Variants on Kuratowski's Closure-Complement Problem

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Comps Term: Winter/Spring 2019

Prerequisites: None.

Problem Description:

The modern abstract definition of a topological space is due to the Polish mathematician Kazimierz Kuratowski, who generalized Hausdorff's ideas in his 1921 thesis. In his seminal paper, he posed the following problem, which may now be found as a (challenging!) exercise in various point-set topology textbooks:

- How many distinct subsets of the real number line $\mathbb{R}$ can one obtain by starting with a chosen subset $A \subseteq \mathbb{R}$, and then applying the topological closure operator and/or taking set complements, repeatedly, in any order?

The correct answer—fourteen!—intially seems somewhat bizarre, which may account for the theorem's lasting appeal. The proof is accomplished in two steps. To see that 14 is an upper bound for the solution, one translates into an algebraic problem: label the closure operator by k and the complement operator by c, and consider the *monoid* (i.e., group without inverses) M generated by the *idempotent* k and the *involution* c; elementary topological arguments establish that M has no more than 14 elements. To get 14 as a lower bound, one writes down her favorite set, namely

$$A = (0,1) \cup (1,2) \cup \{3\} \cup \mathbb{Q} \cap [4,5]$$

and then verifies by hand that each member of M applied to A yields a distinct set.

In the last 100 years, this digestible yet surprising theorem has inspired a small but enthusiastic literature devoted to solving variants of the Kuratowski problem, for instance

- solving the closure-complement problem in topological spaces other than $\mathbb{R}$;
- solving problems where additional set operators like union, intersection, topological boundary, etc. are allowed;
- classifying sets A which are solutions to the closure-complement problem or related set operator problems;
- etc...

Our first goal in this project is to fully understand the ideas involved in the proof of Kuratowski's theorem, and to develop a sense for a larger class of generalized "topological operator problems." This will involve a healthy mixture of elementary techniques from topology and algebra, which can really be learned on the fly without a need for extensive mathematical background.

Our second goal, if time permits, is to try to branch out and ask ourselves, can we pose a variant problem which has an interesting solution? For some initial ideas, we could study problems involving multiple closure operators, corresponding to distinct topologies on $\mathbb{R}$.

It is also interesting to consider the complexity of the solution set A: despite first appearances, the set A we mentioned above is in some sense not very complicated, as it is possible to write it as just a countable union of closed sets. Can one pose an operator problem whose only solution sets are strictly more complicated than this? Can one pose a problem with an arbitrarily complicated solution set?

This project is somewhat open-ended in that there are infinitely many questions to ask, and thus I believe there is potential for genuinely new results—at least half the challenge for us will be to assess which questions are interesting enough to demand an answer!