reason we have the same branches coming out of these two spots is because both the red and black halves of the deck contain face cards. The probabilities that are represented by these two branches are called conditional probabilities. This means that is the probability of an event happening given another thing is true. In this case, the probability of drawing a face card if we already know that the card is red is $\frac{6}{26}$. This is because out of the 26 red cards, 6 of them have faces. When we are discussing probabilities, the things we already know to be true are called "given."



1. What is the probability of drawing a red face card from the deck?
2. What is the probability of drawing a black non-face card from the deck?

How did you go about doing this? Did you add the two probabilities together or maybe multiply them? There is an easy way to remember which way to do this. If you were to go

[13] http://www.rigb.org/christmaslectures08/html/activities/learning-from-probabilities.pdf

through the problem and come across the word "and" explicitly or implicitly, that would mean to multiply the probabilities. In the questions above to find the probability of drawing a card that is red AND a face card is $\frac{1}{2} \times \frac{6}{26} = \frac{6}{52}$. If you were to come across the word "or" in one of these problems, you would add the probabilities of the scenarios that are being compared.

Now you try!

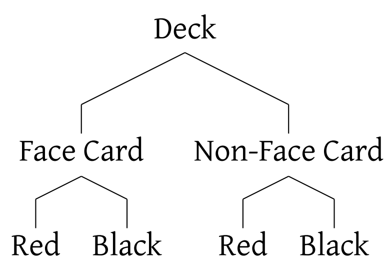
Create your own tree diagram that models the probability of drawing a diamond that is odd. In this case, count the Ace card as an odd card. Use the tree diagram above as your guide. (Hint: There should be two branches coming from the deck)

3. What is the probability of drawing an odd card given it is a diamond?
4. What is the probability of drawing a card that is not an odd diamond?
5. What is the probability of drawing an odd diamond or a card that is not an odd diamond?

What if we have the card?

Now we can think about our probabilities somewhat in reverse. Let's say you draw a face card. What is the probability that card is also red? We can use the tree diagrams we have already created to help answer this question.

6. Redraw the probability tree in the introduction except the first branches will now be Face Card/Non-Face Card and the second will be Red/Black. The basic layout is below.



7. What is the probability of drawing a card that is a Face Card and is Red. How does this compare to the probability found in question 1?

Notation is used often in probability, for example the probability of a card being red is represented by P(Red). When we want to represent the probability of a card being odd given that the card is Red we use P(Odd|Red). The vertical line indicates the information that is given.

8. Label each portion of each tree with the probability it represents. For example going down the leftmost branch of the tree drawn in the introduction we would have P(Red), P(Face|Red), and finally P(Red ∩ Face). The ∩ symbol represents "and", so P(Red ∩ Face) is the probability of drawing a card which is both Red and a Face card.

9. When using the decision tree to answer question one we find that:

   $P(Red \cap Face) = P(Red)P(Face|Red)$

   Write an expression to describe the answer to number 7 using this type of notation.

   We know that the answers to number 1 and 7 are equal so you can set the expression that you wrote equal to the above written expression.

   The expression that we just found can be generalized to Bayes' Theorem,the general form looks like this, $P(A|B)P(B) = P(B|A)P(A)$. The letters A and B represent the outcomes that are described in problems. Looking at the problem that was just worked through, A = Face and B = Red. Bayes' Theorem can be applied in any situation involving conditional probability.

Extra Practice

Before going on vacation for a week, you ask your spacey friend to water your ailing plant. Without water, the plant has a 90 percent chance of dying. Even with proper watering, it has a 20 percent chance of dying. And the probability that your friend will forget to water it is 30 percent.[14]

10. Construct the tree diagram that represents this problem.
11. What is the chance that your plant will survive the week?
12. If it is dead when you return, what is the chance that your friend forgot to water it?
13. If your friend forgot to water it, what is the chance it will be dead when you return?

The reliability of a particular skin test for tuberculosis (TB) is as follows: If the subject has TB, the test comes back positive 98% of the time. If the subject does not have TB, the test comes back negative 99% of the time. In a large population 2 in every 10,000 people have TB.[15]

14. Construct the tree diagram that represents this problem.
15. If a person is chosen at random from the population and tests positive for TB what is the probability that the person actually has TB?
16. If a person is chosen at random from the population and tests negative for TB what is the probability they have TB?

Project Ideas

There are constantly new applications for Bayesian statistics. It is used commonly in medicine, but is now employed in other fields such as economics. There are a few suggestions for research below, but you are encouraged to investigate whatever is interesting to you.

- Bayesian Truth Serum information found at:

  `http://nel.mit.edu/bayesian-truth-serum`

- Pros and cons of Bayesian statistics article found at:

  `http://www.newyorker.com/books/page-turner/what-nate-silver-gets-wrong`

- A New York Times intro to Bayesian statistics found at:

  `http://www.nytimes.com/2014/09/30/science/the-odds-continually-updated.html?_r=0`

[14] http://opinionator.blogs.nytimes.com/2010/04/25/chances-are/?_r=0
[15] https://brilliant.org/discussions/thread/conditional-probability-bayes-theorem/

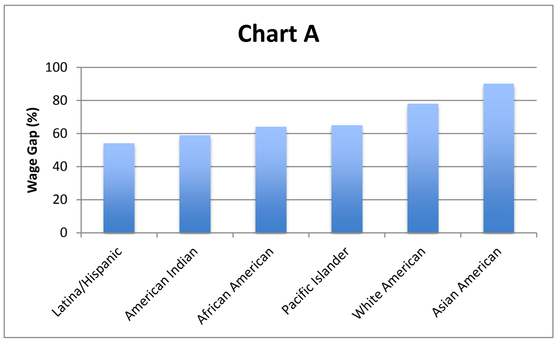
Part VI

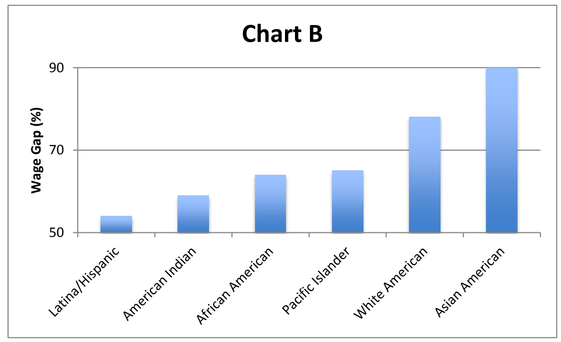
Data

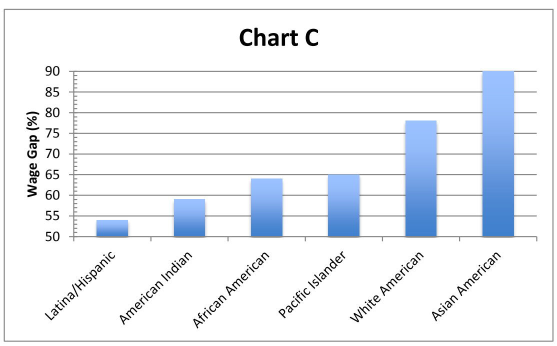
Data

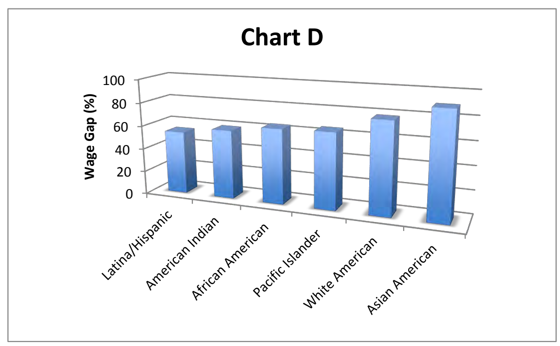
Bad Graphs

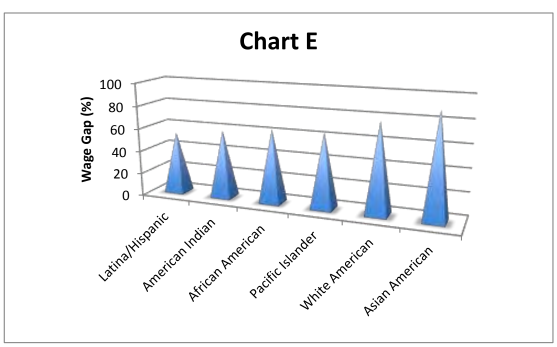
We tend to think of graphs and charts as a way of making information easier to understand. The fact of the matter though, is that the way that we choose to present the numbers can drastically change the message that the graph is trying to convey. Take a look at the following graphs, each of which presents the same data about the wage gap (by showing average income for each group of American women as a percent of average white male income) by race.

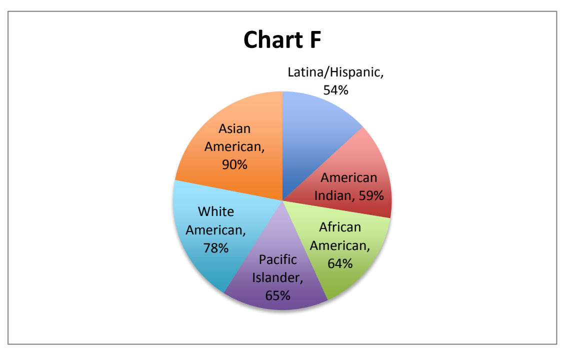












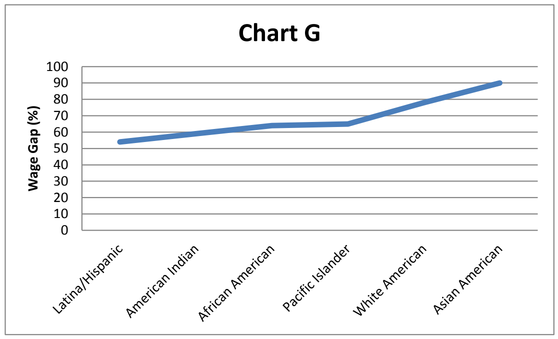


Chart A is a fairly standard bar graph. But it tells a very different story from charts B and C. Those two charts exaggerate the trend shown in the first graph. Because the bottom of those charts start at 50% instead of 0%, the shaded area that shows the wages for Asian American women is about nine times the shaded area corresponding to Hispanic/Latina women. However, as the first chart shows more clearly, it is really only about two times as large - still an important difference, but much less so.

Chart C exaggerates this visual effect to an even greater extent by adding additional subdivisions on the y-axis, tricking your brain into imagining even larger differences between the columns. Compare the columns for African American and Pacific Islander women in chart B. Do you think they are different? Now look at them in chart C; is your answer the same?

The issues of scaling increase when charts are three-dimensional or (even worse) attempt to use non-rectangular pictures to represent the bars. In rectangles, you only change the height of the bar, but to avoid strangely stretched images it is necessary to change their width as well, significantly changing their relative areas. Charts D and E show the error introduced by three-dimensional charts: however exciting it may seem, the three-dimensional representation is only very rarely better.

Charts F and G demonstrate the perils of using the wrong kind of chart. A bar chart is used for comparing data for multiple categories. On the other hand, chart G uses a line, which implies continuity between categories. Chart F uses a pie chart, which in this case is essentially nonsense: a pie chart should represent parts of a whole, so the percentages must sum to 100%.

When we look at a graph, we should think critically about all of these features, and more generally about how the physical picture differs from the conceptual picture given by the data.

Your turn!

Now, let's look at some real examples of bad and misleading charts. For each one, write a couple sentences explaining what's wrong with the picture presented of the data. Remember, there might be more than one problem. Then, take a second to think and write about why the author of the chart might have chosen to make the chart that way. Did they have a specific point they wanted to make? Did they get so caught up in the visual possibilities that they lost the meaning? Did they just mess up?

1. The following graph claims to show the changing purchasing power of the American dollar over the course of the presidencies of six presidents.

Image from University of Auckland
Hint: Does the 44 cents dollar look about half as big as the original dollar?

2. The following infographic compares different washing machines' environmental impact.

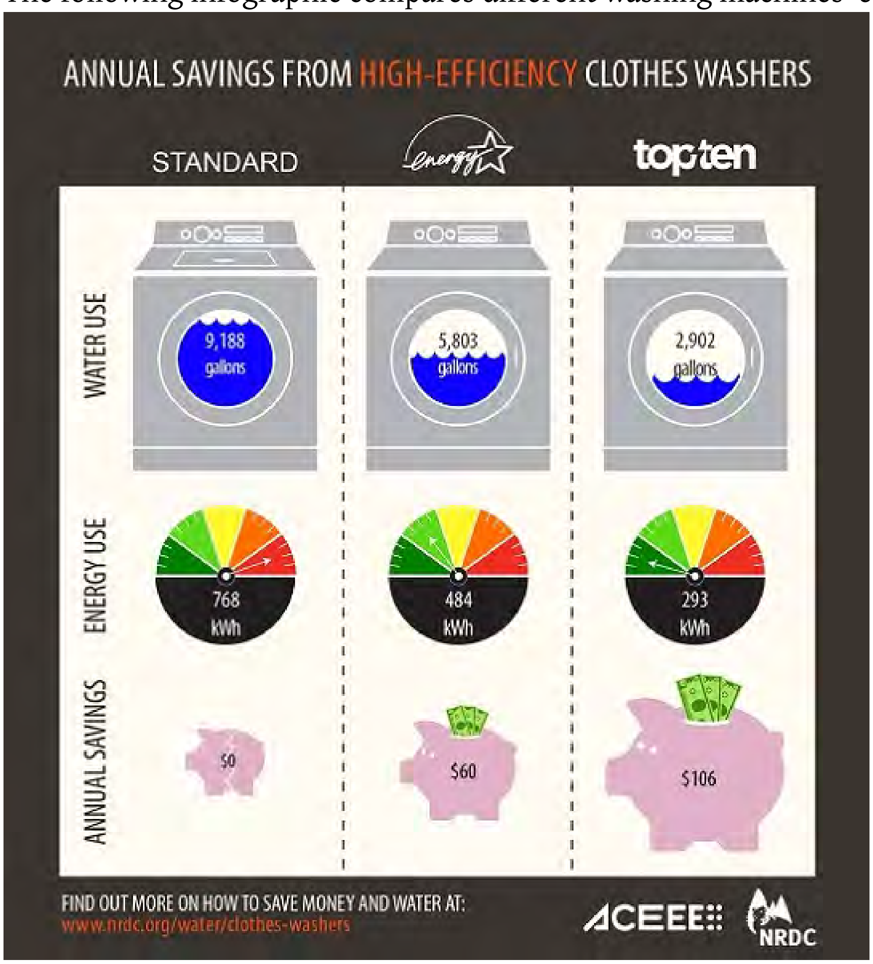


Image from Natural Resources Defense Council

3. The following is a plot of unemployment rate in 2011. Remember, high unemployment is bad.

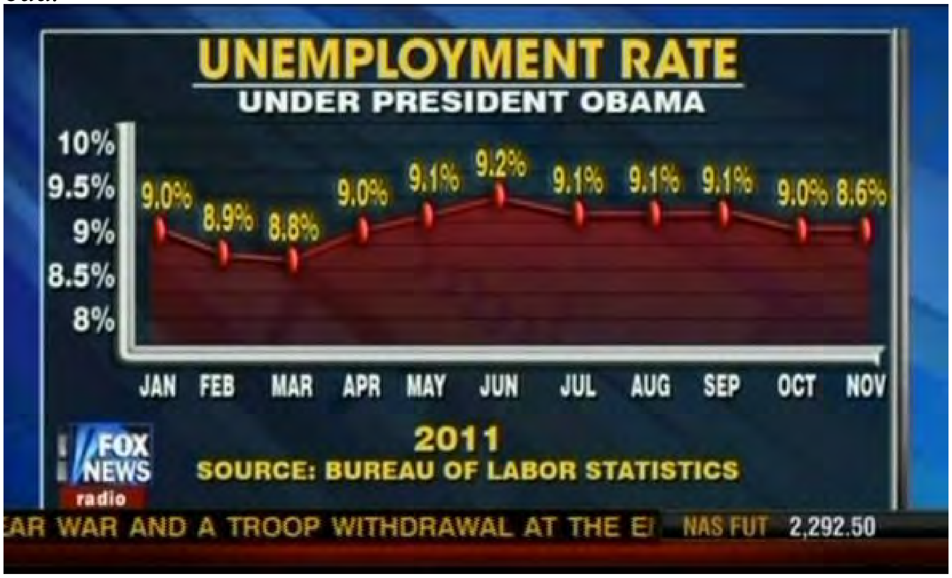


Image from Fox News.

4. The following graph tracks gun deaths in Florida before and after implementation of the "Stand Your Ground" law. This law states that a person who is attacked has the "right to stand his or her ground and meet force with force, including deadly force if he or she reasonably believes it is necessary."

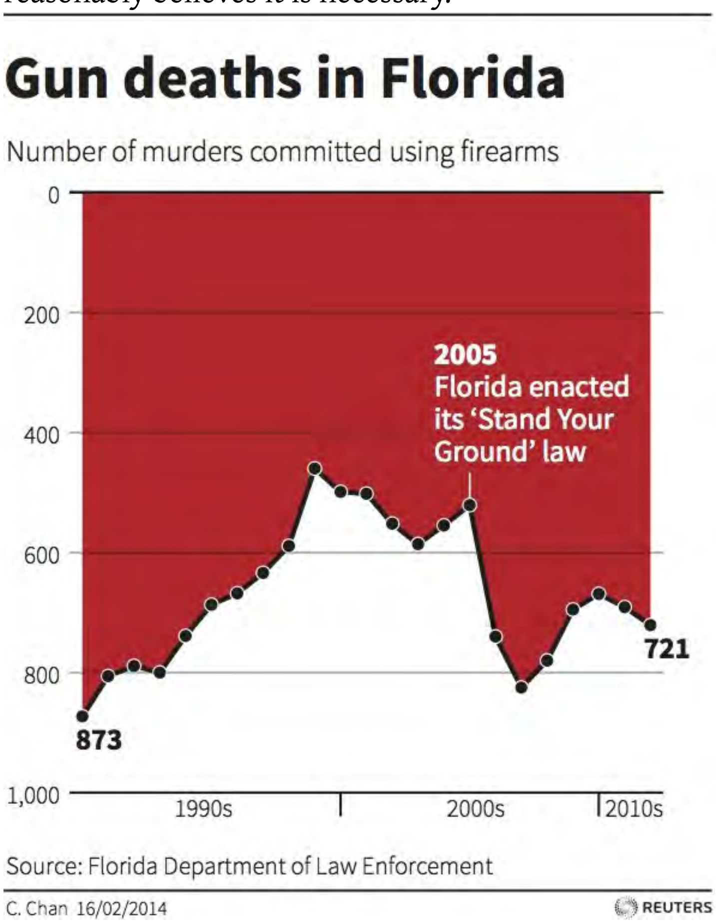


Image from Reuters.

5. The following graphic attempts to convey a variety of information about psychotherapy.

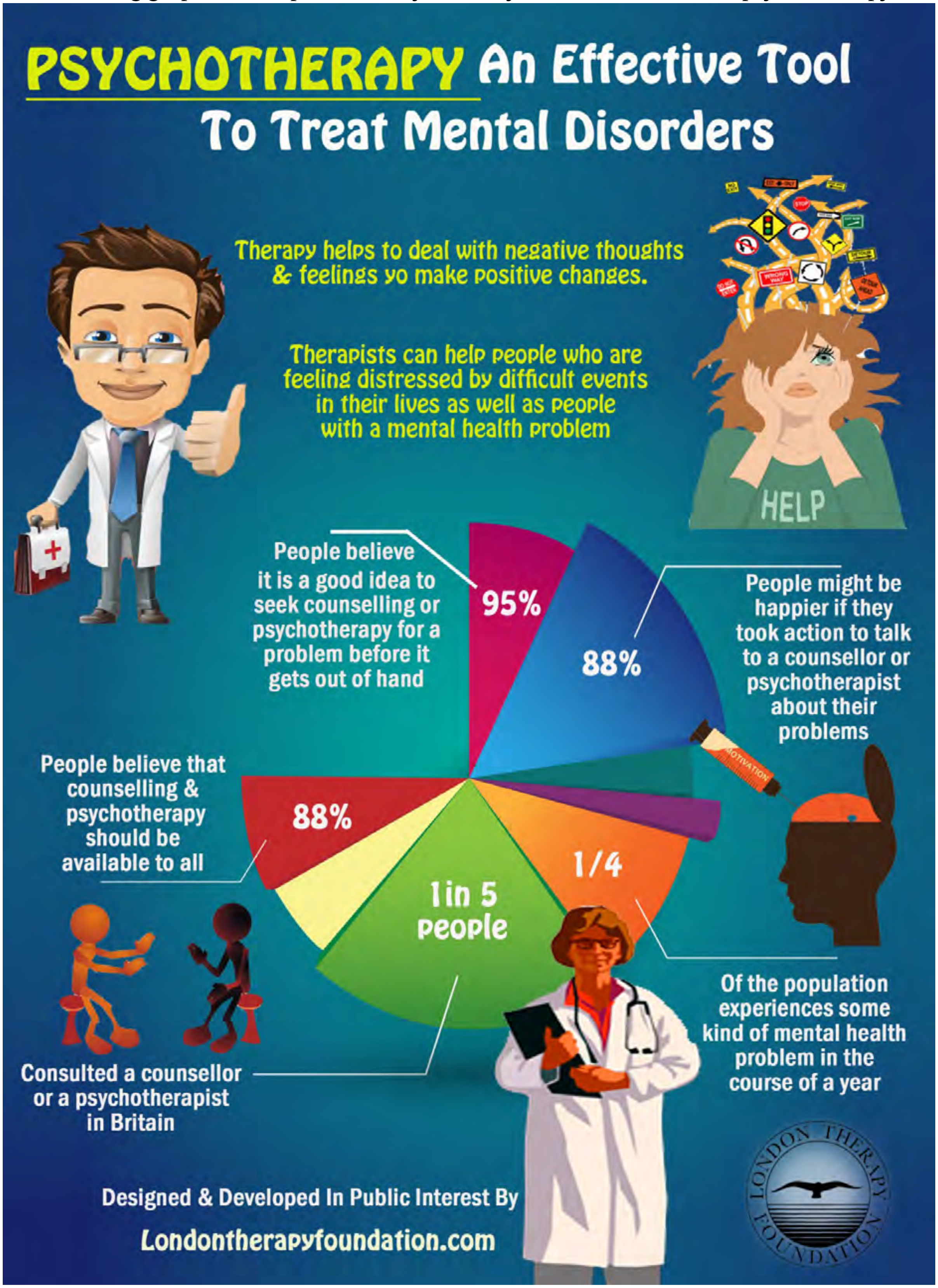


Image from London Therapy Foundation

6. The following graph has so many problems - how many can you find? Even beyond the issues in presentation, it has a flawed premise. Urbanspoon wanted to determine whether Obama was elitist; they decided the best way to measure that was determining whether people that voted for him were elitist (regardless of whether or not people vote for those with identical beliefs). But instead of measuring that (how would you want to compute elitism?) they compared votes to the number of Starbucks in each state.

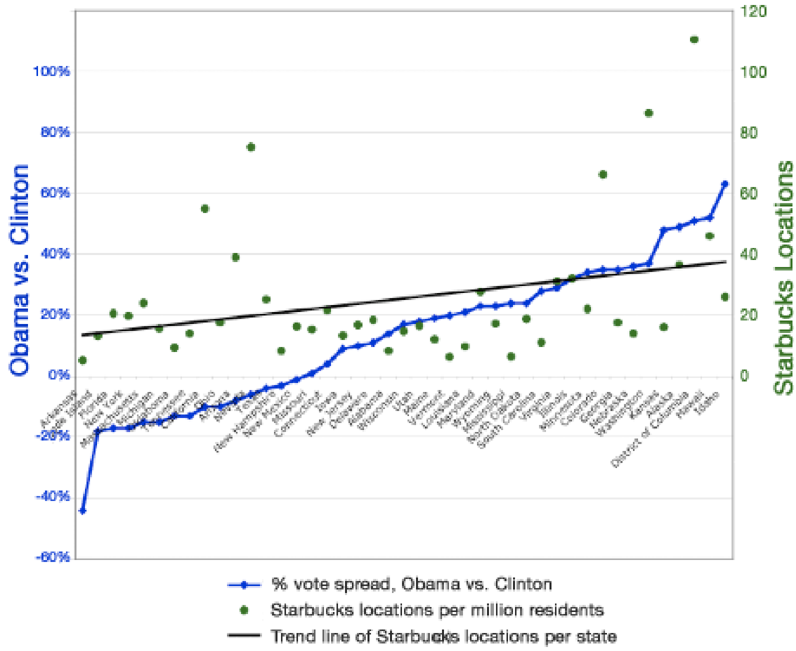


Image from Urbanspoon

Project Ideas

- Go find several examples of bad graphs in the "wild" - newspapers, magazines, advertisements, and so forth. Explain, either in writing or verbally, how they are misleading. How would you represent the same information more clearly?
- Find some data that could be graphed (perhaps survey your class, or find a graph with data you can use). First make it into the best graph you can - think about the best type of graph, watch your axes, and so on. Then think of at least two different reasons someone might want to graph this data. How might they skew the graph to achieve their goals? Make these graphs, and explain what you did and how it is misleading.

Recommended Reading

- `http://www.junkcharts.typepad.com/`
- `https://www.stat.auckland.ac.nz/~ihaka/120/Lectures/lecture03.pdf`
- `http://jwhendy.blogspot.com/2015/01/re-doing-one-of-best-infographics-of.html`

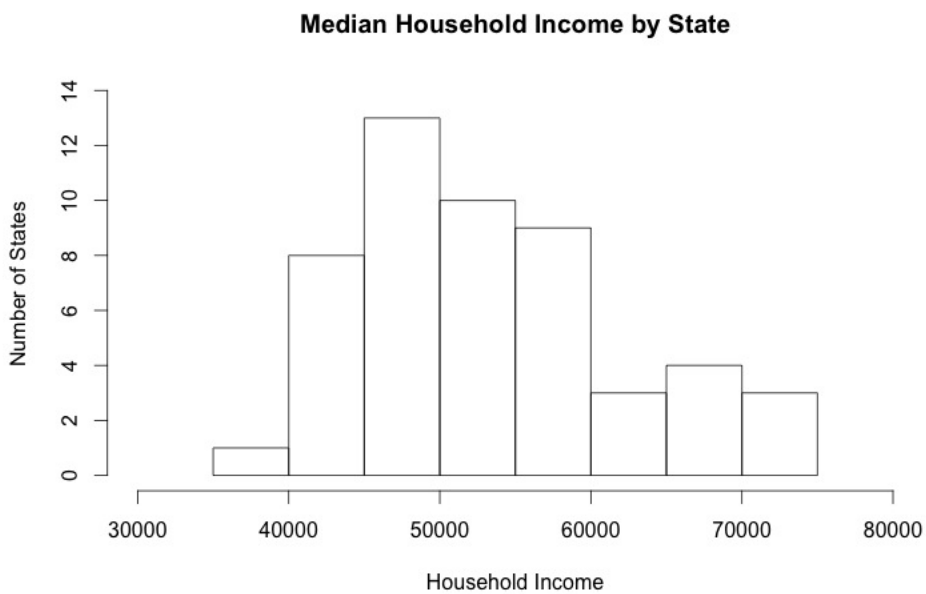
Data

Graphing

Graphs are all around you, even when you don't realize it. Graphs are used constantly to persuade, educate, and even mislead you. Graphs help sell everything from cars to toothpaste and describe everything from the state of world affairs to prom dress trends. In order to understand and learn more about the world around you it is incredibly useful to understand how to read and create graphs.

Histograms

This histogram shows the median income by state from the 2010 census. The data is divided into equal size "bins". That does not mean the same number of states are in each "bin", but rather that each bin covers the same amount of income. In this example each bin is $5,000. Some of these $5,000 ranges have 12 or more states, and for some of the "bins" no states are contained. Each bar in the histogram represents a range of median incomes and the height of the bar represents the number of states in that range. For example, the first bar indicates that exactly one state has a median household income between $35,000 and $40,000.



1. What are the ranges for the lowest and highest bins containing data?
2. How many states have a median income between $40,000 and $45,000?
3. How many states have a median income between $55,000 and $60,000?
4. Which income range has the most states? Why do you think this may be?

5. The median income for the United States as a whole is $55,046. Where does that fit in the histogram? Does that make sense?

6. Which states do you think might be at the higher end? The lower end?

Colony	Connecticut	Delaware	Georgia	Maryland	Massachusettes
Population (in thousands)	184	35	23	203	236
Colony	New Hampshire	New Jersey	New York	North Carolina	Pennsylavania
Population (in thousands)	62	117	163	197	240
Colony	Rhode Island	South Carolina	Virginia		
Population (in thousands)	58	124	447		

The above data (from `http://www2.census.gov/prod2/statcomp/documents/HistoricalStatisticsoftheUnitedStates1789-1945.pdf`) shows the population (in thousands) of the original 13 colonies in 1790. For example, this means that 184,000 individuals lived in Connecticut and were counted by the 1790 census. You will now make a histogram of this data.

The first step in creating a histogram is figuring out what the bins should be. As a rule of thumb, there should be enough bins so that there aren't too many empty ones. One or two empty are fine, but if the bins are too small the graph loses meaning. You also don't want so few bins that all the data is clumped together. After deciding on a number of bins, you have to figure out what values correspond to each bin. Say we had 20 data points spread between 10 and 60 (a range of 50). I might choose to make 5 bins, holding values 10-20, 20-30, 30-40, and so on.

These bins are the labels for the x-axis, and the y-axis is a "frequency count." That is, each column stacks one higher for each value in that bin.

7. What is the range of the data? That is, what are the highest and lowest values, and how far apart are they?

8. Choose a reasonable bin size. Why did you choose this one?

9. Now draw the graph!
   (a) Draw your x-axis, laying out the range divided into your bins.
   (b) Now fill in the columns, plotting each of the 13 colonies into the appropriate bin.
   (c) Finally, remember to label your axes and add a title!

10. Were there any colonies with unusual populations compared to the others?

11. Did more colonies have high or low populations compared to the other colonies?

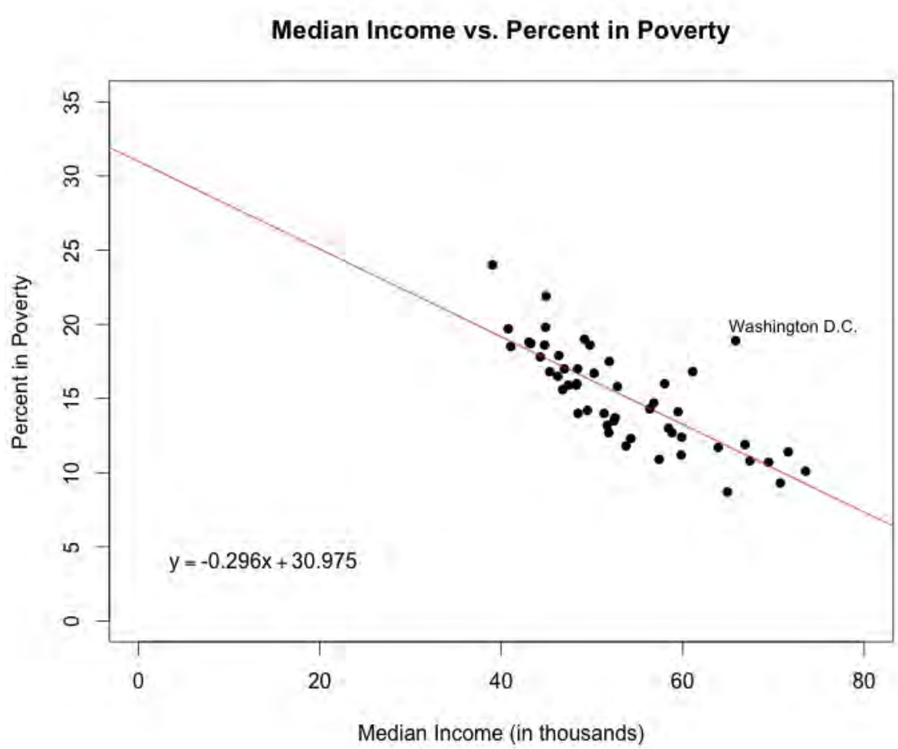
12. Which bin has the most colonies in it? Which colonies fall into that bin? Does this make sense?

13. Do the smaller colonies always have the lowest populations? Why or why not?
14. Now you're going to do this yourself! Think of a survey question to ask your class (heights, number of siblings, pages read in the last month, etc.). Collect data, then make a histogram! Pay attention to reasonable bin sizes.

Scatter Plots

Another type of graph is the scatter plot. This is similar to the xy-graphs you may be used to from math class, and is frequently employed by scientists for a variety of purposes. Both the x and y axes have different data. In this scatter plot each state is represented by one circle; the point's height is determined by the percent of people living in poverty within the state (y-axis) and its horizontal placement is determined by the median income in that state (x-axis). Although there is some randomness in the points, they appear to approximately fall along a line, called the "best fit line," that we can draw in the graph. Graphing software can provide the equation of this line in $y = mx + b$ form (remember, m is the slope of the line and b is the value where the line crosses the vertical axis).

In the graph below, the data is best approximated by the line $y = -0.296x + 30.975$. The slope indicates that an increase in median income of $1,000 is associated with an approximately 0.296% decrease in poverty rate within a state. As indicated by both the equation and the graph, this line intersects with the y-axis at 30.975%, indicating that a theoretical state whose median income was $0 would have a poverty rate of 30.975%.



15. Does it make sense that poverty decreases as median income increases? In other words, does it make sense that the slope is negative? Why?

16. How can you understand a median income of zero? Will that ever really happen?
17. Go to `http://quickfacts.census.gov/qfd/` and look up your town's median income (under the "Income and Poverty" heading). Northfield, MN has a median income of $59,233. What is the predicted percent in poverty for your town?

    Hint : Is median income x or y? Use the equation to find the predicted value!
18. How does that predicted value compare to the actual value from the census?

Although these data points exist pretty close to the line overall, some seem unusually far away. Washington, D.C., for example, has a very high percent of people in poverty for the median income (it is much higher than the line, and all the other data points with similar x-values).

1. Why do you think this may be? Is there anything unusual about Washington, D.C.?
2. Are there any other points that might be outliers? Are they higher or lower than you would expect?

Data

Introduction to Statistics

Have you every watched a commercial that claims 9 out of 10 doctors, mechanics, or moms prefer a product? Do you usually believe the commercials? How do these companies know what kind of toothpaste every single dentist prefers? How could they possibly ask every dentist?

If any of these questions have crossed your mind you have been thinking about statistics. Statistics is a branch of math that focuses on collecting and understanding data. This activity will show you some skills for working with data and some basic statistics. There is a separate activity focused on representing the data visually, or graphing it.

Below is a url for a funny commercial about misleading statistics!

`https://www.youtube.com/watch?v=tXqAyMhgc7I`

Plots

Use the included data file (printed at the end of this activity and available online through the Carleton Math Department webpage containing this book) and a program of your choice (Excel and Google Sheets both work equally well) to create a histogram of the median household income for each state. If you don't have access to graphing software, choose a region (Midwest, East, West) to plot by hand.

1. What does your histogram plot show?
2. Look at the data table - which states have the highest and lowest median household incomes?
3. Why do you think that might be?
4. Do the states with highest median incomes also have the highest per capita incomes and vice versa? Why do you think this may (or may not) be true?

Averages

Taking the average of data is likely familiar, but it's also a really good way to get a general picture of the information. Remember, there are three types of averages: mean, median, and mode. The mean is the sum of each item divided by the total number of items (so the mean of 3 and 15 is $\frac{3+15}{2} = 9$). To find the median you alternate crossing out the highest and lowest numbers and average the middle two if there are an even number (the median of 1, 3, 5, 19, 23 is 5 while the median of 1, 3, 5, 11, 19, 23 is $\frac{5+11}{2} = 8$), and the mode is the number with the most occurrences (the mode of 1, 1, 5, 34 is 1).

5. Are the three different averages always the same? Are they ever the same?

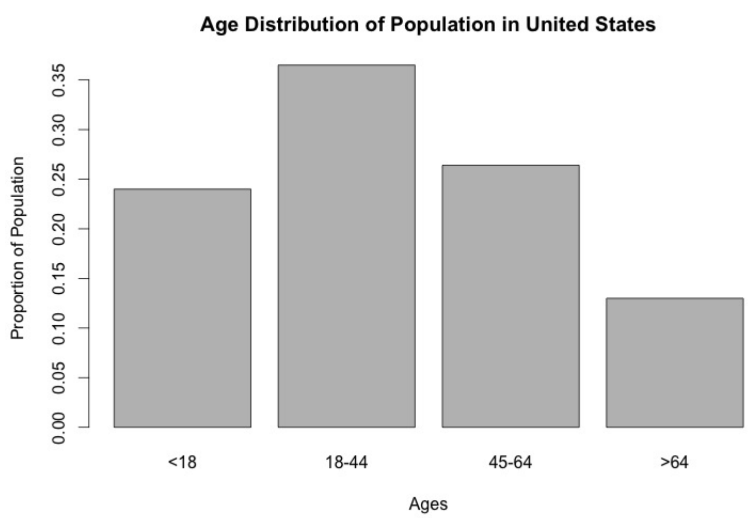
6. Is each type of average an equally good representation of the household income data? Why or why not?

7. Describe a set of data where the mean would be most helpful in describing the sample? The median? The mode?

8. What are the mean, median, and mode of the median household income for each state?

   Tech Help: In Google Docs and Excel, to find the mean you type $= AVERAGE($*values*$)$, while median is $= MEDIAN($*values*$)$ and mode is $= MODE($*values*$)$ where values are identified by highlighting the cells with the data of interest. If you're calculating the mean, median, or mode of a large number of cells all in one column you can fill in $($*values*$)$ with the range using a colon. For instance if you want the mean of the values in column B from row 7 to row 25 you would type $= AVERAGE($*B7:B25*$)$ in the cell where you want the mean to appear. It's important to remember the equal sign!

Is the mean always the middle? Is it always a good representation of the data? Look at the histograms below!



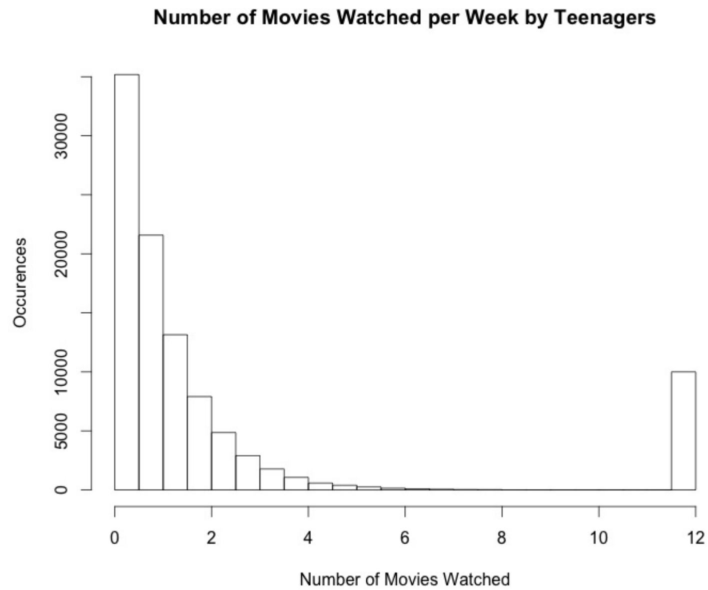
9. In the first histogram (ages), estimate the mean and median. Do these seem to be approximately the same?

10. Do the mean and median seem to be the same in the second histogram?

11. In the second histogram, which value is higher? Which one do you think is a better representation of the data?

Standard Deviation

Another tool statisticians use to describe data is standard deviation. The standard deviation of a set of data measures the "spread" of the data - how far on average a data point is from



the mean, or the middle of the data. The formula sounds confusing, the standard deviation is the square root of the mean of the squares of the differences between each data point and the mean. Mathematically it looks like this $\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$.

Explaining the formula : In the formula μ is the mean of the data and x_i represents the ith data point of the sample. Thus $(x_i - \mu)^2$ is the square of the difference between the ith data point and the mean. The $\sum$ symbol means that we take the square of the difference for each data point and sum them together. N represents the number of data points in the sample so multiplying our sum of squares by $\frac{1}{N}$ finds the mean. Finally taking the square root of this number allows us to compare the standard deviation to our original units, we have a measure of spread in the same units as the data.

For example, we could collect data on how many cousins people had. Our data could look like:

Number of Cousins	2	4	4	4	5	5	7	9

So, three people said they had four cousins each, one person had nine, one only had two cousins, and so on. First, we would find the average of these eight individuals $\frac{2+4+4+4+5+5+7+9}{8} = 5 = \mu$. Next, we calculate how far each point is from the mean, also known as $x_i - \mu$, then square each of them for $(x_i - \mu)^2$.

Number of Cousins	2	4	4	4	5	5	7	9
Distance from Mean $(x_i - \mu)$	(2-5) = -3	(4-5) = -1	(4-5) = -1	(4-5) = -1	(5-5) = 0	5-5) = 0	(7-5) = 2	(9-5) = 4
$(x_i - \mu)^2$	$(-3)^2$ = 9	$(-1)^2$ =1	$(-1)^2$ = 1	$(-1)^2$ = 1	0^2 = 0	0^2 = 0	2^2 = 4	4^2 = 16

Average the squared distances from the mean $\frac{9+1+1+1+0+0+4+16}{8} = \frac{32}{8} = 4$ and now all that's left is taking the square root: $\sqrt{4} = 2 = \sigma$. In other words, this data has a standard deviation of 2.

12. Compute the standard deviation for this data:

Data	70	71	74	74	76	79
Distance from Mean $(x_i - \mu)$						
$(x_i - \mu)^2$						

13. It is also possible to use a computer (Google Docs or Excel) to compute standard deviation with the command $= STDEVP(values)$. Fill in values the same way you did for averages. Find the mean, median, mode, and standard deviations for all the census data provided.

Project Idea

- What do you want to know about your classmates? How many books they read this month, how many pets they have, how tall they are, or something else! Give them a survey, then make graphs and see what the data has to say! Just remember, to use these statistical tools you need your data to be numbers.

State	Median household income (in 2013 dollars), 2009-2013	Per capita income in past 12 months (in 2013 dollars), 2009-2013	Persons in poverty, percent
Alabama	$43,253	$23,680	18.7%
Alaska	$70,760	$32,651	9.3%
Arizona	$49,774	$25,358	18.6%
Arkansas	$40,768	$22,170	19.7%
California	$61,094	$29,527	16.8%
Colorado	$58,433	$31,109	13.0%
Connecticut	$69,461	$37,892	10.7%
Delaware	$59,878	$29,819	12.4%
District of Columbia	$65,830	$45,290	18.9%
Florida	$46,956	$26,236	17.0%
Georgia	$49,179	$25,182	19.0%
Hawaii	$67,402	$29,305	10.8%
Idaho	$46,767	$22,568	15.6%
Illinois	$56,797	$29,666	14.7%
Indiana	$48,248	$24,635	15.9%
Iowa	$51,843	$27,027	12.7%
Kansas	$51,332	$26,929	14.0%
Kentucky	$43,036	$23,462	18.8%
Louisiana	$44,874	$24,442	19.8%
Maine	$48,453	$26,824	14.0%
Maryland	$73,538	$36,354	10.1%
Massachusetts	$66,866	$35,763	11.9%
Michigan	$48,411	$25,681	17.0%
Minnesota	$59,836	$30,913	11.2%
Mississippi	$39,031	$20,618	24.0%
Missouri	$47,380	$25,649	15.9%
Montana	$46,230	$25,373	16.5%
Nebraska	$51,672	$26,899	13.2%
Nevada	$52,800	$26,589	15.8%
New Hampshire	$64,916	$33,134	8.7%
New Jersey	$71,629	$36,027	11.4%
New Mexico	$44,927	$23,763	21.9%
New York	$58,003	$32,382	16.0%
North Carolina	$46,334	$25,284	17.9%
North Dakota	$53,741	$29,732	11.8%
Ohio	$48,308	$26,046	16.0%
Oklahoma	$45,339	$24,208	16.8%
Oregon	$50,229	$26,809	16.7%
Pennsylvania	$52,548	$28,502	13.7%
Rhode Island	$56,361	$30,469	14.3%
South Carolina	$44,779	$23,943	18.6%
South Dakota	$49,495	$25,740	14.2%
Tennessee	$44,298	$24,409	17.8%
Texas	$51,900	$26,019	17.5%
Utah	$58,821	$23,873	12.7%
Vermont	$54,267	$29,167	12.3%
Virginia	$63,907	$33,493	11.7%
Washington	$59,478	$30,742	14.1%
West Virginia	$41,043	$22,966	18.5%
Wisconsin	$52,413	$27,523	13.5%
Wyoming	$57,406	$28,902	10.9%
United States	**$53,046**	**$28,155**	**14.5%**

Part VII

Logic

Logic

Pigeonhole Principle

If I have a bag of gloves, how many do I need to grab before I have a matched set? Do at least two people in Minneapolis-St. Paul have the same number of hairs on their heads? Were two people in your school born in the same day? All these questions and more can be answered using a surprisingly simple tool.

These questions all have to do with sets of objects that we place into different categories or containers, so we can start asking questions about how these containers behave.

1. You and five friends (six total) want to go to the zoo. However, your car only has five seats. Can you all fit in the car without sharing seats?
2. Twelve students are in a gym class. There are ten tennis rackets. Can each person have their own tennis racket? How many more are needed to be sure each student has their own?

The Pigeonhole Principle, which is sometimes called the "drawer principle" is simply a mathematical way of phrasing something you already know from observation, as you can see in these examples. One way of phrasing this is to say that "If we place each one of our set of objects in a container, and there are more objects than containers, then we must have at least one container with more than one object in it." For instance, in the example of your car ride to the zoo, we know that in order for the six of you to fit into five seats (since $6>5$) there must be some seat of the car with more than one person sitting in it. In order to apply the pigeonhole principle, you need to figure out what the objects are, what the categories they are placed into are, and finally show that there are more objects than categories. But the categories or drawers do not necessarily have to be physical locations. Let's look at an example where we have abstract categories rather than physical containers.

Birthdays

3. If there are 25 people in your class, do you know for certain that at least two people were born in the same month?
4. Do you know that at least three people were born in the same month?
5. Do you know which month has multiple birthdays? Do you know which students share a month?
6. Must there only be one month with multiple birthdays?
7. How many people are in your math class? What can you say about their birthdays using the pigeonhole principle? (Do you know that two of you must share a month? Three of you? How about a day of the week? A date?)

8. Find out your classmate's birthdays (go talk to them!). Write a paragraph comparing the actual results to what the pigeonhole principle predicted.
9. How many people are in your school? Do you know that two of them have to share a birth date? Do you know that more than two of them share a birth date? What else can the pigeonhole principle say about their birthdays?

Hair

10. The average number of hairs on a human head is 150,000 hairs, so we're pretty safe in assuming that everyone has between 0 and 1,000,000 hairs on their head. The population of the Minneapolis-St Paul metropolitan area is 3.28 million. Use the pigeonhole principle to show that there must be at least two residents with the same number of hairs on their heads.
11. Do you think there are at least two bald people in the metropolitan area? Can you use the pigeonhole principle to prove it?

Socks

Imagine you keep all your socks in a big drawer - you can reach in to grab one sock at a time, but you can't look to check which sock you're grabbing. You also don't pair your socks before putting them in. On the plus side, each of your socks can go on either foot.

12. If you have four violet socks and sixteen black socks, how many do you need to take out of the drawer to be certain you have two of the same color? (Hint: What are your pigeonholes?)
13. Today you need to dress up in all black, so you don't want to wear violet socks. How many socks do you need to take out to be sure you have at least two black socks?
14. Your friend Zinnia has seven polka dot socks, five striped socks, and ten argyle socks. How many does she need to grab to be sure she has a matched set?
15. Zinnia reads that mismatched socks are stylish, so is happy to wear socks that don't match as well as socks that do. Now how many socks does she need to pull out of the drawer to be sure she has a pair she's willing to wear?
16. The next week, she realizes that argyle clashes with everything but more argyle. She's still willing to mix and match polka dots and stripes, but only with each other. How many socks does she need now to find a pair she will wear?

Party!

Sometimes, it isn't immediately clear when the pigeonhole principle can be applied. Imagine a party with 50 guests, where some of the guests exchange phone numbers. We assume that phone number trades go both ways. That is, if Yasmin has Kayla's number, Kayla also has Yasmin's. We can show that there must be two guests who gave their phone number to the same number of people.

17. In this statement, the objects are the people, and the categories are the number of phone numbers they received. What is the maximum number of people that a guest could swap numbers with? What is the minimum? How many possibilities are there in total?
18. If each one of the 50 guests were to pick a number of guests between 0 and 49 (inclusive) to give their phone number to, would we be able to guarantee that two guests picked the same number?
19. How is the situation with mutual phone number exchange different from the situation where everybody chooses the number of people to give their phone number to? (Hint: Think about the case where someone decides not to exchange their phone number with anyone at the party. How does this limit the other guests' choices if all phone number exchanges are mutual?)
20. Show that there must be two party guests who participated in the same number of phone number exchanges.
21. Is the same statement true for a party with 10 guests? With 100 guests?

Extension

- The generalized Pigeonhole Principle applies when $nk + 1$ items are to be placed in n different categories. Can you figure out what the principle says is true in this case?

Recommended Reading

- `http://www.cut-the-knot.org/do_you_know/pigeon.shtml` Cut the Knot, a website devoted to problem solving, has a nice collection of problems to solve using the Pigeonhole Principle.

Logic

Formal Logic

Fun Fact: There are some logic problems that can't be explained using traditional logic and because of this, many people have tried to invent different forms of logic to account for this. One example of these inexplicable problems is the Liar's Paradox. The Liar's Paradox is simply the sentence that says, "This sentence is false." This paradox is explained more in a later activity.

Formal logic is the study of the veracity of statements. One such statement is a conditional statement, like "If I buy groceries, I can make dinner." Conditional statements have two parts, the antecedent and the implication. The antecedent is the clause that begins with "if," and the implication is the clause that begins with "then."

To evaluate if the conditional statement is true, we have to consider the truth of the two clauses. The easiest way to understand this is by looking at examples.

Logical Implications

The following statement is a logical implication: If you eat four slices of pizza then you will be full. This type of conditional statement is called a logical implication because it is worded as an "if-then" statement.

The antecedent of this statement is: "If you eat four slices of pizza".

The implication of this statement is: "Then you will be full".

- If you ate four slices of pizza (T) and you were full (T) then the statement is true.
- If you ate four slices of pizza (T) and you were not full (F) then the statement is false.
- If you did not eat four slices of pizza (F) and you were full (T) then the statement is true.
- If you did not eat four slices of pizza (F) and you were not full (F) then the statement is true.

The truth values of the last two statements probably seem a little weird. Let's think of it as if your friend told you about going to a pizza party last night. When would you know that it is untrue that if you eat four slices of pizza you will be full? The only time that we know what should happen is when he eats the four slices. If he does not eat the four slices of pizza, there is no way for us to know what would happen next so we assume the statement to be true.

Note: If the above explanation is not satisfying, continue reading. In real life the implication is unknown; we only know what happens when four slices are eaten. We have no idea what happens when fewer than four slices are eaten. However, in formal logic there is only true and false, there is no unknown. Thus it is standard to consider these unknown statements to be true.

When learning formal logic we often forgo the use of verbal conditional statements because we can evaluate the truth of a conditional statement knowing only the truth values of the clauses. When we are using formal logic, P could be any antecedent and Q any implication. Mathematicians use truth tables to formulate sound arguments in mathematical proofs. Below is a table that is partially filled out. Use the sentences above and what you have learned so far to complete the table.

In the table P stands in for the antecedent of the statement and Q represents the implication. P⇒Q is the visual representation of the entire conditional statement, "P implies Q."

P	Q	$P \Rightarrow Q$
T	T	T
T	F	
F	T	
F	F	

The first line is the representation of the first bullet point above. Finish filling out the table.

Logical Disjunction

The following statement is a logical disjunction: The sky is blue or my shoes are tied. If I said this to a friend, when would I be lying?

This statement would only be false if the sky was not blue and my shoes were untied. Then you would know that nothing I was saying was true and that my whole statement was false.

Fill in the blanks for the following circumstances:

- Your shoes are tied (T) or the sky is blue (T) then the statement is ______________.
- Your shoes are untied (F) or the sky is blue (T) then the statement is ______________.
- Your shoes are tied (T) or it is dark out (the sky is black) (F) then the statement is ______________.
- Your shoes are untied (F) or it is dark out (the sky is black) (F) then the statement is ______________.

Fill out the table with the circumstances proposed above and your answers from the blanks. The "or" operator is shown as ∨. Once again the first statement is P and the second Q. Thus P∨Q represents the statement "P or Q."

P	Q	$P \vee Q$
T	T	
T	F	
F	T	
F	F	

Logical Conjunction

The following statement is a logical conjunction: My shirt is green and it is Monday. If I said this to a friend, when would I be lying?

This statement would only be true if both, my shirt is green and it is Monday, are true.

Fill in the blanks for the following circumstances:

- Your shirt is green (T) and it is Monday (T) then the statement is ______________.
- Your shirt is red (F) and it is Monday (T) then the statement is ______________.
- Your shirt is green (T) and it is Friday (not Monday) (F) then the statement is ______________.
- Your shirt is red (F) and it is Friday (not Monday) (F) then the statement is ______________.

Fill out the table with the circumstances proposed above and your answers from the blanks. The "and" operator is shown as $\wedge$. Once again the first statement is P and the second Q. Thus P$\wedge$Q represents the statement "P and Q."

P	Q	P $\wedge$ Q
T	T	
T	F	
F	T	
F	F	

Logical Negation

For every proposition in logic, $P \Rightarrow Q$, there is an opposite statement that is called the negation of $P \Rightarrow Q$, represented by $\sim (P \Rightarrow Q)$. If we use the first proposition as an example, "If you eat four slices of pizza, then you will be full" we can see what a negation looks like. To negate the proposition we claim it is false. For the proposition: "If you eat four slices of pizza then you will be full" the negation is "It is not true that if you eat four slices of pizza then you will be full."

Since our only truth values in formal logic are true and false the negation of true is always false and the negation of false is always true.

Using these rules as well as the ones above fill out the following table. The first row is filled out, but it's up to you to fill out the rest.

P	$\sim$P	Q	$\sim$Q	P $\Rightarrow$ Q	$\sim$P $\Rightarrow$ Q	$\sim$Q $\Rightarrow$ $\sim$P
T	F	T	F	T	T	T
T		F				
F		T				
F		F				

Extensions

In this section there are multiple tables that need to be filled out. They involve the negations of the logical conjunction and disjunction.

P	∼P	Q	∼Q	P ∧ Q	∼(P ∧ Q)	∼P ∧ Q	P ∧ ∼Q	∼P ∧ ∼Q
T		T						
T		F						
F		T						
F		F						

P	∼P	Q	∼Q	P ∨ Q	∼(P ∨ Q)	∼P ∨ Q	P ∨ ∼Q	∼P ∨ ∼Q
T		T						
T		F						
F		T						
F		F						

A final note: If this type of formal logic interests you here is one more brain teaser. In formal logic, two statements are logically equivalent to one another if they take the same truth value in every situation. This means that no matter what values P and Q (or any other propositions) take on, the two statements will always have the same truth value. There are logical equivalences present in the table above; see if you can find the logically equivalent statements.

Logical Paradoxes

Fun Fact: Paradoxical statements and contradictions have led many scholars to try and reinvent traditional logic into a better system that could deal with these types of statements. An example of logic that differs from traditional logic is fuzzy logic.

The language of logic we speak in is two-valued: things are either true or false, but what happens when we come across statements that are both true and false or neither true nor false? To first get your mind thinking about paradoxes, we will introduce you to a few simpler paradoxes before we get into more complicated ones and then we will attempt to find solutions or ways to deal with paradoxes.

Paradoxes

In this section we will introduce you to a few of our favorite paradoxes.

The Liar's Paradox: Say your friend asks you to take a look at this statement, "This sentence is false." Do you tell them it is true or false? If you say that it is true, then you are saying that the sentence is false. If you say that it is false, you are saying that the sentence is true.

The Pop Quiz Paradox: Your teacher tells you on Friday that you're going to have a quiz next week, and it's going to be a complete surprise. You realize this can't be true. After all, if after school on Thursday you hadn't taken the quiz yet you'd know it was coming Friday morning. But it can't be on Friday morning because then it wouldn't be a surprise. Since the test can't be on Friday, and since Monday, Tuesday, and Wednesday have already passed, it can only be on Thursday, so you won't be surprised. In fact, you can argue this way all the way until Monday. You're so satisfied by this reasoning that you are completely surprised by the test on Tuesday.

The Interesting Number Paradox: Take the natural numbers (the counting numbers) and divide them into two sets, interesting numbers and boring numbers defined however you want. Let's take a closer look at the set of boring numbers. If it is a non-empty set of natural numbers that you have deemed boring, it must have a smallest boring number. But the smallest boring number is in fact interesting because it is the smallest boring number. This creates the paradox.

The Dichotomy Paradox: Suppose you are going to a friend's house, before you get to your friend's you must first travel halfway there. Now before you travel the last half to your friend's house you have to travel half that distance. And then again before you can reach your friend's house, you need to travel half the distance that is left. The paradox is created by having to travel half the distance that is left every time. We can represent these distances as a set, {1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$...}. This pattern continues forever and it never reaches 0, so you will never reach your friends house.

These examples might seem a little silly, but in fact paradoxes are very important in the development of math. A paradox is a completely unacceptable logical result– if one is found there must be some inconsistency in the rules of the theory or in the definitions of the concepts.

What Are Paradoxes Good For?

Because paradoxes are a logically unacceptable result, we can make use of them to prove claims. We use what is called a "proof by contradiction"– you show that if your claim is false, it would lead to a paradox, so that you can conclude that the claim has to be true after all. Let's go through an example which dates back to ancient Greece.

Rational numbers are the ones that can be written as a ratio of two whole numbers. That is, a number is rational if there are whole numbers a and b so that the number is equal to $\frac{a}{b}$. Irrational numbers are those which cannot be expressed as a ratio of whole numbers. So to show that $\sqrt{2}$ is irrational, we need to show that for all whole numbers a and b, $\frac{a}{b} \neq \sqrt{2}$.

It seems pretty tricky to go about showing that this is the case for all of the infinitely many combinations of a and b, so instead we will start by assuming that $\sqrt{2}$ is rational, and show that this creates a paradox.

By definition, if $\sqrt{2}$ is rational, then there exist numbers a and b so that $\sqrt{2} = \frac{a}{b}$. We will assume that this fraction is reduced, so that any common factors of the top and bottom of the fraction have been canceled. If we square each side of this equation, we get $2 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$. Then, $2b^2 = a^2$.

1. If the fraction is fully reduced, is it possible that a and b are both even?
2. Is a^2 odd or even? How do you know?
3. Is a odd or even? How do you know?

Now, we know any even number is a multiple of two, so it can be written $2c$ for some whole number c. Then $(2c)^2 = 4c^2$.

4. Since we see that a^2 is even, we know a is even from your reasoning above. We can replace a with $2c$, then $2b^2 = 4c^2$. What can we cancel?
5. Is b^2 odd or even?
6. Is b odd or even?
7. Explain the paradox which is created. Why can't it be the case that $\frac{a}{b} = \sqrt{2}$?

Project Ideas

There are different types of logic out there, mostly in philosophy. These different types of logic have been created to try and solve paradoxes and contradictions like the ones discussed above. This project looks to explore different types of logic that aren't traditional logic and see what they do differently to solve these problems.

- Research a specific type of non-classical logic e.g. fuzzy logic. Determine the similarities and differences between this type of logic and traditional logic. Learn about how they deal with paradoxes and contradictory statements. Also look at some of the problems this type of logic creates and then decide which logical system is better.

- Research Russell's Paradox. What is the paradox? Why is it important in the history of mathematics? What problems did it create and how did mathematicians resolve them?
- What do you make of the above paradoxes? How is the paradox created, and how can it be resolved? Can you think of your own paradoxes?

Logic

Logic Puzzles

Fun Fact: The first person to create logic puzzles was Lewis Carroll, the author of *Alice's Adventures in Wonderland*.

You might think that "formal" logic is only useful in formal situations. However, that couldn't be farther from the truth. Logic is not only helpful in crafting all types of arguments, but can also be used to create logical puzzles. Below are some of our favorite logic puzzles taken from around the internet. We hope you enjoy spending some time on them and stretching your mind! The puzzles in this activity are increasing in difficulty.

An Introduction to Logic Puzzles

If you have never seen a logic puzzle before the images below might look very overwhelming. The following information will walk you through how to fill out the charts. The point of a logic puzzle is to figure out who did what. In Puzzle #1 for example you are figuring out who bought which species of butterfly and how much that butterfly cost. Below are some tricks for doing logic puzzles to get you started.

1. Get your symbols straight.
   - Have a symbol you use for combinations you are excluding, and one for combinations you are certain of. You can use $\times$ for options you have ruled out and $\circ$ for combinations you are certain of.
2. Use clues more than once.
   - For example: Clue 3 in Puzzle #1 reads: Gina's purchase cost $30 less than the peppered butterfly
     - From this clue we can say that Gina's purchase did not cost $75 or $90 because there is not an option that is $30 more than $75 or $30 more than $90. We will put an $\times$ in the "Gina" column in the $75 row and the $90 row.
     - Similarly we know that the peppered butterfly costs $30 more than Gina's so the peppered butterfly cannot cost $45 or $60 because there are no butterflies that cost $30 less than $45 or $30 less than $60. From this we can put an an $\times$ in the "peppered" column in the $45 row and the $60 row.
     - We can also say that Gina did not purchase the peppered butterfly. Thus we put an $\times$ in the "Gina" column in the peppered row.
     - Finally as soon as we eliminate enough possibilities to figure out the cost of either Gina's butterfly or the peppered butterfly we can immediately figure out how much the other butterfly cost because we know the relationship between the two.
3. Make sure you understand all consequences of eliminating a possibility or finding a certain result.

- For example if we have some information about Gina and some information about the chalkhill butterfly and it turns out that Gina bought the chalkhill butterfly anything true about the chalkhill butterfly is also true about Gina and vice versa. We know from above that Gina's butterfly cost either $45 or $60 so the chalkhill butterfly also costs either $45 or $60.

Puzzle #1

		winners				butterflies			
		Gina	Katie	Nick	Yvette	atlas	chalkhill	emperor	peppered
prices	$45								
	$60								
	$75								
	$90								
butterflies	atlas								
	chalkhill								
	emperor								
	peppered								

Active Clues

1. Nick's purchase, the butterfly that sold for $90 and the chalkhill butterfly were all different butterflies.

2. The butterfly that sold for $45 was the atlas butterfly.

3. Gina's purchase cost 30 dollars less than the peppered butterfly.

4. Gina's purchase cost more than Katie's purchase.

Puzzle #2

		collectors				stamps			
		Mel Morton	Pam Parson	Quinn Quade	Ted Tucker	Bull's Dove	Frog's Eye	Inverted Blue	Yellownose
prices	$125,000								
	$150,000								
	$175,000								
	$200,000								
stamps	Bull's Dove								
	Frog's Eye								
	Inverted Blue								
	Yellownose								

Active Clues

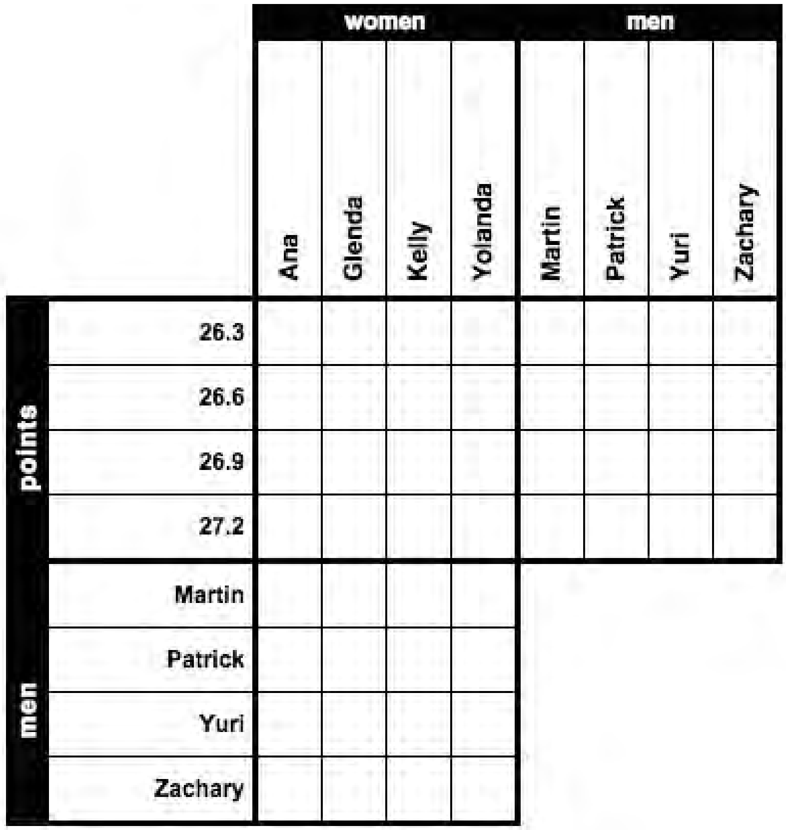
1. Of Pam Parson's stamp and the $150,000 stamp, one was the "Bull's Dove" and the other was the "Yellownose".

2. Mel Morton's stamp sold for $200,000.

3. Ted Tucker's stamp was the "Bull's Dove".

4. The "Frog's Eye", the $125,000 stamp and Mel Morton's stamp were all different stamps.

Puzzle #3



Active Clues

1. Ana's pair scored somewhat higher than Kelly's pair.

2. Patrick's pair scored 6 tenths of a point lower than Zachary's pair.

3. Martin's pair scored somewhat lower than Yolanda's pair.

4. Of Yuri's pair and the pair that scored 26.9 points, one included Kelly and the other included Glenda.

Puzzle #4

Active Clues

1. The movie produced in Finland isn't Harvest Sun.

2. The movie with a running time of 70 minutes wasn't made in Estonia.

3. Lionel Lowe's film is 5 minutes shorter than Arctic Visions.

4. The movie with a running time of 65 minutes wasn't made in Finland.

5. Sid Saunders's film is longer than Arctic Visions.

6. Sid Saunders's film wasn't made in Hungary.

7. Jesse Jimenez's film is 15 minutes longer than the film produced in Iceland.

8. Lionel Lowe's film is Milton Vale.

9. Of Ben Barrera's film and Dreams of July, one has a running time of 60 minutes and the other was made in Estonia.

10. The film produced in Romania is Milton Vale.

11. The movie with a running time of 65 minutes is either Harvest Sun or Jacky Steel.

		directors					titles					countries				
		Ben Barrera	Jesse Jimenez	Lionel Lowe	Sid Saunders	Tim Tucker	Arctic Visions	Dreams of July	Harvest Sun	Jacky Steel	Milton Vale	Estonia	Finland	Hungary	Iceland	Romania
lengths	55 minutes															
	60 minutes															
	65 minutes															
	70 minutes															
	75 minutes															
countries	Estonia															
	Finland															
	Hungary															
	Iceland															
	Romania															
titles	Arctic Visions															
	Dreams of July															
	Harvest Sun															
	Jacky Steel															
	Milton Vale															

Puzzle #5

Active Clues

1. The player who scored 62 points was either Oscar or the player from Oakland Acres.

2. The contestant who scored 48 points wasn't from Worthington.

3. The player who threw the green darts scored 21 points higher than the contestant from Yorktown.

4. Of the player who scored 55 points and the player from Worthington, one threw the red darts and the other was Colin.

5. Neither Greg nor the player who threw the white darts was the contestant who scored 69 points.

6. Jeffrey was from Braddyville.

7. Donald finished 7 points lower than the contestant who threw the black darts.

8. Oscar didn't throw the red darts.

9. Greg threw the green darts.

		players					colors					hometowns				
		Colin	Donald	Greg	Jeffrey	Oscar	black	green	red	violet	white	Braddyville	Epworth	Oakland Acres	Worthington	Yorktown
scores	41															
	48															
	55															
	62															
	69															
hometowns	Braddyville															
	Epworth															
	Oakland Acres															
	Worthington															
	Yorktown															
colors	black															
	green															
	red															
	violet															
	white															

Puzzle #6

Active Clues

1. The structure going to Dallas Center is 15 sq ft larger than the $36,000 structure.

2. The Zimmerman's house didn't sell for $35,000.

3. The home going to Shaver Lake is 15 sq ft larger than the structure going to Fullerton.

4. The Zimmerman's home didn't sell for $27,500.

5. The structure going to Unionville is somewhat larger than the home going to Shaver Lake.

6. The Kirby's house didn't sell for $35,000.

7. The $27,500 house is 15 sq ft smaller than the $38,000 house.

8. The structure going to Laguna Beach didn't sell for $36,000.

9. The $25,000 home is somewhat larger than the Ewing's home.

10. The structure going to Kennebunkport is either the $35,000 home or the $27,500 house.

11. The structure going to Mission Viejo is 45 sq ft smaller than the $27,500 structure.

12. The Kirby's structure is somewhat larger than the house going to Kennebunkport.

13. The $35,000 home is 60 sq ft larger than the Nielsen's home.

14. The Whitehead's house won't be going to Dallas Center.

15. The $29,000 home is 75 sq ft smaller than the $36,000 house.

16. The $25,000 home is either the home going to Unionville or the Ingram's home.

17. The Zimmerman's house won't be going to Unionville.

		customers							prices							cities						
		Ewing	Ingram	Kirby	Nielsen	Pratt	Whitehead	Zimmerman	$25,000	$27,500	$29,000	$32,250	$35,000	$36,000	$38,000	Dallas Center	Fullerton	Kennebunkport	Laguna Beach	Mission Viejo	Shaver Lake	Unionville
sq footage	95 sq ft																					
	110 sq ft																					
	125 sq ft																					
	140 sq ft																					
	155 sq ft																					
	170 sq ft																					
	185 sq ft																					
cities	Dallas Center																					
	Fullerton																					
	Kennebunkport																					
	Laguna Beach																					
	Mission Viejo																					
	Shaver Lake																					
	Unionville																					
prices	$25,000																					
	$27,500																					
	$29,000																					
	$32,250																					
	$35,000																					
	$36,000																					
	$38,000																					

Part VIII

Miscellaneous

Miscellaneous

Map Coloring

Cartography

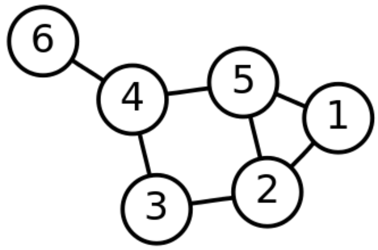
Cartography is the art and science of making maps. Political maps, maps that describe national boundaries, are colored so no two states (or countries, or regions) that share a border are the same color. It could be confusing if adjacent states (or countries, or regions) were the same color - you might think they were actually the same state. In the early days of maps, cartographers and mathematicians became interested in the minimum number of colors needed to shade in political maps.

The last pages of this activity have some blank maps of the United States that you can color. Remember, if two states share an border (for example, Minnesota and Iowa) they cannot be the same color. However, two states that only share a corner (not a complete border), such as Arizona and Colorado, can be the same color if you want. Try coloring the Western United States (only go as far east as Colorado).

1. Why might we be ignoring Alaska (AK) and Hawaii (HI)?
2. How many colors did you use? Do you think you could use fewer?
3. Now look at the South (with TX and OK as the farthest west and KY and VA as the farthest north). Can you color this region using only four colors? Don't worry about the boundaries between the West and the South - Texas (TX) and New Mexico (NM) can be the same color here.
4. Can you color the Midwest (OH is the easternmost state included) using only three colors? Make sure to color both pieces of Michigan (MI) the same color!
5. Now for the Northeast - can you use only two colors?
6. On the other map, look at the West. Can you color it with only three colors? If it isn't possible, how many colors do you need?
7. What is the greatest number of colors you can use to color the continental United States? (Don't actually do this one, just think about it!)

Graph Theory

To make coloring the map with a certain amount of colors a little easier, we can enlist the help of graph theory. Graph theory is a section in math that focuses on the study of graphs. These graphs aren't what you might expect but are used to model the relationships of objects (see Figure 1 on next page). We will use an example of the United States to get a better understanding of what a graph looks like in graph theory.



Simple Graph

Graphs are made up vertices and edges. The vertices are what represent the objects in the graphs and the edges are how we know which objects are related. We will put a vertex to represent each state capitol and the edges to show which states share a boundary. Using Figure 1 as an example, we could rewrite the numbers in the vertices as states: Say 1 = Washington, 2 = Oregon, 5 = Idaho. Then we see that there is an edge drawn from Washington (vertex 1) to Oregon (vertex 2) and another to Idaho (vertex 5). We also see an edge that connects Idaho and Oregon, this is because all three states share edges with each other. To color a graph, each vertex gets a color, and any two vertices that share an edge must be different colors. This is exactly like the map, where any states sharing a border must be different colors!

8. How do you think that representing the United States as a graph will help us with coloring?

We are going to try this but with South America instead! We will be using the countries as vertices instead of states.

9. Use a map of South America to build the graph.

10. Now color on the graph the same way we colored the United States map. How few colors can you use?

Project Idea

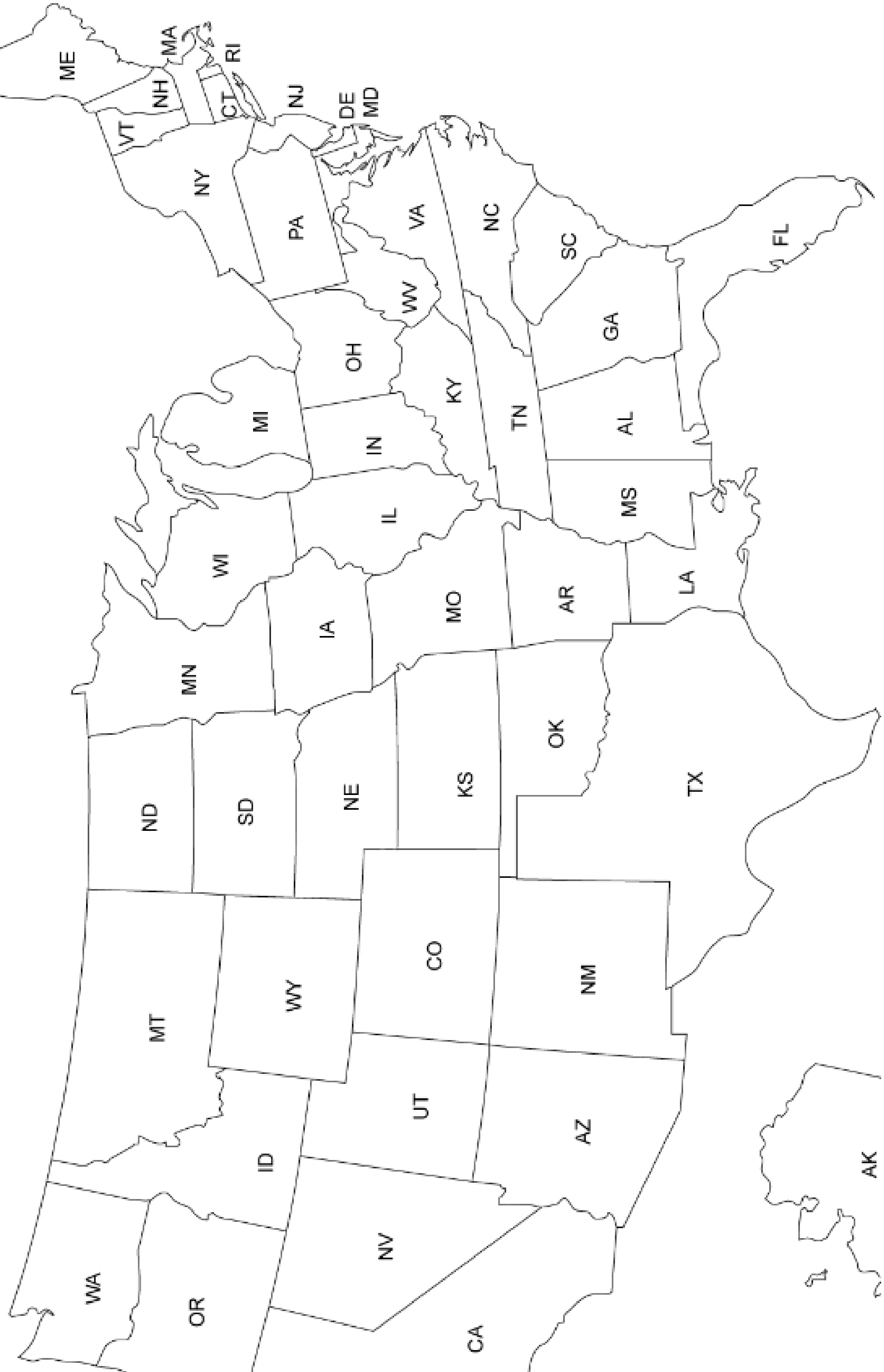
Throughout this activity we kept asking you how many colors are needed to color these maps. It turns out there there are two cool theorems called the "Four Color Theorem" and the "Five Color Theorem." Properly, the five color theorem is implied by the four color version, but is significantly easier to prove - the latter was only able to be proved by computer!

- Learn about these theorems and why and how they were discovered.

- Put together a presentation that shows everything you learned and highlight the differences between the two theorems.
- Find out if there are any special maps related to these theorems.

There is also a special theorem for a map constructed by drawing lines (straight or curvy) all the way across a rectangle. How many colors do you need? Does a map drawn like this ever need more? Can you prove it? Is this the same as a map in which some lines end in the middle of the rectangle? What if there are closed loops entirely contained in the map? Map Source: `www.freeusandworldmaps.com/images/USPrintable/USA522letterBWPrint.jpg`

United States of America



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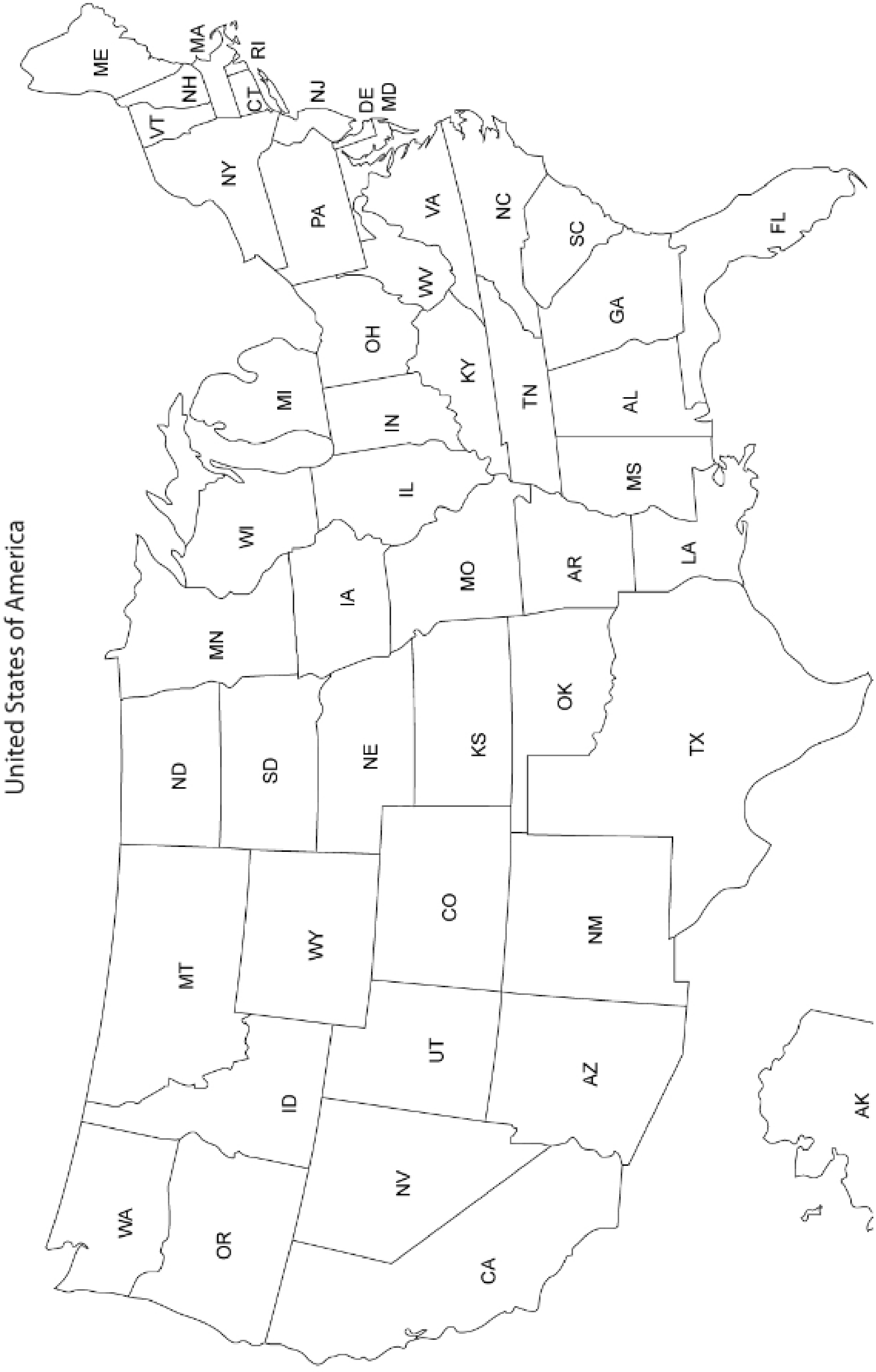
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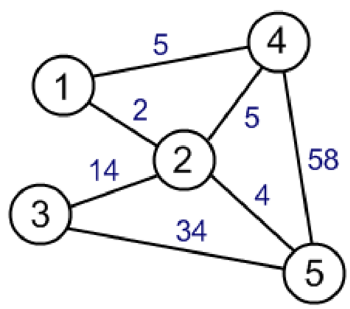
Miscellaneous

Shortest Path Problem

Fun Fact: The world Traveling Salesman problem is a shortest path problem the goal is to find the shortest path that visits each of 1,904,711 cities throughout the world exactly once.

The shortest path problem is a graph theory problem that tries to minimize the length of the path taken between two vertices. A graph consists of vertices and edges (as described in Map Coloring); what makes a graph "weighted" is when the edges have designated lengths attached to them and an unweighted graph is when the edges do not have designated lengths. One way to think about a weighted graph is using cities. Each city acts as a vertex and the straight line that connects each city is the edge. The weight of the edge represents the distance between cities.



An undirected weighted graph [16]

Let's find the shortest path for the undirected weighted graph above.

1. List all possible paths to get from vertex 1 to vertex 5 where you don't visit a vertex more than once. (Hint: There should be 6 total).
2. Calculate the length of each individual path.
3. Which path is the shortest?

As you can probably figure out, listing all paths is not the most efficient way to tackle this problem. Imagine if the map had upwards of 100 vertices, it would take forever to write down all the possible paths and then calculate their lengths. There has to be a shorter way!

Dijkstra's Algorithm

Good news, there is a faster way to calculate the shortest path between two vertices and it is Dijkstra's Algorithm. An algorithm is a set of steps that can be repeated in order to solve a problem. Dijkstra's Algorithm was first published in 1959. It has very few steps, but completing

[16] `http://web.cecs.pdx.edu/~sheard/course/Cs163/Graphics/graph6.png`

them can be a little confusing the first couple times. If you are having difficulty completing the steps, follow this url or type Dijkstra's Algorithm Chart into the Youtube search box, the video below is by user barngrader.

`https://www.youtube.com/watch?v=0nVYi3o161A`

Dijkstra's Algorithm (Using a Chart)

1. Create a chart that has the names of the vertices along the top. The leftmost column should be empty.

2. At the top of the leftmost column write the starting point.

3. The first row of the chart for the problem on the next page has been filled in for you. We are starting at vertex A, the first row shows the distances to all of the vertices that can be accessed from A. Each distance is subscripted with an A to show that the distance is calculated to that vertex from A. The vertices that cannot be visited from A are labeled as ∞ because from the vantage point of A they are "infinitely" far. That is, they cannot be reached directly from A.

4. When working on this problem we will categorize each vertex as either 'solved' or 'unsolved'. If a vertex is solved we have found the shortest path from A to that vertex. Until a vertex is 'solved' we do not yet know if we have found the shortest path from A to that vertex. Vertex A is considered 'solved' when we begin because we know the shortest way to get from A to A is to go a distance of 0.

5. Once the first row is filled in we find the 'unsolved' vertex that has the lowest distance in the chart (besides the starting vertex). In this case that vertex is C. The distance in the chart, 9_A, is the distance from A to C. Since this is the lowest current 'unsolved' number this vertex is now considered 'solved'. We will now look at paths that emanate from C. **Note** : In the chart we never increase the distances written down in a column. We only decrease them or carry our previous paths down. Thus, as soon as a vertex is 'solved' we can fill in its distance in the entire table.

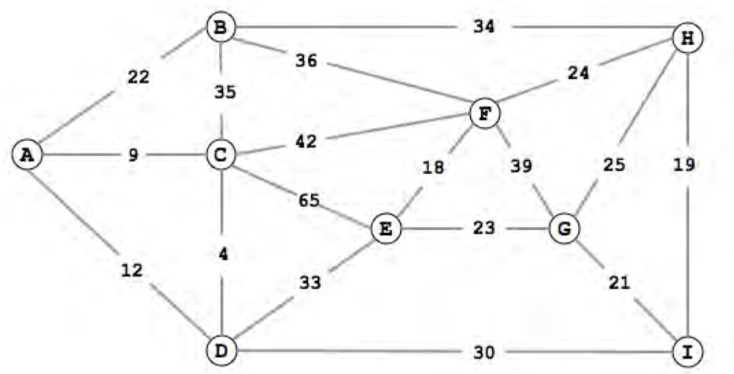
6. We will now look at the paths that emanate from C. We write the name of this vertex in the leftmost column. Using this vertex, assign distances to all vertices that can be accessed from this vertex. We cannot forget to add the distance to C from A, all distances in the table should reflect the total distance from A to the given vertex. If the distance to other vertices via C is longer than the distance already recorded in the chart carry down the distance already recorded in the chart. If it is shorter record the new distance. In this case we can now travel to E and F so we record the distances to E and F, and subscript them C because we arrived at them from C. We can also get to B and D from C but these paths are longer than the ones currently in the table so we rewrite 22_A in the B column and 12_A in the D column.

7. Now our shortest distance in the chart is 12_A, the distance to D. We will consider vertex D 'solved', and look at paths that emanate from there.

8. Continue until vertex H is 'solved'. This is the shortest path from A to H. You can re-trace your path there by following the subscripts. This may not be necessary for such a short path, but would come in handy when running the algorithm on larger or directed graphs.

Again, the video listed above (and many others) walk through this process in more depth with an example. We highly recommend you look at the video before completing the following example.

4. Find the shortest distance between A and H.



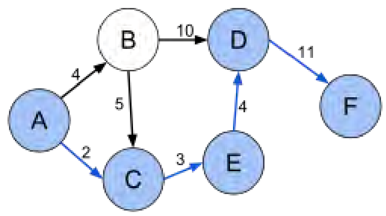
Weighted graph[17]

	A	B	C	D	E	F	G	H	I
A	0	22_A	9_A	12_A	∞	∞	∞	∞	∞
C	0	22_A	9_A	12_A	74_C	51_C	∞	∞	∞
D	0		9_A	12_A					
	0		9_A	12_A					
	0		9_A	12_A					
	0		9_A	12_A					
	0		9_A	12_A					

[17] http://stackoverflow.com/questions/20807286/minimum-sum-weight-of-connecting-3-vertices-in-an-undirected-weighted-graph-wi

A Directed Weighted Graph

Undirected weighted graphs are not the only graphs that exist within this problem. The other type of graph that exists is called a directed weighted graph. The difference between directed and undirected graphs is that directed graphs have a direction associated with their edges. It is helpful to think of directed graphs as a one-way street. You can travel only in the direction of the one-way street, this is exactly how the edges in directed graphs work. We will use the graph in below as our guide.



An undirected weighted graph[18]

If we start at vertex A, the two arrows denote which way we can go, so we can go to vertex B or vertex C. Say we go to vertex C next, the only vertex we can go to after that is vertex E whereas in the undirected graph we could go to vertex B or back to vertex A. So when traveling from vertex to vertex, follow the direction of the arrow.

5. List all possible paths from vertex A to vertex F.
6. Calculate the length of each individual path.
7. Which path is the shortest?
8. Now use Dijkstra's Algorithm to the shortest path.

Project Idea

There is a special shortest path problem that has been studied extensively for many years and still has not yet been completely solved, and it is called the traveling salesman problem. The problem is stated like this: There is a traveling salesman that has to visit x number of cities to make sales. Find the shortest path the salesman has to take to visit all of the cities exactly once and return to his original starting point.

Some ideas for a project relating to this:

- Create your own traveling salesman problem around your hometown. Use your home as your original starting place and then visit 10 places, i.e. your school, grocery store, the park. Find a couple of paths that visit each spot and return home. Then find the optimal route through all these places.
- Create a problem that uses cities from around the United States or look into the World Traveling Problem that is talked about in the fun fact.

[18] http://upload.wikimedia.org/wikipedia/commons/thumb/3/3b/Shortest_path_with_direct_weights.svg/250px-Shortest_path_with_direct_weights.svg.png

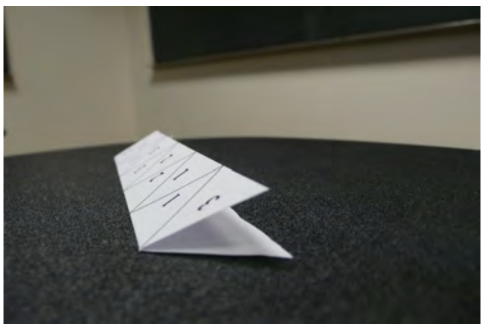
Miscellaneous

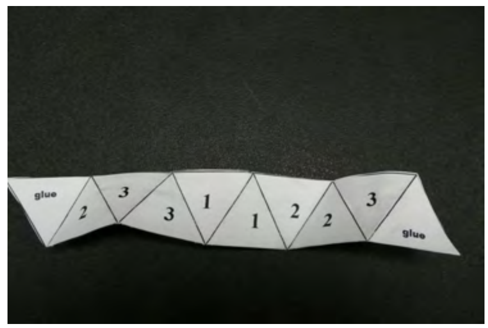
Flexagons

Flexagons are a paper craft with many surprising and interesting features. The best way to learn about flexagons is to play with them. So let's get started on constructing some!

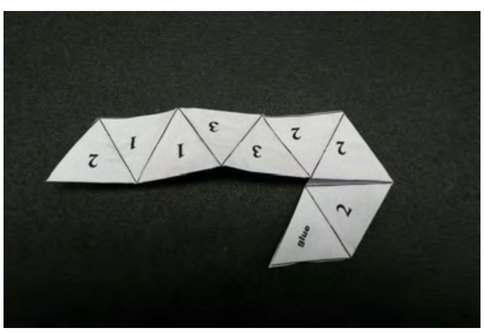
Your First Flexagon

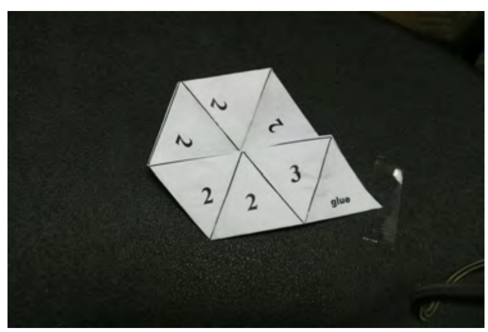
Take a copy of the template called "Trihexaflexagon." Carefully cut it out along the outer line. Next, precisely fold the strip in half along the middle line. Glue the two halves together; this gives you a double-thick row of 10 triangles. Fold back and forth along each edge connecting the triangles, so that the paper can bend both ways.



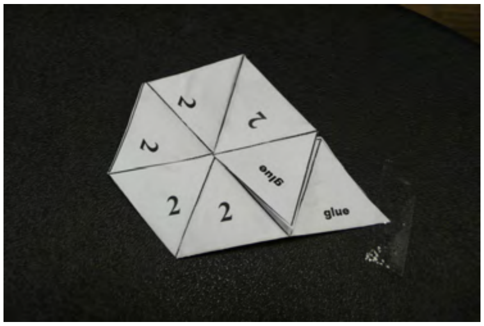


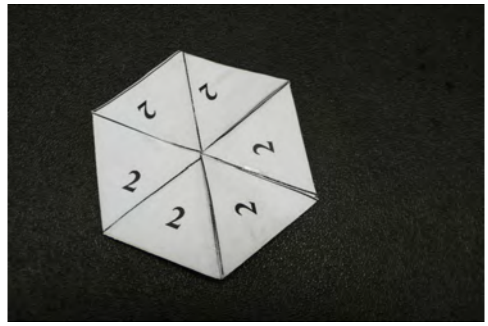
Now, with the side with "glue" labels facing up, fold the strip so that the two adjacent triangles marked with '3's face each other. Now you will have two more adjacent triangles marked '3' facing up. Fold these together as well. You should get a hexagon with five '2's and one '3', with the glue tab extending off of the hexagon.



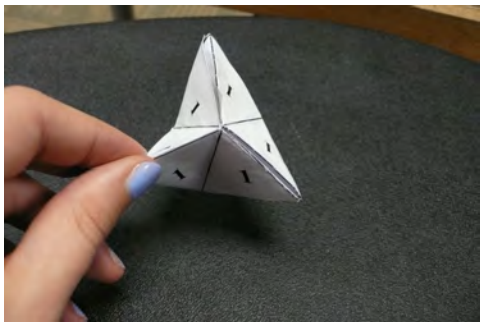
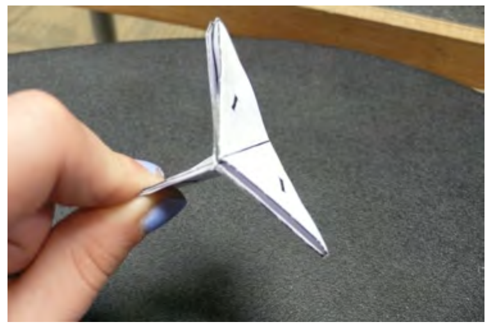
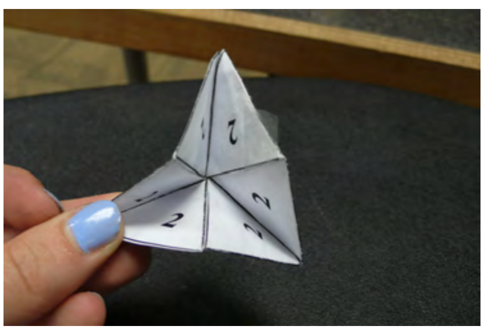
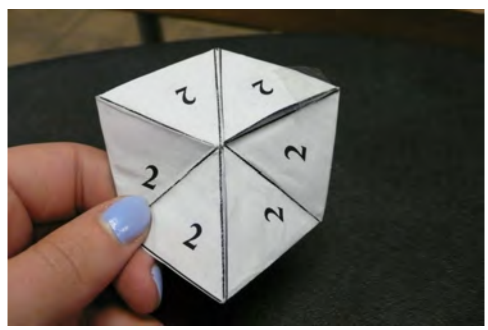


Bring up the tab underneath the triangle marked '3'. Finally, apply a little bit of glue to gluing tabs, and press them together.You should have a hexagon with '1's on one side and '2's on the other. Your flexagon is ready to flex!





To do that, pinch two adjacent triangles together with each hand, making a Y-shape. A new face will emerge from the center of this Y.

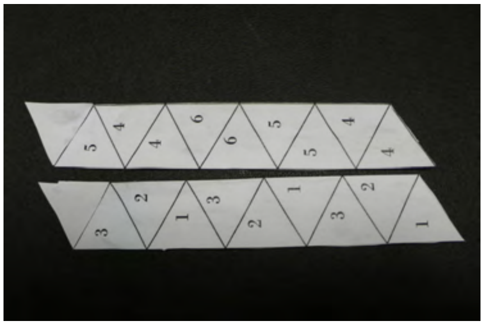





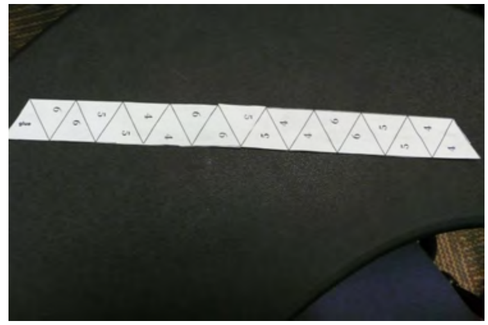
If you can't get a new face out, flatten the hexagon, and try again with a different pair of triangles.It can be tricky, especially if the flexagon is sloppily constructed, but it's easy once you get the hang of it. You can also try finding a video on Youtube with search terms such as "make a flexagon" or "trihexaflexagon".

1. How many faces (each face is identified by the numbers on the flexagon - all the 1's up is a face, for example) can you find?
2. As you flex through faces, what do you notice about the order that the faces appear in?
3. What happens if you flip your flexagon upside down? Does the order change?

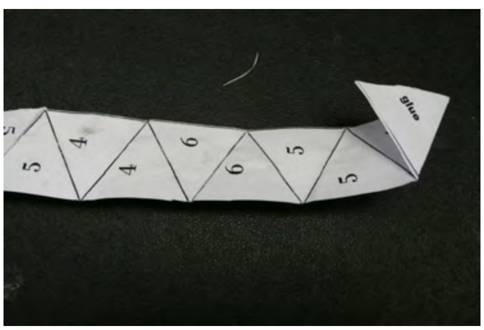
HexaHexaFlexagon

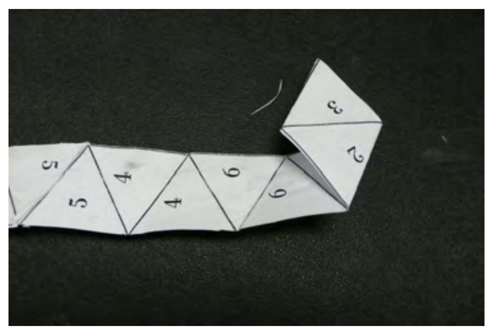
Carefully cut out both halves of the template labeled "Hexahexaflexagon", fold along the center line so that the numbers face out. Glue along the inside, so that the strip is doubled up. Each segment should have a single-sided triangle projecting off one side. Carefully paste together these two triangles, so that you have one long strip of 19 triangles.

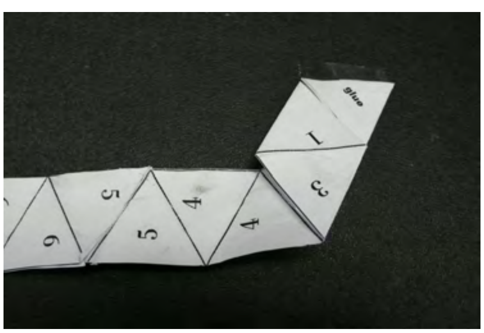


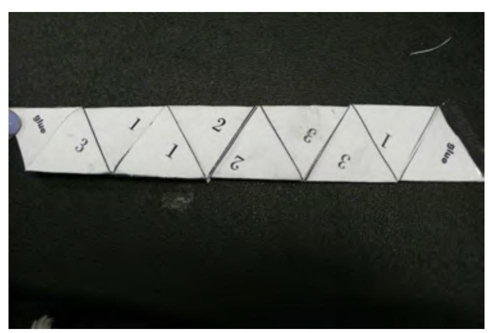


One side should have the numbers 1, 2, and 3, and the other side should have the numbers 4, 5, 6. Now, bring together the two triangles with 4 on them which are at the end of the strip. Then bring together the two ’5’ faces. Next, match the ’6’ triangles together, Continue in this process, coiling the strip around itself, until you have a strip with only 1s, 2s, and 3s showing.

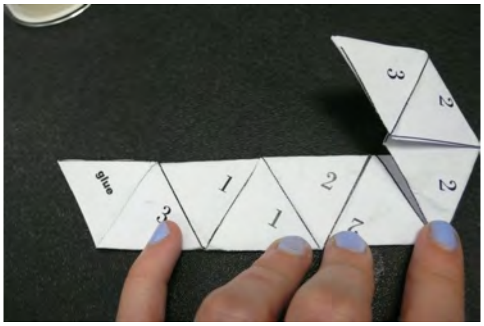


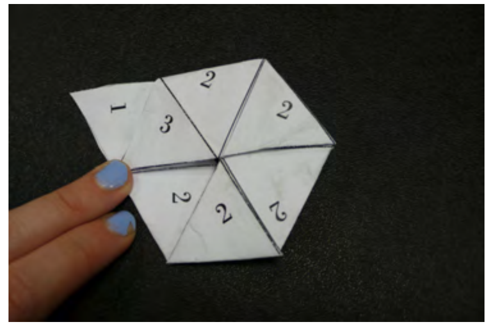


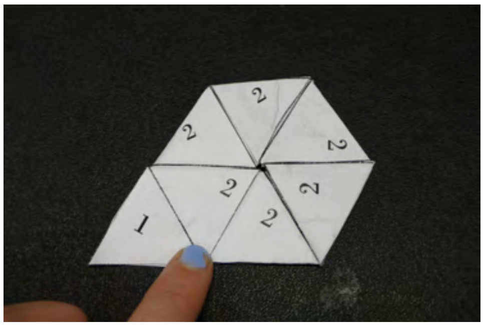


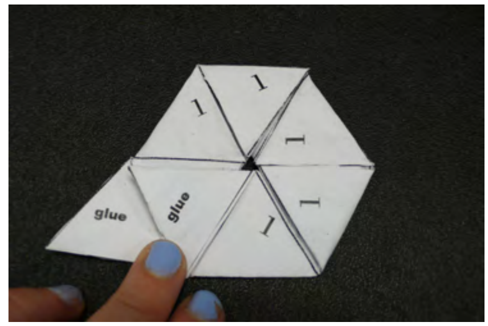


From here it folds exactly the same as a trihexaflexagon– you should end up with a hexagon that has all the numbers matching on both sides. Paste the glue tabs together.









4. Write down a few sentences predicting how the faces of the hexahexaflexagon will appear. Based on the name, how many faces do you think there should be?

   Now start flexing! This one may be a little more complicated to flex.

5. How many faces can you find?
6. What patterns do you start to see?
7. How does this new flexagon behave differently from the trihexaflexagon?
8. Choose one corner to pinch to start your flexing. Then, move to an adjacent corner around the hexagon, and flex again. What pattern do you find? How many faces show up in this cycle?
9. Choose a corner of your flexagon, and flex "at that corner", bringing together the two triangles that share it. Keep careful track of the corner you flexed at, and flex at this same corner again. Continue doing this. You will eventually come to a corner which you cannot flex at. Is this surprising?

If you haven't found all the faces yet, you now have the tools to do so. The Tuckerman Traverse, the fastest way to reveal all 6 faces, consists of flexing at a single corner until you can't flex at that corner anymore, moving to an adjacent corner, and repeating these two steps.

10. When you've reached a corner you can't flex on, how many different ways can you flex that face? Can you find faces that you can flex more than one way?
11. Are there different types of faces?

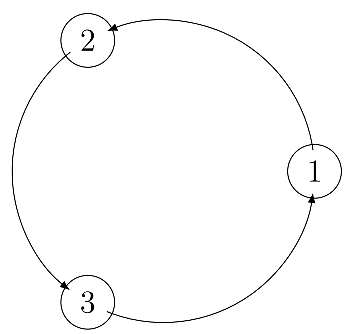
12. Try to make a list of which faces can be flexed in two ways, and which faces can only be flexed one way.

If you're having trouble making this list, that makes sense. Certain faces are sometimes flexible two ways, and sometimes only flexible one way. It seems like there are, for instance, two different 1 faces. But we know from constructing the flexagon that there are only 6 triangles with a particular number.

13. What do you think could be causing this?
14. Try drawing a star or circle around the center of the hexagon, passing over all of the triangles. What happens to this star as you continue to flex? Does this relate to the number of ways the face can be flexed?

Mapping The Flexagon

Consider this illustration of a simple cycle.



15. How does this picture relate to the pattern of faces we saw in the triflexagon?
16. In which of the steps above did you find the same pattern in the hexahexaflexagon?
17. How many different cycles of three faces can you find in the hexahexaflexagon?
18. For each of these simple cycles, mark with a star the faces that can be flexed two different ways.
19. Can you figure out a map for the entire hexahexaflexagon which combines the maps for the four simple cycles? How could you show the fact that some faces may be flexed in two different ways (leading to two different faces)?

Extensions

- Flexagons were first developed at Princeton in the late 30's and early 40's. Research the members of the Princeton Flexagon Committee. Who were they? What did they go on to do? Read about Feynman diagrams. What do they represent? What do they have in common with our diagrams of the hexahexaflexagon?

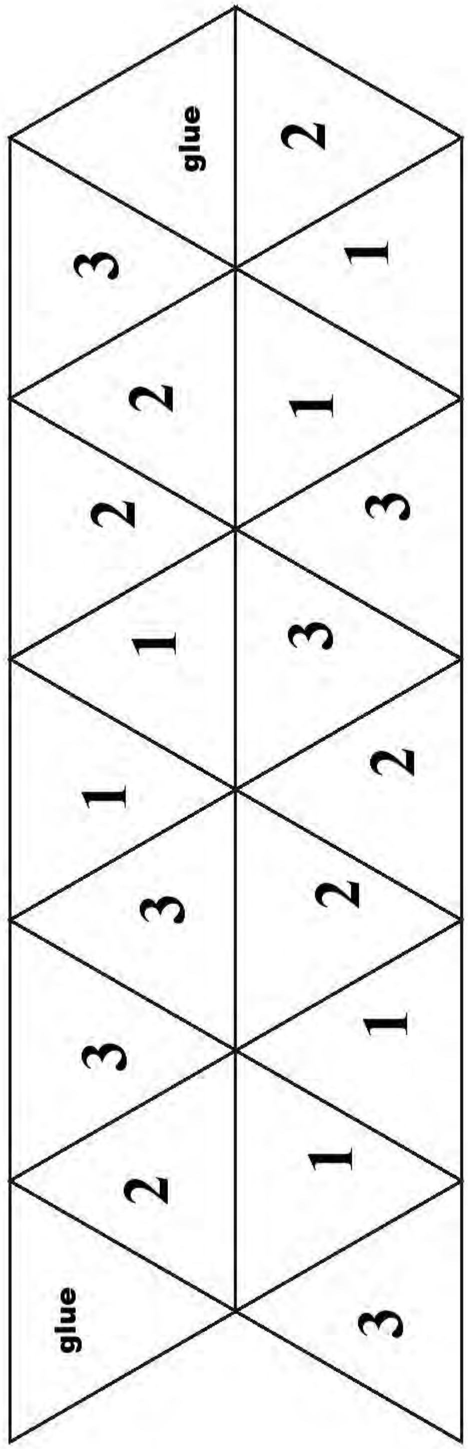
- Flexagons do not have to be hexagonal. Using `www.flexagon.net` (or other books and websites), make some new types of flexagons. Can you draw a diagram showing the faces of these new flexagons?

- We saw that the star drawn in the center of one of the faces was transformed by flexing. What other kinds of designs and patterns can you draw over the face of the flexagons? What happens to them?

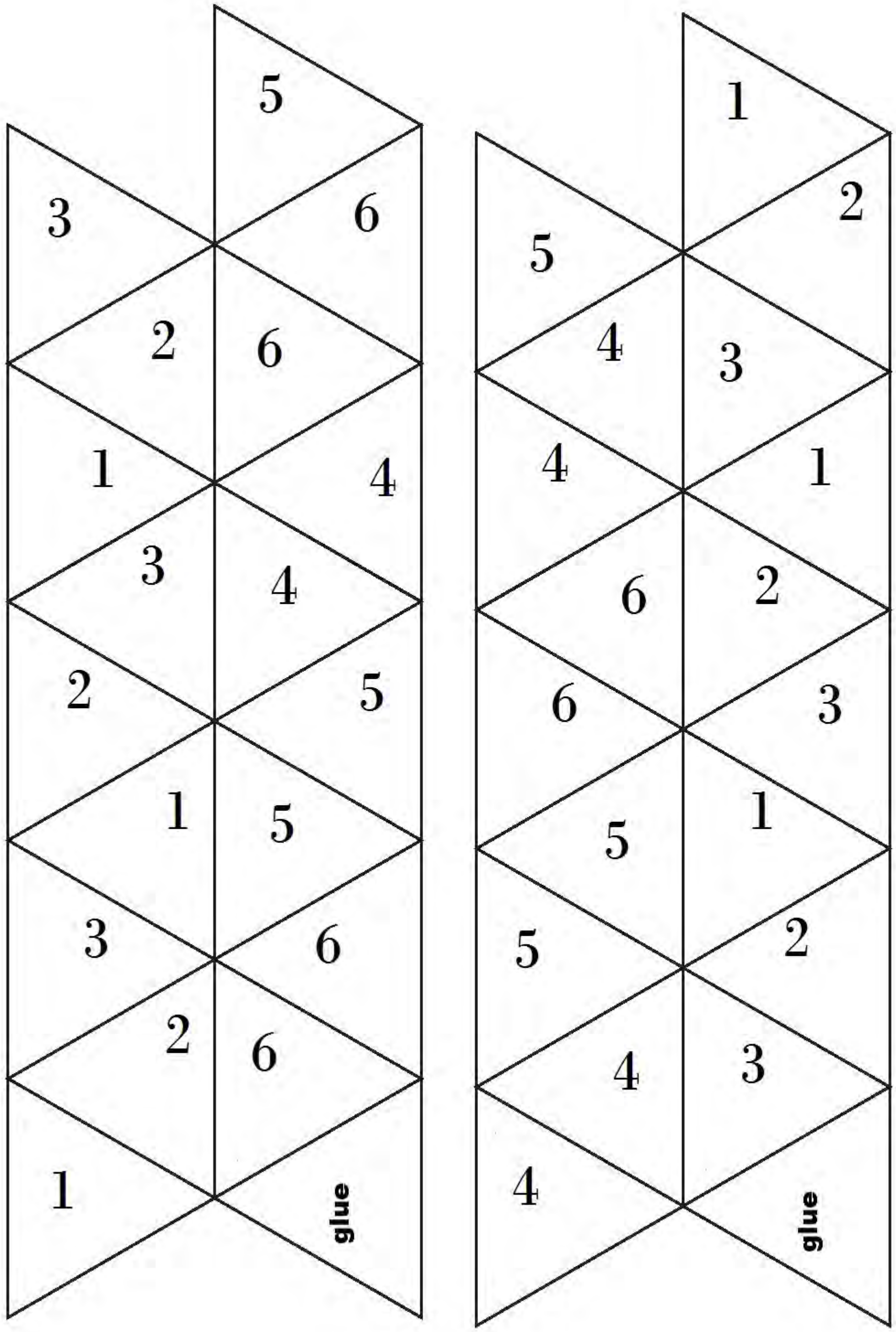
Resources

- `www.flexagon.net` This Flexagon Portal has templates for a variety of different designs and patterns of flexagons.
- Vi Hart's hexaflexagon videos. These videos show the process of making hexaflexagons, as well as providing historical context. `https://www.youtube.com/watch?v=VIVIegSt81k`
- "Hexaflexagons and Other Mathematical Diversions," Martin Gardner. Martin Gardner brought hexaflexagons to the public eye with a column in *Scientific American*. The chapter on hexaflexagons expands on this.

Trihexaflexagon



Hexahexaflexagon



Miscellaneous

Math in Literature

We tend to think of "Math" and "English" as being inherently opposed. But in fact, mathematics has featured in fiction in one way or another in a tradition that's centuries old. Why do writers include this mathematics? What does it add? In this activity, we will zoom in on some of the mathematics featured in and inspired by three different novels: *Flatland* by Edwin A. Abbot, *Alice's Adventures in Wonderland* and *Through The Looking Glass* by Lewis Carroll, and *The Number Devil* by Hans Magnus Enzensberger. They all use mathematics to tell stories, but in very different fashions.

Flatland

> "Imagine a vast sheet of paper on which straight Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, or on the surface, but without the power of rising above or sinking below it, very much like shadows– only hard, and without luminous edges– and you will have a pretty correct notion of my country and countrymen. Alas, a few years ago I should have said "my universe": but now my mind has been opened to higher views of things."
>
> *Flatland: A Romance of Many Dimensions* Edwin A. Abbot (1884)

In this classic work of science-fiction, the author explores what life would be like in the two-dimensional nation inhabited by flat shapes. How could these Flatlanders see? How could they determine who they were talking to, or where they were? In what could almost pass for an extended daydream from geometry class, he gives a vision of the homes, laws, and personal lives of beings who are entirely flat. It also serves as a satire of the social norms and beliefs of the time. There is a strong hierarchy of shapes on the plane, reflective of the classist society of Victorian England.

The novel was extremely radical in its time, and introduced many Victorians to the concept of moving through dimensions as well as the possibility of a fourth dimension. Notably, the main character is a square who is able to "pop" off of the plane, and into our three-dimensional world.

1. Read *Flatland* (it's not that long!) and write a paragraph or two about the math involved and how Abbott uses it as social commentary.

Just as the sphere challenges A. Square's notions of space, we will now pull more familiar two-dimensional concepts into the third dimension. What happens when we think about taking a familiar game played on a flat surface, and think about moving the game into three dimensions? Tic-tac-toe could exist in Flatland (although it might be hard for residents to see the pieces), but what would happen if we moved the game to three dimensions? One way to do this is by putting the game board on the surface of a donut.

It's possible to fold any rectangular flat surface (paper, cloth, etc.) into a donut shape in two simple steps. First connect two opposite edges, giving you a cylinder. Then, although you may not physically be able to do it, you could connect the two circular ends of the cylinder to form a proper donut (what mathematicians call a "torus"). But how do you manage a game on this new board?

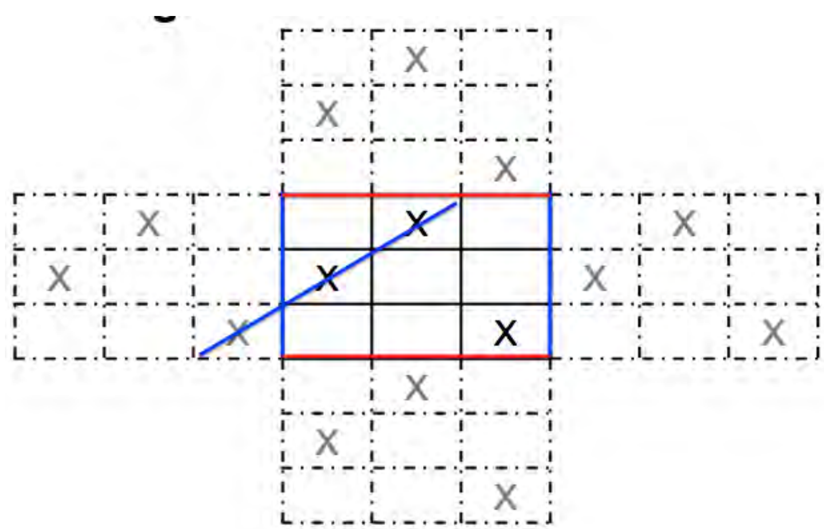
A simpler way to play this new game actually leaves it flat - much easier for A Square and his friends! Instead of folding the board, you consider which squares would be next to each other if you did fold it. In the board pictured below, we think of the solid lines at the top and bottom of the board as being the same line, and think of the dashed lines on the left and right side as being the same line too. This means that if you move one square to the right of our board, you end up (just like Pac-Man!) in the left column. Similarly, the board can be thought of as curving backwards, so that its top and bottom edges connect.

This means that both of the following tic-tac-toe boards are displaying winning moves!

X		
	X	
		X

	X	
X		
		X

The latter configuration can be further illustrated by drawing "echoes" of the game board, showing the connections that would exist if you were actually on the donut. Now it's easy to see the three-in-a-row!



2. You can still win by playing in each of the "standard" tic-tac-toe ways. How many additional ways to win are there?

3. What is similar about all of the new winning positions?

4. Play a few games (at least four) of standard tic-tac-toe with a friend, followed by a few games with the new donut rules. What do you notice? How do the standard games usually end? What about the donut version?

What happens to the game if you add a twist in the donut? Align the solid edges as before, but when you're imagining the connection between the dotted lines, flip one so the bottom connects to the top of the other. On the following board, the two X's are next to each other, but the bottom X remains in the same column as the O, as in the first (untwisted) donut.

X		O
		X

Now the following shows another winning move!

X		
	X	X

5. Now what new winning moves are there?
6. Again, play a few games with your friend (be careful to explain the new rules!). How does the game change?

Project Ideas

- What would happen if you had two twists in your donut? Flip one of the solid sides as well as one of the dotted sides, and think about what happens.
- Tic-tac-toe is a solved game. What does it mean for mathematicians to solve a game? What are some other solved games? Choose one of these games to play with your friends. Does knowing that it's solved change your strategy or opinion of the game? Is tic-tac-toe on a donut solved? Is it possible to tie?
- Flatland introduces the idea of higher dimensions than even the three we live in. How would other three-dimensional shapes appear as they passed through Flatland? Think about cubes, pyramids, rectangular prisms, cylinders, and any other shapes you want. What about irregular shapes like a person, bicycle, guitar, teapot, fork, or others? If you're having trouble imagining it, try making models! Does it matter which part passes into Flatland first? What would these objects look like in Lineland? What about Flatland objects in Lineland: stars, triangles, hexagons, or any curvy blob? What might a fourth dimension cube look like as it passed through our world?
- In *A Wrinkle in Time*, by Madeline L'Engle, the three heroes travel by tesseract. This is explained at the very beginning of chapter 5. How would you explain this concept to Flatlanders?
- Rob Bryanton, through the YouTube channel 10thdim, created a movie (`https://www.youtube.com/watch?v=gg85IH3vghA`) walking through ten whole dimensions! Can you explain higher dimensions to a classmate?

Alice in Wonderland

As you likely know, this novel, written by mathematician Charles Dodgson (under the name Lewis Carroll) in 1865 follows Alice's dream-adventures into a strange land of her imagination. *Through the Looking Glass* takes a slightly older Alice beyond her mirror into Looking-Glass land, further extending Carroll's logical nonsense. You may not have known, however, about all the mathematical concepts Carroll hid in the story!

- Much of Carroll's humor hinges on the seemingly illogical characters and events in Wonderland, but much of the nonsense can be explained through mathematics and formal logic. Look into his logic in sources such as:
  - *The White Knight*, by A. L. Taylor
  - Works of Christopher Pierce
  - *A Tangled Tale* by Lewis Carroll
  - "Logic and the Humor of Lewis Carroll" Peter Alexander (Proceedings of the Leeds Philosophical Society, Vol. 6, May 1951, pgs 551-66)

  Other logical (and illogical) concepts to look into include:
  - Null classes (treating nothing as something)
  - Alice's multiplication in the Pool of Tears chapter
  - Logical contradiction (as in, the hill so large that this hill appears a valley)
  - The logical discussion of the Knight's song and its name
  - Two-valued logic
- Mathematicians have also discussed Alice in Wonderland as an attempt at commentary on then-recent innovations in mathematics, such as: `http://www.newscientist.com/article/mg20427391.600-alices-adventures-in-algebra-wonderland-solved.html?full=true#.VTk8uiFVhBc` and `https://www.maa.org/external_archive/devlin/devlin_03_10.html`

  Investigate one of the scenes discussed in the articles. Does it seem a valid interpretation?
- *Through the Looking Glass* is very explicitly structured as a chess game, with the setup shown at the beginning of the book. It is more properly titled a chess problem, where the initial board layout and a goal are explicitly assigned. If you're unfamiliar with chess, pause now to look up the pieces and how they move. Then try to follow the chess game through Carroll's story! Although the moves aren't always perfectly alternating, he is very good at only letting Alice talk to pieces on squares next to hers!
- Chess puzzles are also a broader category - go find other chess puzzles to solve! Some ask you to achieve a certain position, while in others the goal is to mate the opposing king. Websites such as gameknot.com and chess.com have lots of puzzle beginnings available!

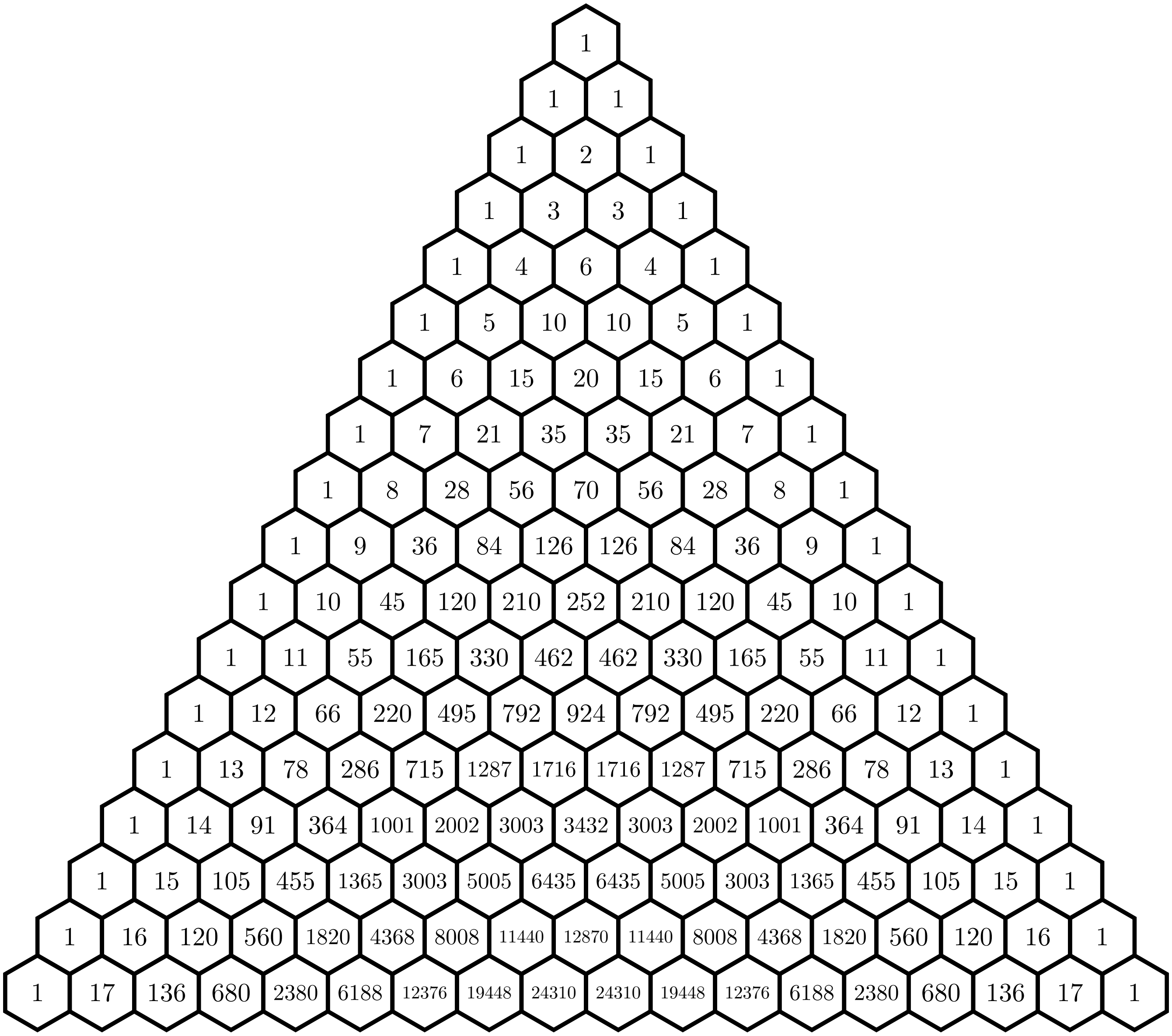
- Another category of chess puzzles takes advantage of the knight's unusual move pattern (something Carroll also pokes fun at by having the knights fall off their horses so frequently). The Knight's Tour challenges the player to cover the board by moving the knight, without revisiting any squares. Knights move in an "L" shape, two squares in any direction then one square sideways. There are almost 10,000 possible tours (routes the knight can take to reach every square) on a 6x6 chessboard. Try playing a few rounds, see if you can find one of them. Then investigate solutions and their connections to mathematics (magic squares!). What happens if you try to do a Knight's Tour on a donut, like the Tic-Tac-Toe in the Flatland section?? Other knight games include Knight's Move, where you switch the position of two knights.
- Through the Looking Glass also plays variously with the idea of symmetry and inversion. Many molecules, as well as more common items like hands are asymmetric, also known as "chiral." Investigate chirality. Why might this cause the looking-glass milk to be unhealthy for Alice? What is the connection between chirality and anti-matter? Screws are also chiral - how does that influence ship and plane construction?

The Number Devil

Inspired by Alice's adventures, this novel focuses on the dreamworld explorations of Robert, a young boy who hates math class. In fact, he hates it so much that it seems to haunt his dreams, in the form of the enigmatic and tricky Number Devil. With lavish illustrations, Enzensberger takes an informal route through some of the most beautiful and beguiling subjects in mathematics. He uses the fiction to make the math a bit easier to swallow, removing it from the stiff atmosphere we associate with math. In one chapter, Robert and the Number Devil decorate a pyramid of cubes with summed numbers. Each entry is determined by the sum of the two numbers above it. We will work with a version of this picture that uses hexagons instead of cubes.

- Using the picture attached, mark each of the even numbers on the pyramid. What pattern do you see?
- Next, try coloring in each of the numbers which is divisible by 5 (remember, numbers divisible by 5 end in either a 5 or a 0). It may be helpful to print out extra copies of the picture.
- In the book, Robert is extremely surprised to see these patterns. Are you surprised? Can you explain these patterns?
- Try adding up all of the numbers in a given row of hexagons. What do you get?
- What other patterns can you discover? Is there one explanation for all of these patterns?
- In fact, this pattern of numbers is known as Pascal's triangle, and these patterns have been investigated before by mathematicians. Why do you think that the author chose to present it without this context?

Diagram for Number Devil Questions



Project Idea

Find a work of fiction which features mathematics– some ideas are listed below. After reading it, write about some of the math that appears in the book. How could you present some of those ideas to your peers? How do they relate to other mathematics you've done?

- *Sphereland* by Dionys Burger, a sequel to Flatland. The sphere returns to Flatland, showing the residents that they live on the flat surface of a sphere, and they are not alone in the universe.
- *Flatterland* by Ian Stewart, a mathematically challenging book inspired by *Flatland* and *Alice in Wonderland* . Explore fractional dimensions, time travel, and escaping a black hole!
- *The Phantom Tollbooth* by Norton Juster, visits the Mathemagician and pokes fun at many familiar mathematical concepts such as averages, as well as harder to grasp ideas of infinity.
- *Cryptonomicon* by Neal Stephenson, is an epic which jumps across history, taking you through decades of cryptographical history- from the code breakers of World War II up through computer security in the modern day.
- *Arcadia* by Tom Stoppard, is a quick-witted play which takes its themes from the mathematical subjects of chaos theory, dynamic systems, and entropy.
- *An Abundance of Katherines* by John Green, whose main character attempts to derive a mathematical formula to describe his relationships. Be sure to read the footnotes and afterword!
- *The Last Universe* by William Sleator, a thriller which incorporates concepts from quantum physics.
- *A Gebra named Al* by Wendy Isdell is another explanatory novel, which aims to make algebra accessible through a zany zebra.