# The Alternating Sign Matrix Conjectures

**Supervisor**: Eric Egge

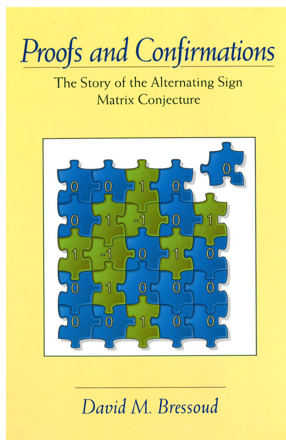
**Terms**: Fall and Winter of 2019-20

**Meeting Times**: Mondays 6a and Thursdays from 1 pm to 2 pm

**Math or Stats?** Math

**Prerequisite**: Previous experience with binomial coefficients, permutations, partitions, and generating functions. Math 333 (Combinatorial Theory) will be sufficient. However, you could also gain this experience by taking certain courses in Budapest, in an REU, or by doing some supervised independent reading.

**Short Description**: We will read the book *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, by David Bressoud, and you will teach me the math involved in the story.




**Longer Description**: An alternating sign matrix (or ASM) is a square matrix all of whose entries are 0, 1, or $-1$, and in which the nonzero entries in every row and every column are alternating in sign and add to one. Here are the seven $3 \times 3$ ASMs.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

In the table below we have the number of $n \times n$ ASMs for $1 \leq n \leq 9$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| number of $n \times n$ ASMs | 1 | 2 | 7 | 42 | 429 | 7436 | 218348 | 10850216 | 911835460 |

In the early 1980s William Mills, David Robbins, and Howard Rumsey conjectured that the number of $n \times n$ ASMs is given by the formula

$$\prod_{k=1}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

In the process of trying to prove their conjecture, Mills, Robbins, and Rumsey discovered connections between ASMs and plane partitions, along with interesting structures within the set of ASMs itself. They were even able to use their insights to prove a longstanding conjecture about cyclically symmetric plane partitions. But they were not able to prove their formula for the number of $n \times n$ ASMs.

The Mills-Robbins-Rumsey conjecture was shrouded in mystery until 1993, when Doron Zeilberger gave a long technical proof of it. Shortly after that, Greg Kuperberg discovered a connection between ASMs and the square ice model for ordinary ice from statistical physics, which he was able to use to give a much shorter proof of Mills, Robbins, and Rumsey's conjecture. Finally, in 1999 David Bressoud published a book that tells the story of the Mills-Robbins-Rumsey conjecture and its proof, along with related combinatorial stories. Our goal in this project will be for you to read Bressoud's book, and give twice-weekly presentations in which you teach me the math involved in this story. If we have time then we may also investigate some open problems involving pattern-avoiding ASMs, and how ASMs can be used to generalize Latin squares.