Special Sets of Vertices in Paley Graphs

Clara Buck

Carleton College
Collaborators: Emily Barranca and Lauren Hartmann

Northfield Undergraduate Mathematics Symposium
October 2018

Overview

Introduction

Definition

A **graph** is a collection of vertices and edges, where each edge is composed of exactly two vertices.

Introduction

Definition

A **graph** is a collection of vertices and edges, where each edge is composed of exactly two vertices.

Definition

A graph G is **strongly regular** if for parameters (v, k, λ, μ): there exist **v** vertices each adjacent to **k** others, such that any 2 adjacent vertices share λ common neighbors and any 2 nonadjacent vertices share μ common neighbors.

Definition

The **adjacency matrix** of a graph is a 0-1 matrix indexed by the graph's vertices that keeps track of which vertices are adjacent in the graph.

The adjacency matrix for a strongly regular graph will have exactly 3 eigenvalues k, θ_1, θ_2 that give us information about the structure of the graphs.

Fields

Vaguely, a **finite field**, $\mathbb{F}_q$, is a set of q elements in which addition, multiplication, subtraction, and division are defined and have properties similar to the real numbers.

- All fields have order p^k where p is prime. Fields of the same order are isomorphic.
- $\mathbb{Z}_{13}$, the integers 1 to 13, are a field.
- $\mathbb{Z}_9$, the integers 1 to 9, do not form a field, although $\mathbb{F}_9$ does exist.

The Paley Graph

Definition

A **Paley graph P(q)** is a graph

- with vertex set $\mathbb{F}_q$ where $q = p^n \equiv 1 \pmod 4$ for prime p
- vertices u,v are adjacent iff $(u - v)$ is a nonzero perfect square in $\mathbb{F}_q$

The Paley Graph

Definition

A **Paley graph P(q)** is a graph

- with vertex set $\mathbb{F}_q$ where $q = p^n \equiv 1 \pmod 4$ for prime p
- vertices u,v are adjacent iff $(u - v)$ is a nonzero perfect square in $\mathbb{F}_q$

Paley graphs are **strongly regular** with parameters:

$$\left(v, k, \lambda, \mu\right) = \left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right)$$

and eigenvalues:

$$\left(k, \theta_1, \theta_2\right) = \left(\frac{q-1}{2}, \frac{-1+\sqrt{q}}{2}, \frac{-1-\sqrt{q}}{2}\right)$$

Example: P(13)

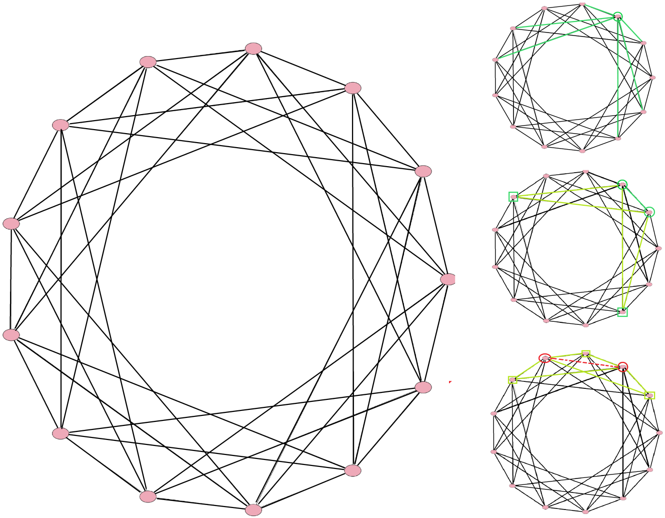


Figure 1: P(13) is a strongly regular graph with parameters (13,6,2,3).

Example: P(81)

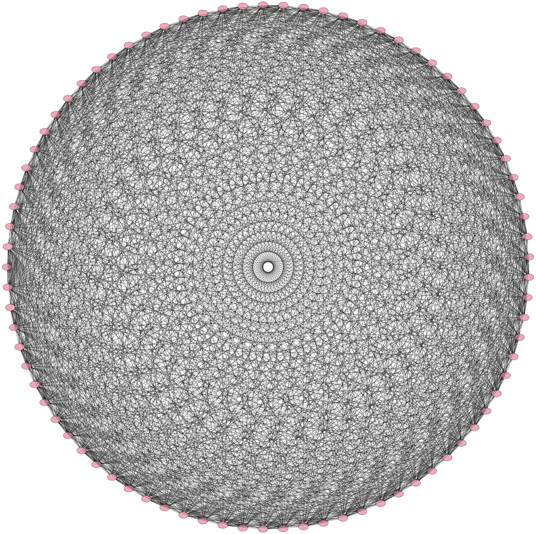


Figure 2: P(81) has a total of 1620 edges

Cliques and Independent Sets

Definition

A **clique** is a subset of a graph's vertices such that every pair of vertices in the subset are adjacent.

Cliques and Independent Sets

Definition

A **clique** is a subset of a graph's vertices such that every pair of vertices in the subset are adjacent.

Definition

An **independent set** is a subset of a graphs vertices such that no two vertices within the subset are adjacent

Cliques and Independent Sets

Definition

A **clique** is a subset of a graph's vertices such that every pair of vertices in the subset are adjacent.

Definition

An **independent set** is a subset of a graphs vertices such that no two vertices within the subset are adjacent

Paley graphs can be partitioned by their cliques or independent sets.

Cliques and Clique Decompositions Example

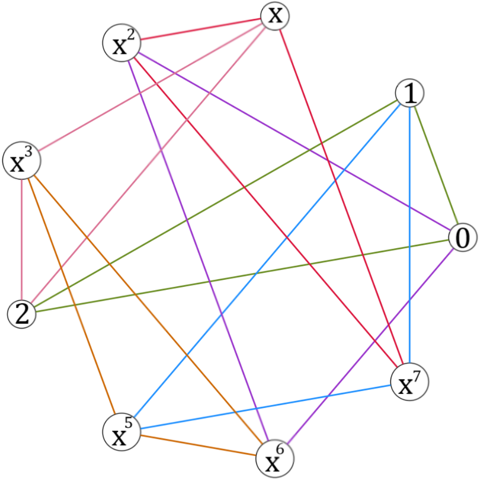


Figure 3: The clique decomposition made up of 6 cliques of order 3 in P(9), where x is a generator for the field.

Independent Sets Example

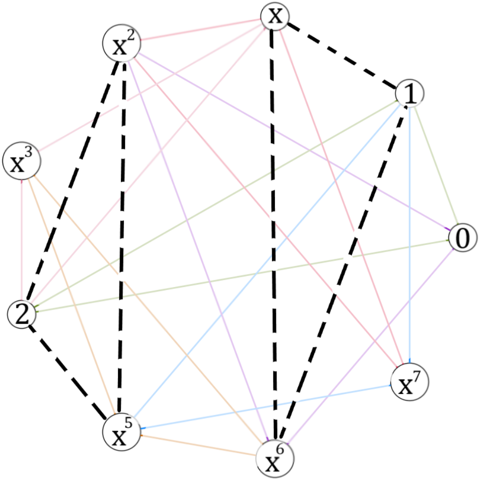


Figure 4: P(9) with two independent sets of order 3 highlighted

The eigenvalues of a graph, $k > \theta_1 \geq 0 > \theta_2$, give bounds on the order of cliques and independent sets. If $\mathbf{T}$ is a set of vertices such that each vertex is adjacent to α others, we have

$$\theta_2 + \frac{(k-\theta_2)|\mathbf{T}|}{v} \leq \alpha \leq \theta_1 + \frac{(k-\theta_1)|\mathbf{T}|}{v}$$

The eigenvalues of a graph, $k > \theta_1 \geq 0 > \theta_2$, give bounds on the order of cliques and independent sets. If $\mathbf{T}$ is a set of vertices such that each vertex is adjacent to α others, we have

$$\theta_2 + \frac{(k - \theta_2)|\mathbf{T}|}{v} \leq \alpha \leq \theta_1 + \frac{(k - \theta_1)|\mathbf{T}|}{v}$$

If $\alpha = 0$ then $\mathbf{T}$ is an independent set and if $\alpha = |\mathbf{T}| - 1$ then $\mathbf{T}$ is a clique.

The eigenvalues of a graph, $k > \theta_1 \geq 0 > \theta_2$, give bounds on the order of cliques and independent sets. If $\mathbf{T}$ is a set of vertices such that each vertex is adjacent to α others, we have

$$\theta_2 + \frac{(k - \theta_2)|\mathbf{T}|}{v} \leq \ \alpha \leq \ \theta_1 + \frac{(k - \theta_1)|\mathbf{T}|}{v}$$

If $\alpha = 0$ then $\mathbf{T}$ is an independent set and if $\alpha = |\mathbf{T}| - 1$ then $\mathbf{T}$ is a clique.

Not all cliques and independent sets meet these bounds. Those which do meet the bounds have special properties.

Motivating Tight Sets

Given a "special" clique in a strongly regular graph, any vertex outside the clique will be adjacent to α' vertices inside the clique, where α' is a constant.

Motivating Tight Sets

Given a "special" clique in a strongly regular graph, any vertex outside the clique will be adjacent to α' vertices inside the clique, where α' is a constant.

Can these "special" cliques and independent sets be generalized?

Cliques Revisited

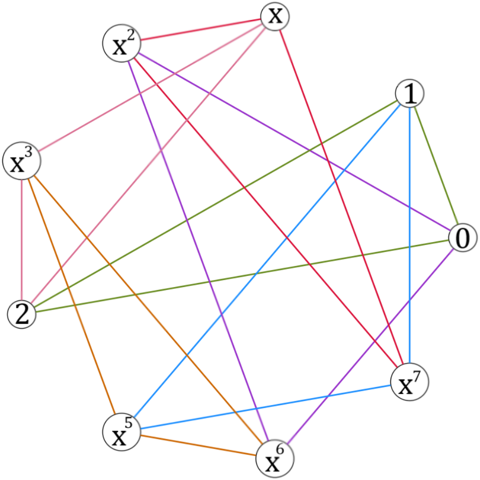


Figure 5: Cliques of order 3 in P(9) are "special", with $\alpha' = 1$.

Motivating Tight Sets

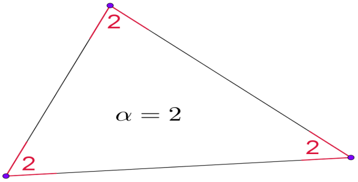
Definition

Given a vertex set **V** and a subset **T** $\subseteq$ **V**, where on average each vertex in the set is adjacent to α others in the set and each vertex not in the set is adjacent to α' vertices in the set, we call α the **interior intersection number** and α' the **exterior intersection number**.

Motivating Tight Sets

Definition

Given a vertex set **V** and a subset **T** $\subseteq$ **V**, where on average each vertex in the set is adjacent to α others in the set and each vertex not in the set is adjacent to α' vertices in the set, we call α the **interior intersection number** and α' the **exterior intersection number**.



Example: In a clique with 3 vertices, $\alpha = 2$ because each vertex is connected to 2 others.

Introduction to Tight Sets

The adjacency matrix of a strongly regular graph (SRG) has 3 distinct eigenvalues where $k > \theta_1 \geq 0 > \theta_2$. Given a subset $\mathbf{T}$ of the vertices with intersection number α, we obtain the following:

$$\theta_2 + \frac{(k - \theta_2)|\mathbf{T}|}{v} \leq \alpha \leq \theta_1 + \frac{(k - \theta_1)|\mathbf{T}|}{v}$$

Introduction to Tight Sets

The adjacency matrix of a strongly regular graph (SRG) has 3 distinct eigenvalues where $k > \theta_1 \geq 0 > \theta_2$. Given a subset $\mathbf{T}$ of the vertices with intersection number α, we obtain the following:

$$\theta_2 + \frac{(k-\theta_2)|\mathbf{T}|}{v} \leq \alpha \leq \theta_1 + \frac{(k-\theta_1)|\mathbf{T}|}{v}$$

For the Paley Graph $P(q^2)$,

$$\frac{1}{2}(q-1)(\frac{|\mathbf{T}|}{q}+1) \leq \alpha \leq \frac{1}{2}(q-1)(\frac{|\mathbf{T}|}{q}+1)$$

Introduction to Tight Sets

The adjacency matrix of a strongly regular graph (SRG) has 3 distinct eigenvalues where $k > \theta_1 \geq 0 > \theta_2$. Given a subset $\mathbf{T}$ of the vertices with intersection number α, we obtain the following:

$$\theta_2 + \frac{(k-\theta_2)|\mathbf{T}|}{v} \leq \alpha \leq \theta_1 + \frac{(k-\theta_1)|\mathbf{T}|}{v}$$

For the Paley Graph $P(q^2)$,

$$\tfrac{1}{2}(q-1)(\tfrac{|\mathbf{T}|}{q}+1) \leq \alpha \leq \tfrac{1}{2}(q-1)(\tfrac{|\mathbf{T}|}{q}+1)$$

If the upper or lower bound on α is achieved, we have a "tight interlacing" and $\mathbf{T}$ is a tight set.

The subgraph induced by a tight set is always α-regular (for SRGs).

Tight Sets

Because of their eigenvalues, only Paley graphs of order q^2 where $q \in \mathbb{Z}$ contain tight sets, so we will refer to $P(q^2)$.

Definition

A set $\mathbf{T}$ of vertices in $P(q^2)$ is a **tight set Type I** if each vertex of $\mathbf{T}$ is adjacent to exactly $\alpha = \frac{1}{2}(q+1)(\frac{|\mathbf{T}|}{q} - 1)$ other elements of $\mathbf{T}$.

Tight Sets

Because of their eigenvalues, only Paley graphs of order q^2 where $q \in \mathbb{Z}$ contain tight sets, so we will refer to $P(q^2)$.

Definition

A set $\mathbf{T}$ of vertices in $P(q^2)$ is a **tight set Type I** if each vertex of $\mathbf{T}$ is adjacent to exactly $\alpha = \frac{1}{2}(q+1)(\frac{|\mathbf{T}|}{q} - 1)$ other elements of $\mathbf{T}$.

Definition

A set $\mathbf{T}$ of vertices in $P(q^2)$ is a **tight set Type II** if each vertex of $\mathbf{T}$ is adjacent to exactly $\alpha = \frac{1}{2}(q-1)(\frac{|\mathbf{T}|}{q} + 1)$ other elements of $\mathbf{T}$.

Type I generalizes tight independent sets and Type II generalizes tight cliques.

Parameters of Tight Sets

Theorem

If $\mathbf{T}$ is a tight set in $P(q^2)$, $|\mathbf{T}| = cq$ for some $c \in \mathbb{Z}$ where $1 \leq c \leq q$.

Definition

We refer to a tight set of order cq as a tight set of **parameter c**.

Example: In $P(q^2) = P(25)$, a tight set of order 5 is of parameter 1, whereas a tight set of order $2 * 5 = 10$ is of parameter 2.

Tight Sets: Example

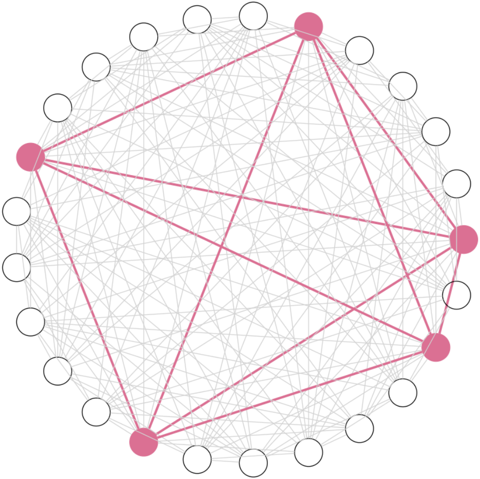


Figure 6: A tight set of Type II of parameter 1 in P(25). Note: This is a tight set and a clique, so it is a tight clique.

Finding Tight Sets in Paley Graphs

- Method 1:
  - Search through all possible subgraphs G of size cq where $c \in \mathbb{Z}$ and test if:
    $\forall s \in G, \alpha = |N(s) \cap \mathbf{T}| = \theta + \frac{(k-\theta)|T|}{v}$

Finding Tight Sets in Paley Graphs

- Method 1:
  - Search through all possible subgraphs G of size cq where $c \in \mathbb{Z}$ and test if:
    $\forall s \in G, \alpha = |N(s) \cap \mathbf{T}| = \theta + \frac{(k-\theta)|T|}{v}$
- Method 2:
  - Search for characteristic vectors in the eigenspace of the graph's adjacency matrix.

Finding Tight Sets in Paley Graphs

Definition

A **characteristic vector** of a graph is a 0-1 vector that corresponds to a subset of vertices.

Remark: For a graph with v vertices, characteristic vectors live in the vector space $\mathbb{R}^v$.

For a strongly regular graph, $\mathbb{R}^v = E_k \bigoplus E_{\theta_1} \bigoplus E_{\theta_2}$

Finding Tight Sets in Paley Graphs

Definition

A **characteristic vector** of a graph is a 0-1 vector that corresponds to a subset of vertices.

Remark: For a graph with v vertices, characteristic vectors live in the vector space $\mathbb{R}^v$.

For a strongly regular graph, $\mathbb{R}^v = E_k \bigoplus E_{\theta_1} \bigoplus E_{\theta_2}$

We can find characteristic vectors for:

- a tight set of Type I in the basis for $E_k \bigoplus E_{\theta_2}$
- a tight set of Type II in the basis for $E_k \bigoplus E_{\theta_1}$

Tight Sets in Paley Graphs

P(9)	Type II
Parameter	**Number of Sets**
3	1
2	6
1	6
0	1
Total	14

P(25)	Type II
Parameter	**Number of Sets**
5	1
4	15
3	130
2	130
1	15
0	1
Total	292

Recall: For a tight set $\mathbf{T}$ of parameter c, $|\mathbf{T}| = cq$.

Tight Sets in Paley Graphs

P(49)	Type II
Parameter	**Number of Sets**
7	1
6	28
5	672
4	5,726
3	5,726
2	672
1	28
0	1
Total	12,854

P(81)	Type II
Parameter	**Number of Sets**
9	1
8	45
7	4500
6	141540
5	1106550
4	1106550
3	141540
2	4500
1	45
0	1
Total	2,505,272

Recall: For a tight set $\mathbf{T}$ of parameter c, $|\mathbf{T}| = cq$.

Preliminary Results

Theorem

$P(q^2)$ has the same number of Type I and Type II tight sets of each parameter.

Preliminary Results

Theorem

$P(q^2)$ has the same number of Type I and Type II tight sets of each parameter.

Theorem

In $P(q^2)$ there are the same number of tight sets (type I or II) of parameter m ($m \neq q$) as there are of parameter $q - m$.

Tight Sets in Paley Graphs Revisited

P(9)	Type II
Parameter	**Number of Sets**
3	1
2	6
1	6
0	1
Total	14

P(25)	Type II
Parameter	**Number of Sets**
5	1
4	15
3	130
2	130
1	15
0	1
Total	292

Recall: For a tight set **T** of parameter c, $|\mathbf{T}| = cq$.

Affine Planes

Definition

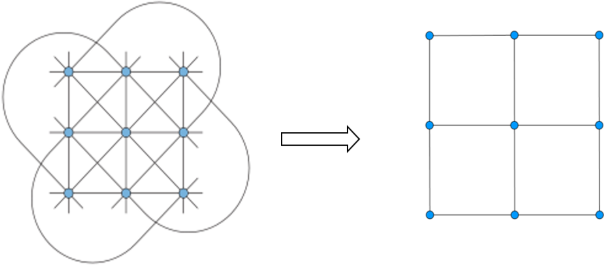
The **affine plane** is a linear space with at least three noncollinear points, in which any given point p and line ℓ not containing p there is exactly one line m through p which does not meet ℓ.

Affine Planes

Definition

The **affine plane** is a linear space with at least three noncollinear points, in which any given point p and line ℓ not containing p there is exactly one line m through p which does not meet ℓ.

Remark: We use partial affine planes of $AG(2, q)$ with q^2 points, which include half of the lines of a full affine plane.



The Paley Graph in the Affine Plane

$P(q^2)$ can be represented geometrically in a partial affine plane:

- Two points are on a line in the partial $AG(2, q)$ if and only if they are adjacent in $P(q^2)$.
- Consequently, lines in the partial plane are cliques in the graph

The Paley Graph in the Affine Plane: P(9)

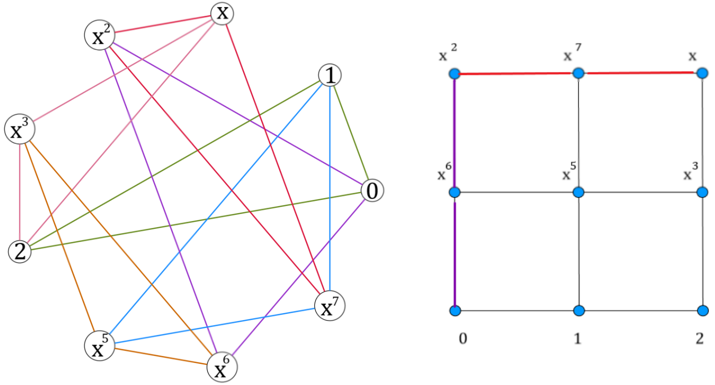


Figure 7: The graph of P(9) and P(9) as a partial affine plane. Each line in this partial affine plane is a clique in the graph.

Describing Tight Sets

Definition

A tight set is **indecomposable** if it is not the union of smaller disjoint tight sets.

Definition

An **isomorphism class** is a class of tight sets under an edge preserving bijection (their graphs look the same).

The Paley Graph in the Affine Plane: P(25)

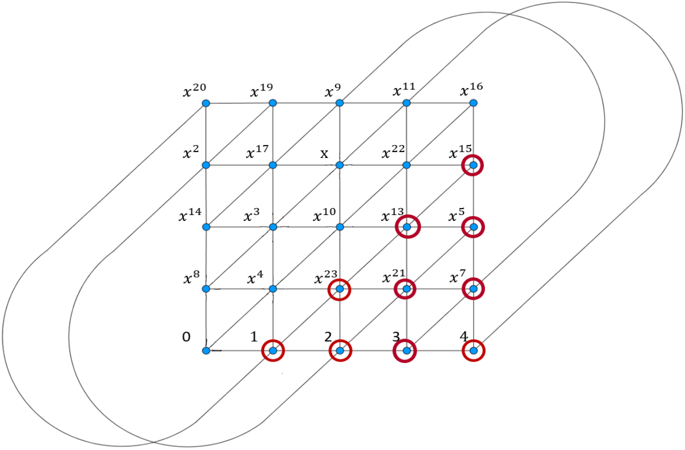


Figure 8: Affine P(25) with a tight set Type II of parameter 2 highlighted

The Paley Graph in the Affine Plane: P(25)

Observations:

- There are $\binom{5}{2} * 3 = 30$ tight sets of parameter 2 which are the union of two disjoint cliques.

The Paley Graph in the Affine Plane: P(25)

Observations:

- There are $\binom{5}{2} * 3 = 30$ tight sets of parameter 2 which are the union of two disjoint cliques.
- There are 100 tight sets of parameter 2 which are *indecomposable*.
- These indecomposable tight sets of parameter 2 in $P(25)$ are all isomorphic.

P(25) Tight Set Data Revisited

P(25)	Type II		
Parameter	**Number of Sets**	**Isomorphism Classes**	**Indecomposable**
5	1	1	0
4	15	1	0
3	130	2	0
2	130	2	1
1	15	1	1
0	1	1	0
Total	292	8	2

The Paley Graph in the Affine Plane: P(49)

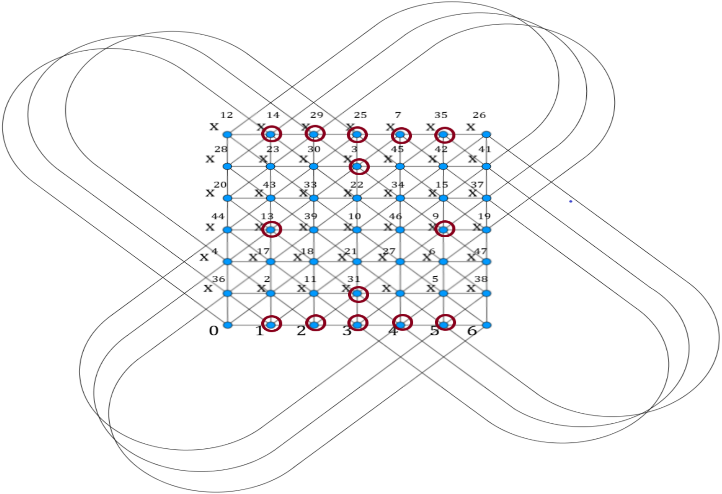


Figure 9: A parameter 2 tight set of P(49) highlighted in the affine plane

The Paley Graph in the Affine Plane: P(49)

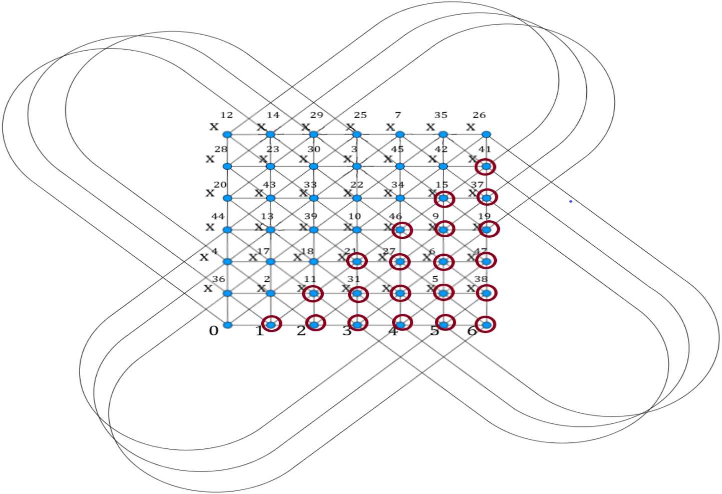


Figure 10: A staircase parameter 3 tight set of P(49)

The Paley Graph in the Affine Plane: P(49)

Observations:

- There are $\binom{7}{2} * 4 = 84$ tight sets of parameter 2 which are the union of 2 cliques (decomposable).

The Paley Graph in the Affine Plane: P(49)

Observations:

- There are $\binom{7}{2} * 4 = 84$ tight sets of parameter 2 which are the union of 2 cliques (decomposable).
- There are 588 indecomposable parameter 2 tight sets, all of which are isomorphic.

The Paley Graph in the Affine Plane: P(49)

Observations:

- There are $\binom{7}{2} * 4 = 84$ tight sets of parameter 2 which are the union of 2 cliques (decomposable).
- There are 588 indecomposable parameter 2 tight sets, all of which are isomorphic.
- For parameter 3, there are 3,668 decomposable and 2,058 indecomposable tight sets.

The Paley Graph in the Affine Plane: P(49)

Observations:

- There are $\binom{7}{2} * 4 = 84$ tight sets of parameter 2 which are the union of 2 cliques (decomposable).
- There are 588 indecomposable parameter 2 tight sets, all of which are isomorphic.
- For parameter 3, there are 3,668 decomposable and 2,058 indecomposable tight sets.
- The 2,058 indecomposable parameter 3 tight sets can be partitioned into 3 classes where all sets within a class are isomorphic.

P(49) Tight Set Data Revisited

P(49)	Type II		
Parameter	**Number of Sets**	**Isomorphism Classes**	**Indecomposable**
7	1	1	0
6	28	1	0
5	672	2	0
4	5,726	8	0
3	5,726	8	3
2	672	2	1
1	28	1	1
0	1	1	0
Total	12,854	24	5

P(81) Tight Set Data Revisited

P(81)	Type II		
Parameter	**Number of Sets**	**Isomorphism Classes**	**Indecomposable**
9	1	1	0
8	45	1	0
7	4500	3	0
6	141540	26	?
5	1106550	?	?
4	1106550	?	?
3	141540	26	17
2	4500	3	2
1	45	1	1
0	1	1	0
Total	2,505,272	?	?

Surprising Results in P(121) and P(169)

- We found computationally that all parameter 2 tight sets in P(121) are a union of 2 cliques. Thus, there are 0 indecomposable tight sets.
- Similarly, in P(169) there are no indecomposable parameter 2 tight sets.

Surprising Results in P(121) and P(169)

- We found computationally that all parameter 2 tight sets in P(121) are a union of 2 cliques. Thus, there are 0 indecomposable tight sets.
- Similarly, in P(169) there are no indecomposable parameter 2 tight sets.
- For parameter 3 in P(121), there are 2 isomorphism classes of decomposable tight sets and 10 isomorphism classes of indecomposable tight sets.

Conjectures

- **Conjecture:** For any $P(q^2)$ where q is prime, the indecomposable tight sets of parameter 2 are all isomorphic.

Conjectures

- **Conjecture:** For any $P(q^2)$ where q is prime, the indecomposable tight sets of parameter 2 are all isomorphic.
- **Conjecture:** For tight set T in $P(q^2)$, either T or T^c will be decomposable (where T^c, "T complement", is the set of all points not in T).

Conjectures

- **Conjecture:** For any $P(q^2)$ where q is prime, the indecomposable tight sets of parameter 2 are all isomorphic.
- **Conjecture:** For tight set T in $P(q^2)$, either T or T^c will be decomposable (where T^c, "T complement", is the set of all points not in T).
- **Conjecture:** For every $P(q^2)$ where q is prime, there exists an isomorphism class of parameter $\frac{q-1}{2}$ tight sets which follow the "staircase pattern"

Open Questions

- Do all tight sets exhibit either symmetry or a staircase patter in some parallel class, as we have seen?

Open Questions

- Do all tight sets exhibit either symmetry or a staircase patter in some parallel class, as we have seen?
- Do tight sets in $P(q^2)$ always behave differently when q is composite, as we have seen in P(81)?

Open Questions

- Do all tight sets exhibit either symmetry or a staircase patter in some parallel class, as we have seen?
- Do tight sets in $P(q^2)$ always behave differently when q is composite, as we have seen in P(81)?
- Not very many strongly regular graphs have been studied for tight sets outside the context of finite geometry. Do any of the patterns we observed in the Paley graphs generalize?

Acknowledgements

I would like to thank:

- Collaborators Emily Barranca and Lauren Hartmann, as well as our research mentor, Dr. Morgan Rodgers.
- The National Science Foundation for their financial support (NSF Grant #DMS-1460151)
- California State University, Fresno, and the CSU Fresno Mathematics Department and REU program.
- Rafe Jones and my advisor Erice Egge for their support and for helping me apply to the REU.
- Carleton College Mathematics and Statistics department and the NUMS organizers.

THANK YOU!