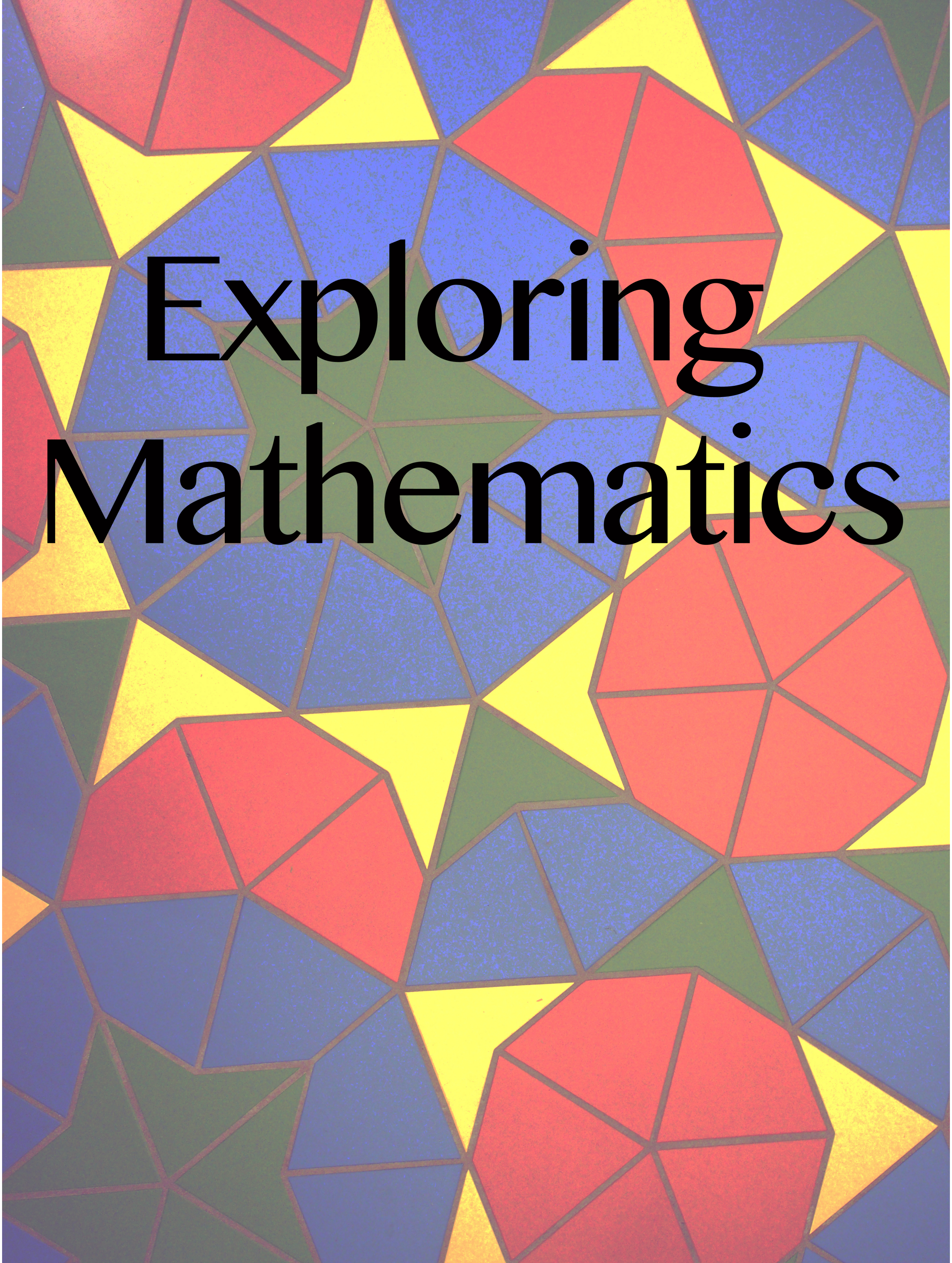
Exploring Mathematics



A Teacher's Guide to Exploring Mathematics

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Introduction

Part I

Number Sense

Number Sense

Counting Basics

Goals

Learn basic skills for counting possible choices, estimating number of possible outcomes

Supplies

N/A

Prior Knowledge

N/A

9. How many ways are there for the word "light?" How many ways can you rearrange the word "happy" and end up with "happy"?
10. Think about using the shepherd's principle. The sheep are the number of circular arrangements, and their legs seem to be linear arrangements of the same number of people. How many legs on each sheep?
11. Think about a picky eater who only eats one flavor of ice cream. How many different cones can he or she buy? How many dishes? Why does this matter?

Answers

1. We make one of 5 choices of paper, and then one of 3 choices of model: $5 \times 3 = 15$ possibilities.
2. 12 possibilities, the order does not matter in this case (because the possible results are distinct for the coin and for the die).
3. $3 \times 5 \times 2$ ways
4. If we had n objects, and wanted to arrange them all in a row, we would have n choices for the first object in the row, $n - 1$ for the second object, and so on. Thus $n!$ is the number of arrangements (or "permutations") of n objects.
5. Continued in exercise.
6. (Total outfits)-(forbidden outfits) $(2 \times 3 \times 3) - (1 \times 3 \times 1) = 15$

7. (total ways)-(ways with Arnold driving and Betsy shotgun) $= (4 \times 3 \times 2) - (2) = 22$

8. We need to know that we are overcounting consistently– if we have a 3-legged sheep (that is, if the pieces we break our set of items into aren't all the same size) then we cannot use it.

9. There are $5!$ ways to rearrange the letters, but this count will include both "ppyah" and "ppyah". We can't distinguish between the two p's so we are overcounting by a factor of 2. So there are $5!/2 = 60$ distinct rearrangements. Similarly for "mammal", for a given arrangement there are $3!$ ways to swap around the m's without changing anything, so we need to divide by 6. Furthermore, the two a's can also be swapped, so there are $(6!/3!)/2 = 120$ distinct rearrangements.

10. For any circular seating arrangement, we can find a different linear arrangement (permutation) depending on who we start at. For example, ADCB =DCBA = CBAD = BADC. If there are n people at the table, then I have n different choices for who to start recording at. Thus the number of circular arrangements is $n!/n$. (How can we simplify this expression?)

11. Your friend is wrong! Specifically, if a cone has more than one scoop of a given flavor, then there are fewer than 6 ways to rearrange them. This means that we are not over-counting by the same amount for each cone. You could rehabilitate her argument by breaking it down into cases by the number of different flavors.

Number Sense

Grocery Shopping

Goals

To develop estimation (and percentage) skills within a real life context.

Supplies

Do NOT provide calculator. This activity is meant to strengthen computation and mental math skills.

Prior Knowledge

N/A

Hints

N/A

Answers

1. One possible answer:

   2 Oreos, 4 fruit roll-ups, 1 bag of chips, 2 cans of pop, 1 gum, 1 bread, 1 mrilk, 1 block cheese, 1 package of meat.

2. The estimate should be between $31 and $34, $32.70(exact).
3. The estimate should be between $32 and $37, it's nearly $35.97(exact). If students are far below encourage them to overestimate each item to the nearest 10 cents.
4. One possible answer: 1 Oreos, 4 fruit roll-ups, 1 bag of chips, 2 cans of pop, 1 gum, 1 bread, 1 milk, 1 block cheese, 1 package of meat.
5. The sticker prices should add up to $18.18 ($1.1 \times x = 20$).
6. One possible answer: 2 Oreos, 4 fruit roll-ups, 1 bag of chips, 1 cans of pop, 1 bread, 1 milk, 1 block cheese, 1 package of meat.

Number Sense

Units

Goals

Practice estimating and converting with different units.

Supplies

N/A

Prior Knowledge

Some algebra is useful.

Answers

1. $0°C = 32°F$, x + 32 = 32
2. Multiply by 1.8.
3. Shift first: F = 1.8(C + 32)
4. $39.2°F$ and $69.8°F$
5. C = F/1.8 - 32
6. $79.4°C$ and $36.6°C$
7. Answers vary, but might include (in order): swimsuit, pants and a T-shirt, winter coat, and coat and hat and scarf.
8. London: Cool (and probably rainy) but not freezing - about $50°F$; St. Petersburg: Quite cold, near $20°F$; Dubai: Very hot, over $100°F$! Remember when you converted human body temperature?
9. London $45°F$, St. Petersburg $14°F$, Dubai $104°F$.
10. It is the same in Fahrenheit and Celsius! You could use algebra to show that there are no other temperatures (solve 1.8(x+32) = x/1.8-32).

The rest of the items have varying answers so are not included.

Number Sense

Modular Arithmetic

Goals

Discover the connection between modular arithmetic and familiar time concepts.

Supplies

N/A

Prior Knowledge

N/A

Answers

Clocks

1. 2 A.M.
2. 9 A.M.
3. 7 A.M.
4. 7
5. 7 P.M.
6. In the latter, you implicitly calculate the mod since 11 + 8 = 19 and clocks are mod 12.
7. 7
8. 3
9. Midnight.

Military Time

10. 4 (4 P.M.)
11. 2 A.M.
12. 24:00 or midnight

Other Bases

13. 5
14. There are seven days in the week!
15. Answers may include 60 seconds in a minute or minutes in an hour or 30 (approximately) days in a month.
16. 1
17. 0
18. Only two.
19. Odd numbers are 1 mod 2, while even numbers are 0 mod 2.
20. 2

Doomsday

1. July 4, 1776 was a Thursday.
2. October 12, 1492 was a Friday.
3. December 7, 1941 was a Sunday.
4. July 20, 1969 was a Sunday.

Number Sense

Logarithms and Exponents

Goals

Provide either an elementary introduction or a refresher on exponents and logarithms

Supplies

A calculator which can compute logarithms in different bases

Prior Knowledge

N/A

Answers

1. If they work out the math they should choose the doubling scheme. If they pick the $100 scheme you should ask them to think about the end of the month with the doubling scheme.
2. $30 \times \$100 = \$3000$
3.

Day	Formula	Amount
1	.01	.01
2	$.01 \times 2$	.02
3	$(.01 \times 2) \times 2$	.04
4	$(.01 \times 2 \times 2) \times 2$	.08
5	$(.01 \times 2 \times 2 \times 2) \times 2$	.16
6	$(.01 \times 2 \times 2 \times 2 \times 2) \times 2$	.32
7	$(.01 \times 2 \times 2 \times 2 \times 2 \times 2) \times 2$	.64
8	$(.01 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times 2$	1.28
9	$(.01 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times 2$	2.56
10	$(.01 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times 2$	5.12
11	$(.01 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times 2$	10.24
12	$(.01 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times 2$	20.48
13	$(.01 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times 2$	40.96
14	$(.01 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times 2$	81.92

4. Doubling .01 29 times gives us \$5,368,709.12
5. On day 15, you will double to \$163.84, more than \$100.
6. In these cases, the total amount you get is $30 \times \$1000 = \$30,000$ or $30 \times \$10,000 = \$300,000$, clearly less than the doubling option gives you even on just the final day. We would need to make more than \$200,00 each day to make it worth taking the flat option.
7. $log_2(8) = 3$ and $log_2(64) = 6$.
8. Allowance on a day = $.01 \times 2^{n-1}$
9. $log_2(\text{allowance on a day/.01}) + 1 = n$. In other words, we can use logarithms to figure out on which day a certain amount of money is received.
10. One day in the month, you will receive \$163.84. Can you use logarithms to figure out which day? As you may have noticed with the allowance example above, exponents can help us show very large numbers succinctly. They can also help us display growth in simple terms. Above we have 2^n in our formula.

    For example if we wanted to figure out on which day we receive \$163.84 in allowance we would set up the following equation. $\$163.84 = .01 \times 2^{(x-1)}$. The unknown x is the exponent, contrast that with the following equation $169 = x^2$ where we are solving for the base. You probably know how to "undo" the square with a square root in the second equation, but you might not know how to solve the first equation. This is where logarithms come in. When you use a logarithm you get to decide first what the base is and then find the exponent. Most often you will need a calculator to find the logarithm.

    Solving $\$163.84 = .01 \times 2^{(x-1)}$

    We will begin solving this problem by dividing both sides by .01 . $\frac{\$163.84}{.01} = \frac{.01 \times 2^{(x-1)}}{.01}$

    $\$16,384 = 2^{(x-1)}$

    This is where the logarithm comes in. We will take the logarithm (base 2) of 16,384 to find x-1.

    $log_2(\$16,384) = log_2(2^{(x-1)})$

    $log_2(2^x) = x - 1$ because we are figuring out what exponent we have to raise 2 to in order to get $2^{(x-1)}$. The exponent in this case is x-1. The log_2 "undoes" the 2 raised to x-1.

    $log_2(\$16,384) = x - 1$ So how do we figure out $log_2(\$16,384)$? In this case we must use a calculator.

    $$14 = x - 1$$

    $$15 = x$$

11. What is the first day you will recieve more than $\$6000$? As above, we want $\$6,000 = .01 \times 2^{x-1}$

    Taking $log_2(600000)$ gives us 19.19 days, so after we add 1 we get 20.19 But we need a whole number of days, since the allowance is doubled once a day. So we would round up, and say that the first day you receive more than six thousand dollars is the 21st day.

12.

Year	Formula	Amount
0	\$100	\$100
1	$\$100\times(.10)+\$100$	\$110
2	$((\$100\times(.10)+\$100)\times.10)+(\$100\times(.10)+\$100)$	\$121
3	$\$100\times1.10^3$	\$133.10
4	$\$100\times1.10^4$	\$146.41
5	$\$100\times1.10^5$	\$161.05
6	$\$100\times1.10^6$	\$177.16
7	$\$100\times1.10^7$	\$194.87
8	$\$100\times1.10^8$	\$214.36

13. It will be between 7 and 8 years before your investment doubles.

14. $A = 100 \times (1.10)^t$

15. $A = P \times (1 + r)^t$

16. Your interest rate was 7.5%.

17. Your interest rate was 8.0%.

18. Your interest rate was about 9.05%.

19. The 5 year scheme will deliver more money. The effective interest rate for those five years is about 7.7%.

Number Sense

Changing Bases

Goals

Explore different bases.

Supplies

N/A

Prior Knowledge

Logs and Exponents activity (or other familiarity.)

Answers

$1 \leftarrow 2$ Machines

1. Now keep going with the dots and fill out this table!

1	2	3	4	5	6	7	8	9	10
1	10	11	100	101	110	111	1000	1001	1010

2. The code for 13 is 1100.

3. The code for 21 is 10101.

4. The completed table looks like this:

128	64	32	16	8	4	2	1

5. These numbers are doubling– that is, they are powers of 2. The rightmost box is 2^0, the next box to the left is 2^1, and so on.

6. $2^9 = 512$, so 511 only needs 9 boxes but 512 needs 10 boxes.

$1 \leftarrow 3$ Machines

7. Fill out the table for the $1 \leftarrow 3$ machine:

1	2	3	4	5	6	7	8	9	10
1	2	10	11	12	20	21	22	100	101

8. The code for 13 is 111. The code for 21 is 1101.

9. Table below.

2187	729	243	81	27	9	3	1

10. Now the numbers are powers of 3, 3^x, x starts counting at 0 on the right.

11. 4 boxes since $3^3 = 27 < 35 < 81 = 3^4$

12. $3^9 = 19683$, so 19682 only needs 9 boxes but 19683 needs 10 boxes.

$1 \leftarrow 10$ Machines

13. $1 \leftarrow 10$ machine:

1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10

14. In order: $13, 21, 42, 100, 9387$.

15. Table below.

1000000	100000	10000	1000	100	10	1

16. $10^0, 10^1, 10^2$, etc.

17. This machine rewrites numbers exactly the same as they were input.

Part II

Geometry

Geometry

Construction

Goals

Learn and practice some basic compass and straight edge constructions, reinforcing basic geometry skills.

Supplies

Compass, straight edge (a ruler will work, but a straight-edge without length markings would be better), something round.

Prior Knowledge

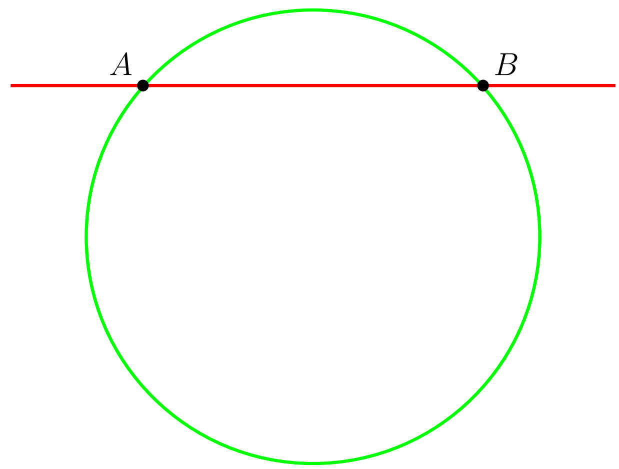
Geometry, in particular perpendicular bisectors, quadrilaterals, triangle equality (SAS and SSS), and circles (especially chords).

Hints

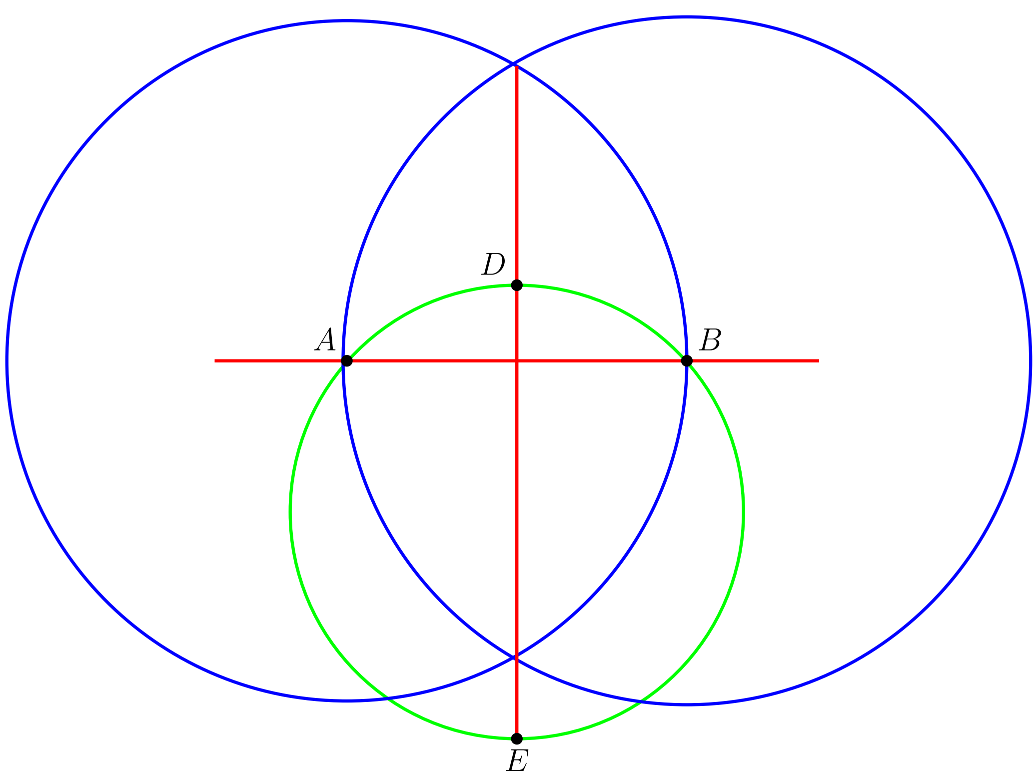
- Equilateral triangle: Remember that all radii of a circle are the same length.
- Perpendicular lines: Can you show that the angles we want to be right are equivalent? If so, what does that imply about their size?
- Center of circle: Any chord of a circle is perpendicularly bisected by a diameter of that circle.
- Square: All radii of a circle are the same length! Also, remember that a square is defined as having four right angles and four equal sides.

Answers

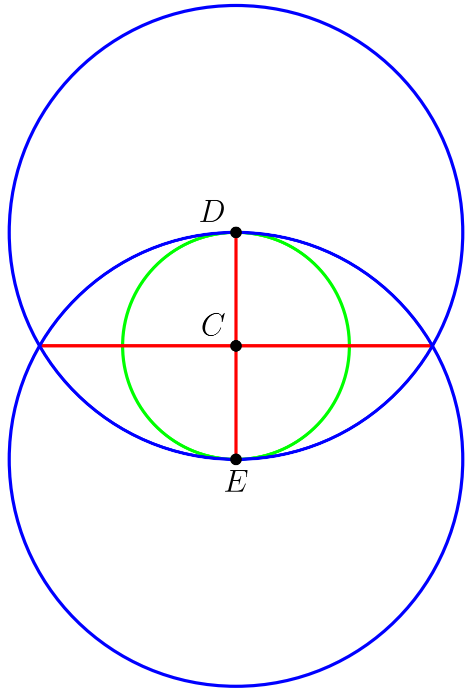
Constructing a circle's center :
Draw some line intersecting the circle in two points, such as chord AB.



Construct its perpendicular bisector DE as before; make sure it's long enough to cross the entire circle as a diameter.



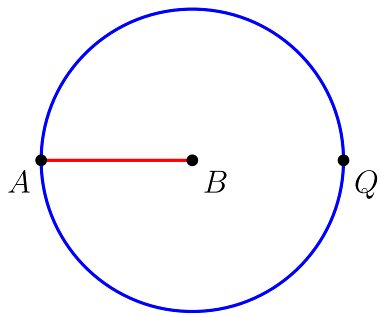
Using the same method, draw the perpendicular bisector of DE Every diameter is divided in half by the circle's center, so the perpendicular bisector to DE will intersect DE at the center of the circle.



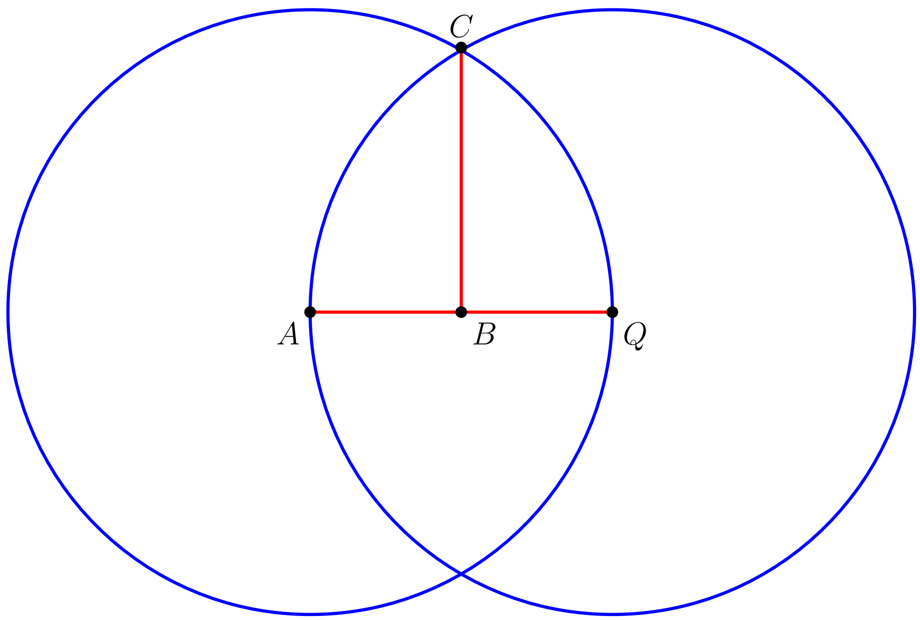
D
C
E

Constructing the Square:

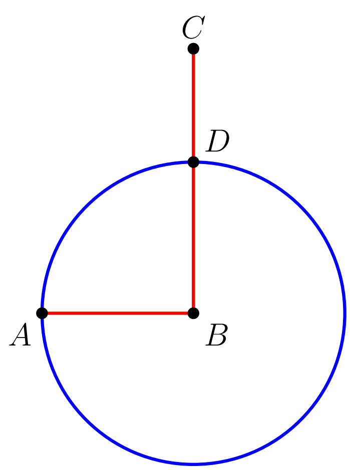
Use a compass to draw a circle centered at B that goes through A. Then we can find a point Q so that B is halfway between A and Q:



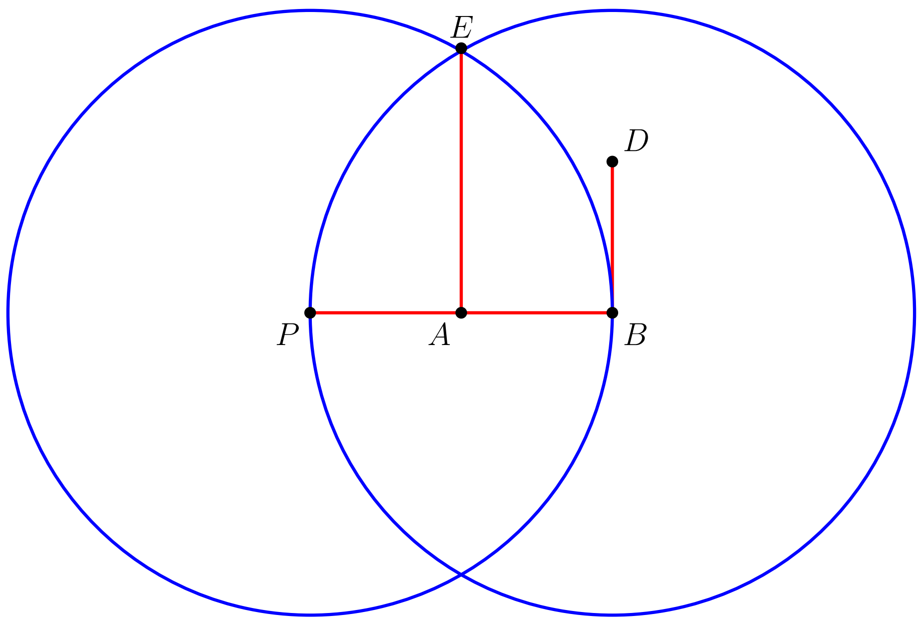
We then construct BC at a right angle to AB as before:



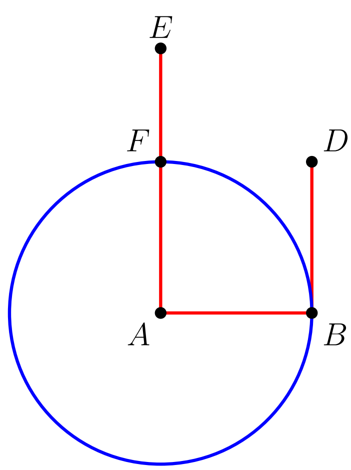
We now draw a circle with center B going through point A, and mark that circle's intersection point with line BC as point D.



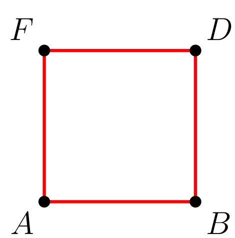
We repeat this process to get another corner, again ensuring a perpendicular angle.



Again, this line is trimmed to a length equal to the distance between A and B



Repeat the same steps one more time to construct a perpendicular line at point F, and cut it off at point D.



Geometry

Construction II

Goals

Learn and practice some further compass and straight edge constructions, reinforcing basic geometry skills. Students will also encounter the idea of unsolvable problems.

Supplies

Compass, straight edge (a ruler will work, but a straight-edge without length markings would be better), square paper.

Prior Knowledge

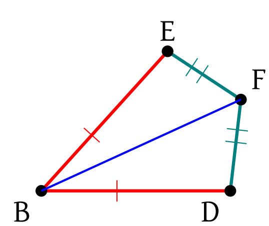
This should be completed after the first construction activity.

Hints

- Bisecting: One valuable way to prove that angles are equal is through triangle equality. Do you have any triangles? How can you know if they are equal?
- Marked ruler: think about creating an isosceles triangle by marking or extending the sides of the angle. The midpoint of the last side of the triangle (which is easy to find with a ruler!) will be on the angle bisector.

Answers

- Bisecting: The triangles formed, BDF and BEF are congruent by SSS. In the construction, BD and BE are radii of the same circle, so BD = BE. Similarly, DF and EF were constructed as radii of two equivalent circles, since the compass was not changed between drawing those circles. Thus DF = EF. Finally, BF = BF so the triangles are congruent. As corresponding parts of congruent triangles are congruent, angles DBF and EBF are congruent, and BF bisects the angle.



Geometry

Möbius Strips

Goals

Explore an unexpected result, investigating properties of Möbius strips and beyond.

Supplies

Tape, long thin strips of paper (looseleaf or printer paper cut long-ways works); butcher paper (optional)

Prior Knowledge

N/A

Hints and Answers

1. Without twists, there are two sides.
2. Each strip has two edges.
3. The line on the strip with a twist goes on the entire sheet of paper - there is no inside or outside, as the line covers both apparent sides.
4. How many pieces of paper are there? How many twists are in them? Are they Möbius strips too?
5. You may need to start over with a wider strip if it's too narrow to cut.
6. What happens when you wrap a rubber band around a bottle once? What if you try to wrap it twice? Also think about where the twists come from. Can the scissors cause twists where they separate the paper?
7. Think about what you saw before!
8. Again, think about where the twists came from!
9. Results in the table!

Number of half-twists	number of sides	result of cutting in half
0	2	two cylinders (zero twists)
1	1	one strip, 4 half twists
2	2	two strips, 2 half twists each
3	1	one strip, 8 half twists
4	2	two strips, 4 half twists each
5	1	one strip, 12 half twists

Geometry

Pythagorean Theorem

Goals

Find two ways to prove the Pythagorean Theorem.

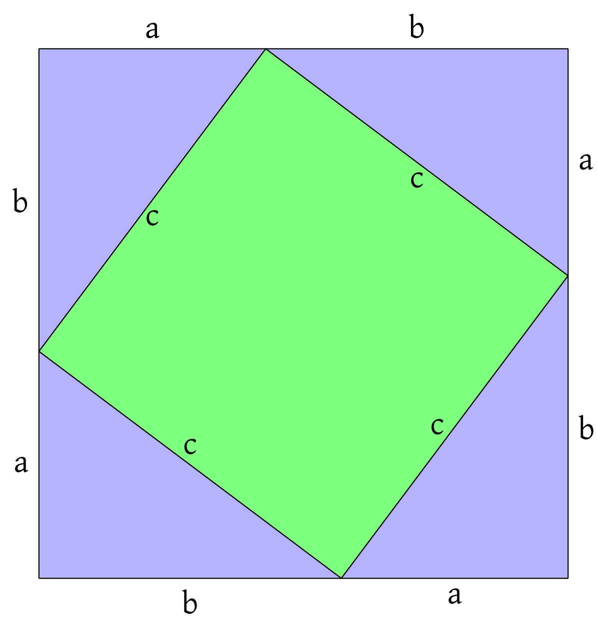
Supplies

Ruler, Protractor, Scissors, Paper.

Prior Knowledge

N/A

An Algebraic Proof



Answers

5. The first equation for the area of the square is $(a+b)^2$ because each side is represented by adding up the sides of the blue triangles so s = $(a+b)$. Expanded this equals $a^2+2ab+b^2$.

6. The second way to represent the area would be to add up the area of the blue triangles plus the area of the green square. This looks like $4(1/2ab) + c^2$ = $c^2 + 2ab$.
7. The two equations together would look like $a^2 + 2ab + b^2$ = $c^2 + 2ab$. It is easy to see that both equations share a common term $2ab$. To simplify the equations subtract this term from both sides. This should result in the simplified equation $a^2 + b^2 = c^2$. This equals the Pythagorean Theorem and therefore the proof is done.

Geometry

How Far Can a Robot Reach?

Goals

Learn how to solve for the area of washers.

Supplies

Ruler.

Prior Knowledge

N/A

Hints

1. This drawing should look like a circle with radius 3. The set of points that can be reached by the fully extended robot arm is all the points on the edge of the circle.
2. This diagram should be the same as the larger circle in part 1 but now there should a smaller circle of radius 1, centered at the end point of the 2 inch segment.
3. The final stage of the diagram should look like a CD. With an outer ring of the circle shaded in.

Answers

3a. The area of this small circle is π inches squared.

3b. The two areas combine to form a circle. The area of this circle is 9π inches squared.

3c. The area of the washer is the full circle minus the area of the inner circle. So the area would be equal to 8π inches squared.

Hints for the Extensions

Does Order Matter?

The whole point of this exercise is to show that the order of the arms does not matter and because of this, the area of the two washers will be the same. If the student is having a difficult time with this problem, it might be because the diagram is a little tricky to draw. When drawing this diagram there is going to be overlap because the longer arm is now on the outside. This might be alarming at first, but when the diagram is finished the resulting washer should look exactly the same as the washer in the original problem. From these pictures, the student should observe this fact and see that the areas are the same.

Is it always a washer?

Now with two segments of equal length there will not be a washer that comes out of the diagram. The entire circle will be covered and it is shown that with arm segments of the equal length the reachablility region will be the entire circle.

What about More?

This problem seems more difficult than it actually is. Have the students label where the segments connect as points. Point 1 is where the outer two segments connect, Point 3 is the center point where the arm is fixed and Point 2 is where the inner segments connect.

Now that we have the points labeled it is easier to imagine the rotation. This first rotation will happen around Point 1 while the inner two segments act as a single segment. This should create a washer that looks similar to the our previous washers. The next rotation will happen around Point 2 with the outer two segments acting as a single segment. This rotation should look similar to the first washer in the Extensions Portion.

The end result should be a washer that is the same as the previous problems and the area is easy to calculate.

Geometry

An Introduction to Spherical Geometry

Goals

Gain a basic understanding of the properties of spherical geometry. Especially the similarities and differences between Euclidean and spherical geometries.

Supplies

Ball, marker, protractor, ruler and string.

Prior Knowledge

Basic Euclidean geometry. Basic information about triangles and lines.

Hints

In questions 1-4 we are trying to bring the student towards the conclusion that geometry on a sphere in 3-dimensions is inherently different than 2-dimensional Euclidean geometry. We guide them towards this conclusion by asking them to draw a 'line' on a sphere and think about whether this is the shortest distance between two points.

Questions 5-11 focus on the question of distance on a sphere. We hope that they begin to understand that the shortest distance on a sphere can be deceiving. Ultimately we would like them to approach the idea of curved shortest distances before reading about great circles.

Answers

N/A

Part III

Sets

Working with Sets

De Morgan's Laws

Goals

Build basic set vocabulary, derive relations between set operations.

Supplies

Colored pencils or crayons.

Prior Knowledge

N/A

Hints

There are actually two such rules combining intersection, union, and complement. Set up a new picture that is the same as before; a box with a red and a blue circle. How could you represent (Red $\cap$ Blue)^c in this picture?
Hint: Remember that operations in parentheses happen first, then try coloring your diagram!

Can you think of another way to represent this expression by using a union instead?
Hint: To create the first law you wrote a union in terms of an intersection by removing the parentheses. What happens if you try a union without any parentheses? Where should you take a complement? How many complements do you need? Use a drawing!

How would you write each of these laws with three colors? (Hint: Draw a three-way Venn Diagram!)
Hint: Go back to the way you wrote the laws. On each side, think about when it makes sense to add another color. Should you add it inside or outside the parentheses? the color itself or its complement?

Just as we can use our set vocabulary to talk about things besides colored circles, De Morgan's Laws have applications far beyond the diagrams we have used to derive them. In our example about coin collections, how would you understand $(A \cup B)^c$? Can you think of a way to express that same idea using a $\cap$ using De Morgan's Laws?
Hint: Think about the physical meanings of A and B, the coin collections. Also, just as in the case for union, intersection, and complement don't only apply to colors, De Morgan's Laws can describe sets defined by coins just as well as the colorful circles we started with.

Answers

1. The purple area is the intersection of the red and blue circles. Represented in symbols, this would be Red $\cap$ Blue.
2. The combination of the three areas is the union of the blue and red circles. Represented in symbols, this would be Red $\cup$ Blue.
3. The white area would be described as the complement of the union of the red and blue circles.
4. Using symbols to represent the above answer, (Red $\cup$ Blue)^c.
5. The area just shaded is the complement of the blue circle, or Blue^c.
6. The brown (orange and green) area is an intersection.
7. Blue^c $\cap$ Red^c.
8. The brown area is equal to the white area in the first diagram.
9. (Red $\cup$ Blue)^c = Blue^c $\cap$ Red^c

Set up a new picture that is the same as before; a box with a red and a blue circle. How could you represent (Red $\cap$ Blue)^c in this picture?

Can you think of another way to represent this expression by using a union instead?
Answer: First color in Red $\cap$ Blue (this should be purple), then look at everything that is not purple. As a union, this can be expressed as Red^c $\cup$ Blue^c.

How would you write each of these laws with three colors? (Hint: Draw a three-way Venn Diagram!)
1: (Red $\cup$ Blue $\cup$ Yellow)^c = Red^c $\cap$ Blue^c $\cap$ Yellow^c
2: (Red $\cap$ Blue $\cap$ Yellow)^c = Red^c $\cup$ Blue^c $\cup$ Yellow^c

In our example about coin collections, how would you understand $(A \cup B)^c$? Can you think of a way to express that same idea using a $\cap$ using De Morgan's laws?
$(A \cup B)^c$: First, think about all the coins in both Alice and Bob's collections, then take the complement to have every coin that is in neither Alice nor Bob's collection.
De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

Sets

Building Sets

Goals

Continue building set vocabulary, increasing mathematical conception of sets.

Supplies

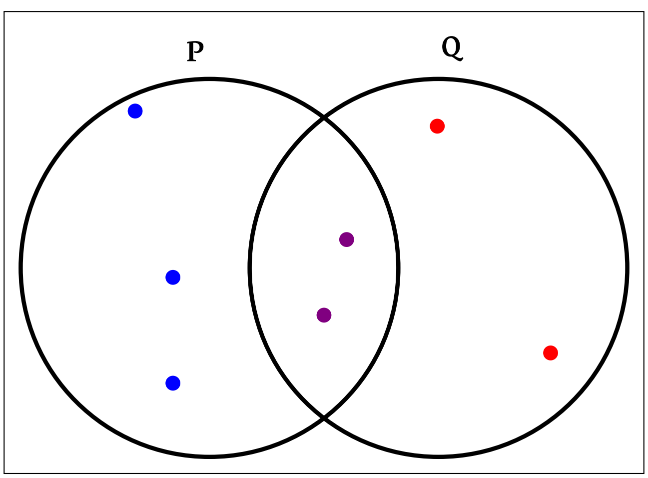
N/A

Prior Knowledge

Union and intersection (the De Morgan's Laws activity introduces these concepts).

Possible Solutions

1. Penelope's collection may involve items such as a teddy bear, a stuffed broccoli, and a beanie baby octopus. Quinn's collection could include Harry's wand, copies of the books and movies, LEGO Hogwarts sets, and Chocolate Frog cards.
2. Penelope and Quinn may both have a stuffed Hedwig (Harry's owl), among other items.
3. What would the Venn Diagram look like?



In this diagram, purple dots could represent the stuffed Hedwig or other Harry Potter stuffed animals. The blue dots would be items which are only in Penelope's collection, such as a stuffed broccoli. The red dots represent Harry Potter memorabilia which are only in Quinn's collection, like copies of the books.

4. Four items - remember that repeats don't count.
5. Those sets are {flowers} and {A,B,C,D,E}
6. Various; may include tenth graders, high school students, girls, redheads, people wearing blue, and people in your immediate family.
7. Various; may include 90-year-olds, people who like anchovies, Boy Scouts, or people who have been on television.
8. Various; one example is the set of all students in your class (that's one description) are also all high school students (there's two) who likely all live in the same town (there's a third!).
9. The set of cats is a subset of mammals.
10. House cats are a subset of cats, dogs are a subset of mammals but do not intersect cats, while animals kept as pets will intersect with mammals, cats, and dogs (but is not a subset of any).
11. Subsets may include seniors, middle school students, and people who like acting.
12. Tessa's collection is smaller than (or the same size as) Grace's.
13. {bat,tab,baa,at}
14. {bath,path}
15. Some of the elements of this set are "nominates","innate", "inmate", "nations", "tension", "tin", "at","an", "to".
16. Some of the elements of this set are "cat", "bat", "rat", "bed", "die", "lie", "sky", "eye", and "rye" To think about the size, can it go on forever? There are a finite number of choices of letter, and only three positions to arrange them in. So while this set is very large, it is still finite.
17. Five: {dwarf,dwell,dwindle,dweeb,dwine}, not including variations (such as dwelling, dwarvish, etc.)
18. There are a lot of these words, but there can only be finitely many - they are all contained by the dictionary.
19. This is finite, as they are all contained within the finite Earth.
20. There are none.
21. There are none.

Sets

Subsets

Goals

Extend knowledge of sets, introduce Inclusion-Exclusion Principle.

Supplies

Colored pencils or crayons.

Prior Knowledge

De Morgan's Law activity, basic set vocabulary.

Answers:

1. There are 4 committees possible: no members, just Allison, just Brendan, and both members.
2. Each sign-up sheet would have to have either a yes or a no added to it with Charlene's name. So each single sign-up sheet has two possibilities when a new member is added. So there are 8 possible committees. If a fourth member joined, then by the same logic there are 16 possible committees.
3. In general, adding one new member to the club means that each of the existing possible committees could have that new member in it or not. This will double the number of possible committees.
4. There are 2^n possible committees.
5. There are 16 subsets: $\{\}$, $\{2\}$, $\{4\}$, $\{7\}$, $\{15\}$, $\{2,4\}$, $\{2,7\}$,$\{2,15\}$, $\{4,7\}$, $\{4,15\}$, $\{7,15\}$, $\{2,4,7\}$, $\{2,4,15\}$, $\{2,7,15\}$, $\{4,7,15\}$, and $\{2,4,7,15\}$
6. The size of the power set is 2^n.
7. The size of B is equal to or less than the size of A.
8. The two sets must be equal since they have exactly the same elements.
9. They are both subsets of C because all of the elements will be chosen from C.
10. $|A \cup B| = |A| + |B| - |A \cap B|$

11. (99/3) = 33, (100/5) = 20, (90/15)=6, so $33 + 20 - 6 = 47$ numbers are divisible by 3 or 5.

12. (16+20)-30 = 6 houses

13. Let's call our three sets A, B, and C. We could start by simply adding together the number of items in each set. But then, as in the case of 2 sets, we overcount items which are in two sets. So we need to subtract out the number of elements which are in A and B, and the number of elements which are in A and C, and the number of elements which are in B and C. But we're not quite done yet. Consider an element which is in all three sets! It got counted three times (since it is in A and B and C) but then it is subtracted out three times, because it is in each of the intersections of two sets. So we need to add one final term to our sum to account for any elements which are in all three sets.

    In set vocabulary, this says that

    $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

    .

Sets

Types of Infinity

Goals

Understand countable infinity and the existence of others, introduce new ways of thinking about infinity.

Supplies

N/A

Prior Knowledge

Basic set vocabulary and understanding elements of sets. This activity extends ideas from earlier ones in this chapter.

Answers

5. The function described is $f(x) = (2x + 1)$.
6. Two other functions that are easy to define: $f(x) = 2x$ and $f(x) = 2x - 1$

Part IV

Probability

Probability

Probability vs. Reality

Goals

Explore the implications of probability in real life and introduce the concept of expected value.

Supplies

Package M&M's, a coin to flip.

Prior Knowledge

Basic probability, familiarity with mean.

Answers

5. If we choose by the expected value, box A is better. However, the answer depends on your attitude toward risk.

6. $E(A) = .01 \times 1,000,000 + .99 \times 0 = \$10,000$ and $E(B) = .95 \times 100 + .05 \times 0 = \$95$. So box A has a much higher expected value. Answers will vary for the second question, but box B seems like a much safer bet.

7. $.50 \times 0 + .25 \times 5 + .20 \times 10 + .05 \times 100 = 0 + 1.25 + 2 + 5 = \$8.25$

8. E[time] $= (.60)(60) + (.40)(30) = 36 + 12 = 48$ minutes.

9. Each of the outcomes (TTT, TTH, THT, HTT, HHT, HTH, THH, HHH) has an equal probability. So we can see that the expected number of heads is equal to

$$(0 \times \frac{1}{8}) + (1 \times \frac{1}{8}) + (1 \times \frac{1}{8}) + (1 \times \frac{1}{8}) + (2 \times \frac{1}{8}) + (2 \times \frac{1}{8}) + (2 \times \frac{1}{8}) + (3 \times \frac{1}{8}) = \frac{12}{8} = \frac{3}{2}$$

10. $\frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$, so the expected number of heads is $\frac{1}{2}$.

11. For each individual flip, the expected number of heads is $\frac{1}{2}$. The total number of heads is the sum of the number of heads obtained in each individual flip. So each flip adds $\frac{1}{2}$ to the expected number of heads for the sequence of n flips. Thus the expected number of heads is $n\frac{1}{2}$.

12. If the probability is .4, the expected number of heads is $.4 \times n$. In general, for probability p, we would have the expected number of heads as $p \times n$.

M&M Probability

To fill out these tables, we can calculate in a very similar way, using the formula $p \times n$ for each probability and sample size. For each color, we can find the expected value of that color for one candy, and then multiply by the number of candies in each sample.

Based on equal probability: (answers may be slightly different depending on rounding)

Color	Sample of 10 candies	25 candies	50 candies
Blue	1.6	4.16	8.3
Brown	1.6	4.16	8.3
Green	1.6	4.16	8.3
Orange	1.6	4.16	8.3
Red	1.6	4.16	8.3
Yellow	1.6	4.16	8.3

Based on manufacturer statistics:

Color	Sample of 10 candies	25 candies	50 candies
Blue	2.4	6	12
Brown	2	5	10
Green	1.6	4	8
Orange	1.4	3.5	7
Red	1.3	3.25	6.5
Yellow	1.3	3.25	6.5

Answers for the final table will vary.

13. Answers will vary.
14. This depends on results, but in general we would expect to see the number of candies get closer and closer to the expected values given by the manufacturer statistics.
15. Answers will vary. Possible topics include the size of the sample compared to the total number of M&Ms that are made each year, the method the sample was gathered in, etc.

Probability

Game Theory

Goals

To gain an elementary understanding of game theory including Nash Equilibria and mixed strategies.

Supplies

N/A

Prior Knowledge

Basic probability, possibly Bayes' theorem exercise.

Answers

1. Confess and betray.
2. Confess and betray.
3. Confess and betray, no matter what the other prisoner does, it is best for each to confess and betray.
4. Do not buy gift.
5. Buy gift.
6. We don't know. There are two Nash equilibria.
7. Nash = confess, confess
8. Don't buy, Don't buy and Buy, Buy
9. 25%
10. 25%
11. 25%
12. 25%
13. 16%

14. 16%
15. 64%
16. 4%
17. The probability acts as a weight to help us understand how much that outcome should change our expectation.
18. 2.5 + 1.25 + 2.5 + 5 = $11.25
19. 2.5 + 1.25 + 2.5 + 5 = $11.25
20. $.16 \times 10 + .16 \times 20 + .64 \times 10 + .04 \times 5 = \$11.40$

Probability

Bayes’ Theorem

Goals

To derive Bayes’ theorem through an investigative exercise.

Supplies

N/A

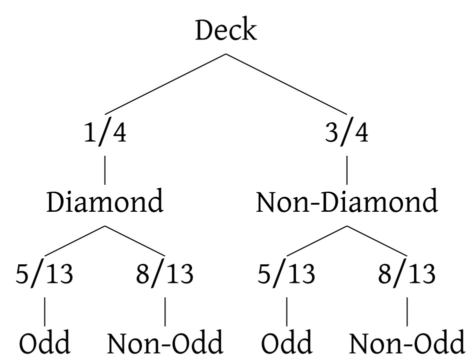
Prior Knowledge

Basic fraction arithmetic, understanding ’randomness’ of pulling cards.

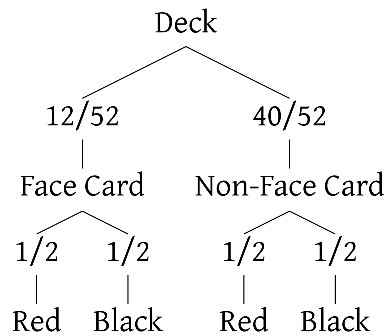
Answers

1. $\frac{6}{52}$, we multiply down the branch that represents our chosen outcome $(\frac{1}{2} \times \frac{6}{26})$.
2. $\frac{20}{52}$, we multiply down the branch that represents our chosen outcome $(\frac{1}{2} \times \frac{20}{26})$.

Use the tree below to help with these questions, we grouped face cards and even numbered cards together in non-odd probabilities.

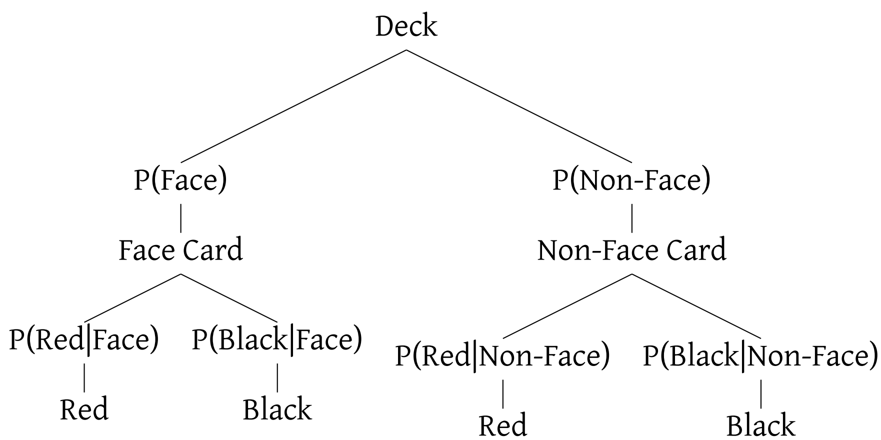


3. $\frac{5}{13}$, here we know that we have diamond, out of the 13 diamonds 5 are odd (including the Ace).
4. $\frac{47}{52}$, we get this by adding up all of the branches excluding the odd diamond branch.
   $(\frac{1}{4} \times \frac{8}{13}) + (\frac{3}{4} \times \frac{5}{13}) + (\frac{3}{4} \times \frac{8}{13})$
5. 1, this encompasses all possibilities for drawing a card. Every card is either an odd diamond or not an odd diamond.

6. 

7. $\frac{6}{52}$, once again we multiply through the branch representing the described outcome. This is equal to the probability found in question 1. If their answer is $\frac{12}{104}$ encourage the students to think about simplifying.

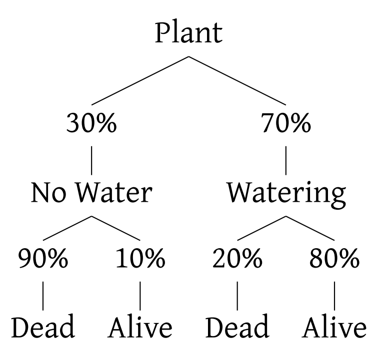
   $\frac{12}{52} \times \frac{1}{2}$

8. 

9. P(Face$\cap$ Red)=P(Face)P(Red|Face), this is very similar to the expression given in the activity except that Red and Face are switched in each position.

   Hopefully they will find Bayes law:

$$\text{P(Face} \cap \text{Red)=P(Face)P(Red|Face)=P(Red} \cap \text{Face)=P(Red)P(Face|Red).}$$

10. 

11. There is a 59% chance the plant will survive. This is $(.30 \times .10) + (.70 \times .80)$. This represents the chance your friend forgets to water the plant and it survives plus the chance your friend remembers to water the plant and it survives.

12. We want P(watering|dead). This problem is challenging, the students needs to generalize the answer to number nine and use Bayes' Theorem.

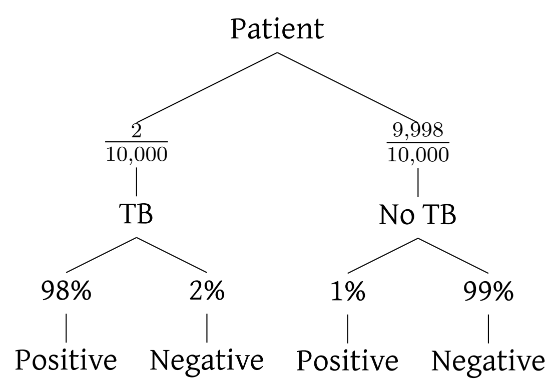
    We will use P(dead|watering)$\times$P(watering)=P(watering|dead)$\times$P(dead).

    We then divide both sides by P(Dead) to find P(watering|dead). We know P(Dead) because we found P(Alive) and P(Dead)+P(Alive)=1, because we know with certainty the plant is either dead or alive. Thus P(Dead)=1-.59=.41

    With numbers this equates to $\frac{.20 \times .70}{.41} = .3415$

    Thus there is a 34.15% chance that your friend forgot to water your plant if you come home and find it dead.

13. If your friend forgot to water it there is a 90% chance that it will be dead. This is given in the tree and the question.

14. 

15. We want P(TB|Positive). This problem is challenging, once again the student needs to generalize the answer to number nine and use Bayes' Theorem.

    We will use P(Positive|TB)$\times$P(TB)=P(TB|Positive)$\times$P(Positive).

    We then divide both sides by P(Positive) to find P(TB|Positive).

    To find P(Positive) we add up the probability of having TB and getting a positive result and not having TB and getting a positive result.

    $(.98 \times .0002) + (.9998 \times .01) = .010194$

    This means within the entire population there is about a 1.02% chance of testing positive for TB.

    Writing the expression with the numerical values, it is equal to $\frac{.98 \times .0002}{.010194} = .019227$.

    Thus there is only about a 2% chance that you have TB even if you test positive for TB.

16. We want P(TB|Negative). This problem is very similar to the above problem.

    We will use P(Negative|TB)$\times$P(TB)=P(TB|Negative)$\times$P(Negative).

    We then divide both sides by P(Negative) to find P(TB|Negative).

To find P(Negative) we can use our answer from above because we know every test is either positive or negative. Thus P(Negative)+P(Positive)=1 and therefore P(Negative)=1-.010194=.9898

This means within the entire population there is about a 99% chance of testing negative for TB.

With numbers the entire expression becomes $\frac{.02\times .0002}{.9898} = .00000404$.

Thus there is only less than a hundredth of a percent chance that you have TB if you test negative for TB. To put this in numbers of people, for every million people there are four who have TB and will test negative.

Part V

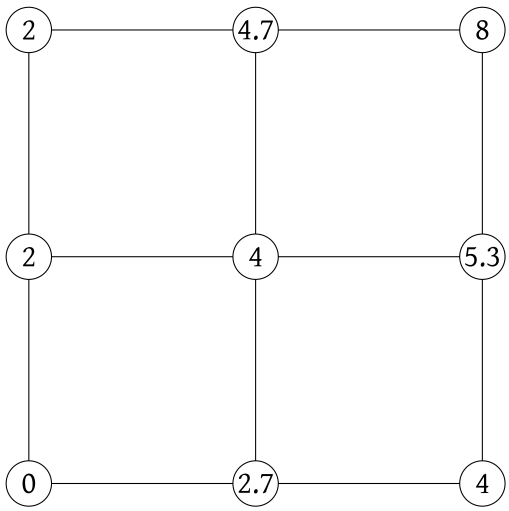
Patterns

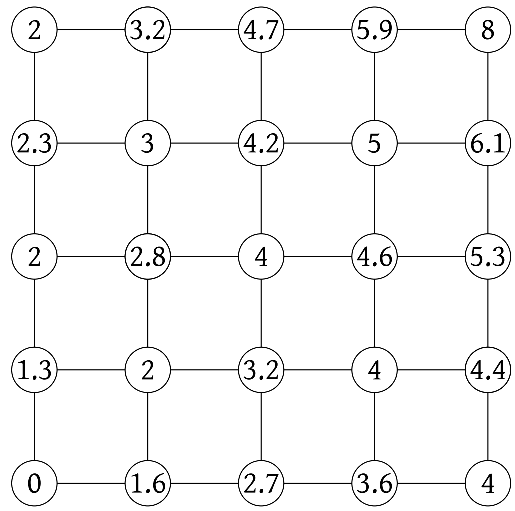
Patterns

Math in Nature

Fractals

1. They should see at least three levels.
2. Trees, ice crystals, and leaf veins are three possible answers.
3. Bubbles, pine cones, and seashell spirals are some examples.
4. Answers could vary.





Voronoi

5. Answers vary.
6. Answers vary but may consider the cell nucleus or source of pigmentation as the points defining the Voronoi cells.
7. Drying mud, turtle shells, leaf cells, foam, territories of territorial animals, etc.
8. Snow constructed the Voronoi cell around the Broad Street pump, showing the region for which that was the closest water source. Most of the cases of cholera were located within that cell, supporting his theory that the pump spread the disease. They then removed the pump handle (forcing people to get their water elsewhere), and the outbreak ceased.
9. On a sphere, you could map the nearest airport (`https://www.jasondavies.com/maps/voronoi/airports/`) or consider sunspots, which also show Voronoi patterns, among other options. Defining distance on a sphere is more complicated than on a plane.

Patterns

Fibonacci Numbers in Nature

Goals

To begin recognizing the mathematical underpinnings of our everyday existence.

Supplies

N/A

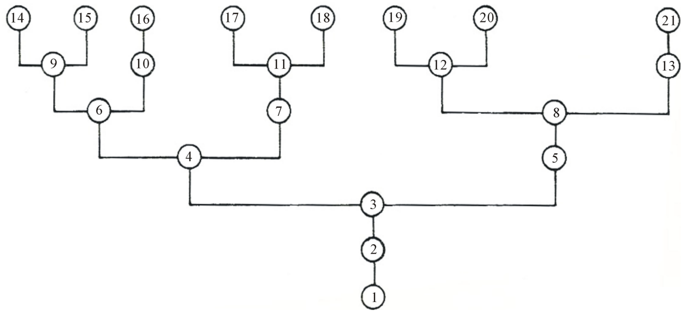
Prior Knowledge

N/A

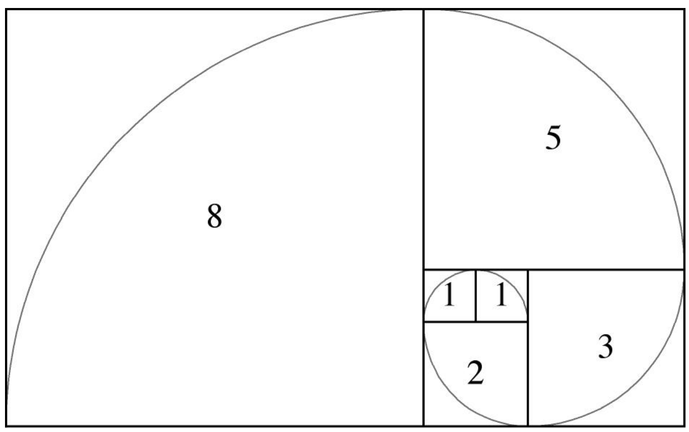
Answers

1. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.
2. There are 34 'seeds' in the outermost ring. It is the ninth number in the Fibonacci sequence.
3. There are 13 seeds in each arc. This is relatively difficult to see so it is not an issue if they are incorrect.
4. Pine cones and pineapples are two good examples.
5. Flowers have developed to do this because it is the most efficient way to place the seeds. In this configuration, there is the least amount of empty space between seeds so the plant can fit more.
6. The most efficient number will be .618 plus any integer value, 1.618, 2.618 etc.

8-9. This is just the beginning of the diagram, the full answer should be a larger tree.



For the Fibonacci spiral, the students picture should look like the figure below.



Patterns

Counting Patterns

Goals

Discover more occurrences of the Fibonacci numbers.

Supplies

N/A

Prior Knowledge

Fibonacci in Nature activity (or other prior encounter with Fibonacci numbers).

Answers

1. It is not always possible to use only rectangles - whenever there are an odd number of squares, at least one square tile is needed.

2.

Number of Squares on Board	Number of Ways to tile
1	1
2	2
3	3
4	5
5	8
6	13

3. They seem to be growing pretty quickly - I wouldn't want to fill out many more!

4. They are the Fibonacci numbers! Where have you seen those before?

5. Answers vary.

6. All the tilings in the left column end with a square tile, while all those in the right column end with a rectangular tile. There are 5 in the left and 3 in the right.

7.

Squares on Board	Number of Ways to tile	# ending in a rectangle	# ending in a square
1	1	0	1
2	2	1	1
3	3	1	2
4	5	2	3
5	8	3	5
6	13	5	8

8. Each column is counting up the Fibonacci numbers (except for the very first), they just start at slightly different places!

9. If there are n squares, the nth Fibonacci number will tell you how many tilings there are, the (n-1)th will tell you how many of those end in a square, and the (n-2)th will tell you how many end in a rectangle.

10. You can think about creating the new tilings for the larger board by starting with smaller boards. If you are trying to tile a board with, say, 8 squares, you could take each possible tiling of a 7-board and stick a square on the end to get 8 tiles, and also each tiling of a 6-board with a rectangle stuck on the end. You don't have to worry about any smaller boards because they need multiple pieces to get to 8, and would be caught by either the 6's or 7's. Then you just have to add both of those numbers together - exactly how you calculate the Fibonacci numbers. This applies to any board length!

11. There are 8 ways, and they can be written exactly the same way we wrote out our tilings. It is possible to write out the 12 stair version, but it would take a long time because the Fibonacci numbers grow really quickly.

12. Yes! It is exactly the same pattern, but you look at whether the top step is stepped on or skipped.

13. Yes, using the Fibonacci numbers is much easier - just keep adding! There are 233 different ways!

14. It probably is surprising, but math is funny like that.

Geometry

Tiling

Goals

Explore different patterns that cover the plane, using repeated polygons and symmetry descriptions.

Supplies

Scissors, coloring supplies, research material.

Prior Knowledge

N/A

Answers

1. Regular tilings only contain one type of polygon. If there were multiple types, not every edge could be the same.
2. There are three different regular tilings - triangles, squares, and hexagons.
3. Six triangles, four squares, three hexagons.
4. It is semiregular, as some edges have two dodecahedra and some have one dodecahedron and one triangle. Tilings can be described by their vertices; this one is triangle-dodecahedron-dodecahedron or 3-12-12, using each number to represent the sides on the polygons that are present.
5. Tilings with only one type of polygon cannot be semiregular without being regular - you cannot have different edges with only one shape present.
6. There are eight different semiregular tilings.
7. The pattern with four triangles and a hexagon is different when reflected - that makes a ninth pattern!
8. Answers include (but are not limited to) Reflection: butterflies, faces (approximately), the letters A, H, U, and M, the numbers 8 and 3. Rotation: flowers, the letters Z, S, and N. Translation: All the regular and semiregular tesselations, stripes, a row of desks.
9. Many items with rotational symmetry, such as flowers, also have reflectional symmetry. Anything can have translational symmetry if it is repeated.

10. All the tesselations show translational symmetry; many also show rotational and reflectional symmetry as each polygon is symmetric in several directions.
11. Answers vary.
12. Answers vary; both are possible.
13. Answers vary.
14. It has translational symmetry only if you are moving vertically (or if your repeating unit is an entire V).
15. When the repeating unit is only \, this pattern has symmetry from a glide rotation.

Part VI

Data

Data

Bad Graphs

Hints

Now, let's look at some real examples of bad and misleading charts. For each one, write a couple sentences explaining what's wrong with the picture presented of the data. Remember, there might be more than one problem. Then, take a second to think and write about why the author of the chart might have chosen to make the chart that way. Did they have a specific point they wanted to make? Did they get so caught up in the visual possibilities that they lost the meaning? Did they just mess up?

1. Does the bill representing 44 cents look about half as big as the original dollar?
2. Check where each point is plotted.
3. Which direction does the y axis normally go?
4. Is this an appropriate type of graph for the data presented?

Answers

1. This graph scales the length of the dollar with purchasing power. However, since they keep the height proportional to width, this means that the difference in area is squared. So the bill representing 44 cents, which should be about half the size of the original bill (100 cents), ends up looking about a quarter of the size! This exaggerates the decrease in spending power. The author of this graph was perhaps intending to do so to prove a political point, or to suggest that the trend is more scary or important.
2. The energy meters and washing machines do not scale correctly to the numbers they intend to represent. Additionally, the piggy banks suffer from a similar problem as the dollar bills above, failing to reflect the value in the area, a problem made worse by distractions like the dollar bills sticking out. Additionally, even though the first piggy bank represents zero it doesn't have zero size (in any dimension). The graph's creators perhaps hoped to sell TopTen brand washing machines to an environmentally conscious audience.
3. First of all, this graph has one point (November) plotted in the wrong place! 8.6% is the lowest value on the graph, yet it appears level with the 9% tickmark. This has the effect of showing a flattened curve. Furthermore, we see that the graph is plotted within a very small range, to give the appearance of a sharp increase in unemployment, when in fact it has remained fairly stable, only moving up and down a few tenths of a percent. This graph might have been created to make President Obama look bad.

4. This graph has an extremely misleading y-axis: the highest numbers are at the bottom. Also, the use of color can be misleading as well. This is the graph flipped right-side up, telling a completely different story! The creators of the first graph were likely interested in showing that the "stand your ground" law didn't increase gun murders.

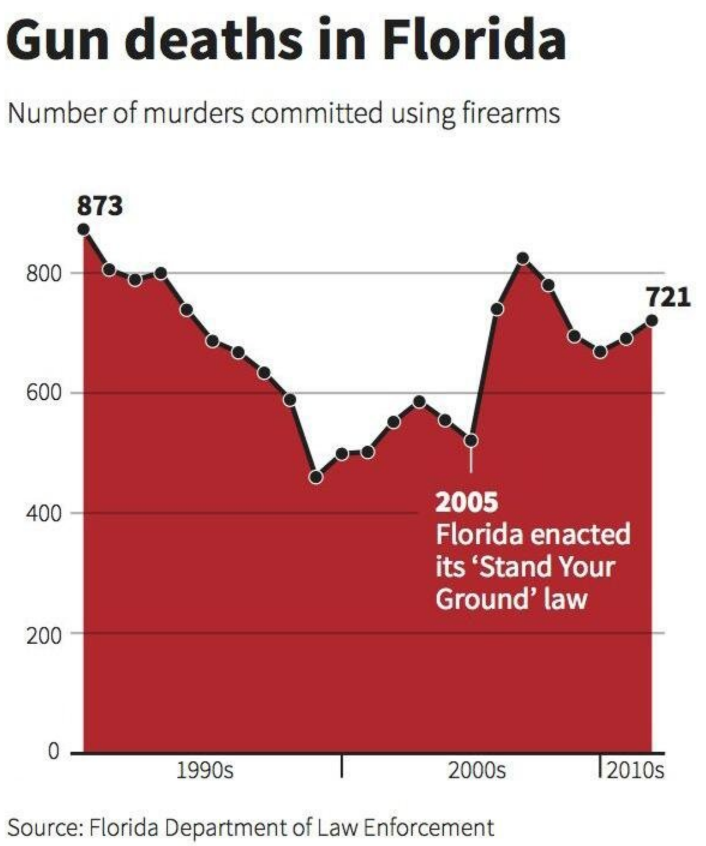


Image from Business Insider.

5. Yikes! In addition to the typos and unhelpful illustrations, this looks something like a pie chart, but it does not form a complete circle, and the percentages are not parts of a whole. Not all of the wedges are labeled, not all of the labeled wedges are percents or other comparable measures, and the size of the wedge in no way reflects the value it represents. Look at the two 88% wedges! In short, a pie chart is not at all the correct representation for this data, and the chart they made is a particularly poor example of a pie chart. This chart also doesn't cite any sources for the data, casting doubt on numbers such as "88% of people might be happier if they took action to talk to a counselor or psychotherapist about their problems." This graph may have been intended to encourage people to visit psychotherapists, in particular hoping to convey the scientific nature of the field through use of a graph.

6. This is not the appropriate use of a best-fit line (a line added to show a relationship between variables)! Not only does the data being fit not appear linear (maybe a curved quadratic fit would be better?), the x-axis is not arranged along a numeric scale (the states are organized by increasing votes, but regularly spaced). Additionally, there are two different y-axes, and the Obama/Clinton axis is not well explained.

Data

Graphing

Goals

Understand making and reading histograms and scatter plots.

Supplies

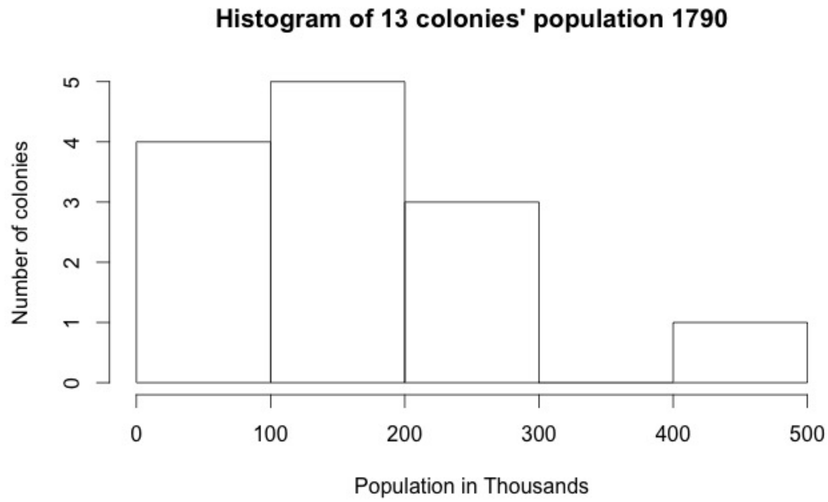
Graph paper is useful but not necessary.

Prior Knowledge

Basic algebra may be helpful.

Answers

1. The lowest bin with at least one state is \$35,000-\$40,000 is, and \$70,000-\$75,000 is the highest occupied bin.
2. Eight states.
3. Nine states.
4. The \$45,000 to \$50,000 bin; reasons may vary.
5. It would fall in the \$55,000 to \$60,000 bin, near the majority of the states.
6. Answers vary. States at the high end include Washington, D.C., Alaska, Maryland, and Connecticut. Mississippi is the lowest, and West Virginia, Arkansas, and Alabama are also on the low end.
7. The populations spread between 23 and 447 thousand, a range of 424.
8. Since the data covers almost 500 numbers (0-450), five bins of 100 each seems reasonable.

9. 

10. Virginia has a particularly high population – notice the empty bin separating it from the other twelve colonies.

11. The populations seem to cluster at the lower end (nearer zero).

12. There are the most colonies between 100,000 and 200,000 people. Connecticut, New Jersey, New York, North Carolina and South Carolina were all in that bin. This is a reasonable middle of most of the colonies' populations.

13. No they do not. For example, Connecticut is pretty small but very well populated. Also, remember that the original 13 colonies did not have the same shape as their modern counterparts. Massachusetts included most of what is now Maine!

14. Answers vary.

15. Yes, it makes sense - a higher median income tends to indicate that fewer people have particularly low incomes, or else they would skew the income lower.

16. It is highly unlikely that any place will have a median income of zero, but is an important theoretical extension of the line.

17. Median income is x, and remember that your x is in thousands. So for Northfield $y = -0.296 \times 59.233 + 30.975 = 13.4\%$.

18. Answers vary, but for Northfield the Census reports 10.0% poverty, lower than the equation predicts.

19. Unlike any other state, Washington, D.C. is really only a city without any rural or suburban areas to balance either the median income or poverty rate. Many cities are likely comparable to D.C. with high economic disparity.

20. All the way to the left, Mississippi might be considered an outlier, as it has particularly high poverty.

Data

Introduction to Statistics

Goals

Provide basic background and knowledge in analysis of data.

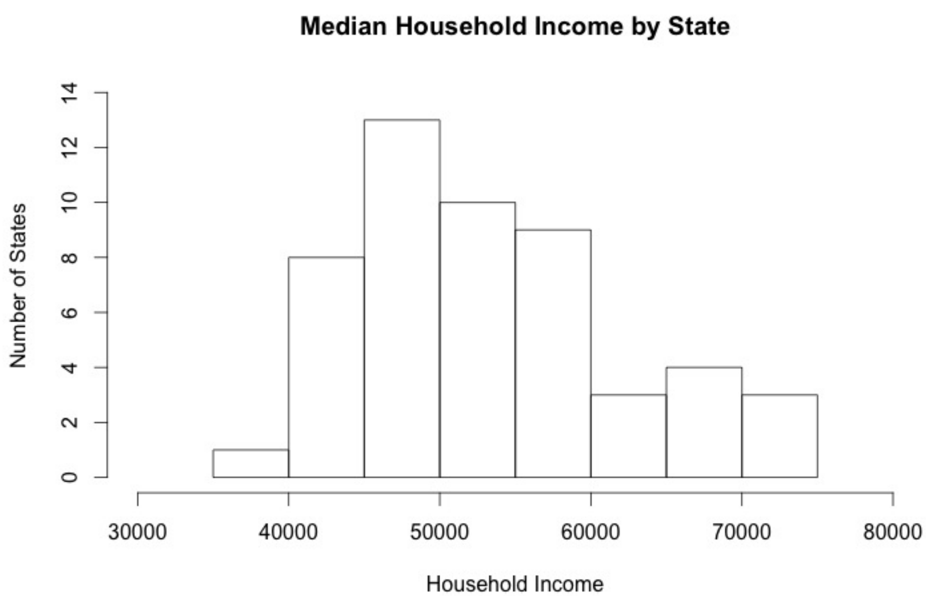
Supplies

Graph paper is useful but not necessary. For some parts of the activity access to Excel/Google sheets is necessary.

Prior Knowledge

Basic algebra may be helpful. Must complete graphing activity or be familiar with histograms before completing this activity.

Answers



1. My histogram shows that the median household income in the United States (by state) is most often between \$45,000 and \$50,000. There are a few states with median household incomes higher than \$65,000 but it is not common.

2. Mississippi has the lowest median income ($39,031) and Maryland is the highest ($73,538).
3. Mississippi has a lot of poverty and is not thriving economically. Maryland is relatively small and prosperous. (Possibly other economic explanations–research could provide.)
4. There is an association between high per capita income and high median household income, however it is not always true. That is to say the order of median household income from highest to lowest is not exactly the order of per capita income from highest to lowest. If the students do any investigation on per capita income they will realize it is a mean-type average and can be heavily influenced by outliers.
5. The three types of averages can be the same but are not necessarily the same. A data set that only contains one unique value will always have the same mean, median and mode.
6. The mean can be skewed by outliers and it is unlikely two households will have exactly the same income so mean and mode seem like they would be inappropriate measures for household income. Median is the most appropriate because it gets rid of outliers and does not depend on identical data points.
7. A set of data with a few outliers and a large amount of one number would be ideal for the mean. A varied sample with outliers calls for the median, while a sample with few extreme outliers is a good use for the mean.
8. Mean:$53,530

   Median:$51,843

   Mode:The mode does not exist, all values are unique.
9. The median definitely seems to be between 18 and 24 in the United States. The mean is less clear although also is probably between 18 and 24.
10. The mean and median are definitely not the same in the second histogram.
11. The extreme outliers on the right of the histogram suggest that the mean will be skewed and too high to effectively represent the data. The median should more appropriately represent the data. The mean will be higher than the median.
12. The standard deviation is 3.

Data	70	71	74	74	76	79
Distance from Mean (74) $(x_i - \mu)$	-4	-3	0	0	2	5
$(x_i - \mu)^2$	16	9	0	0	4	25

13. **Median Household Income**:

    Mean: $53,530

    Median:$51,843

Mode:N/A

Standard Deviation: $8,694

Per Capita Income:

Mean: $28,054

Median: $26,824

Mode:N/A

Standard Deviation:$4,613

Persons in Poverty (percent):

Mean: 15.16%

Median: 15.60%

Mode: 18.6%

Standard Deviation: 3.33%

Part VII

Logic

Logic

Pigeonhole Principle

Goals

Introduce and explore the pigeonhole principle and its applications.

Supplies

N/A

Prior Knowledge

N/A

Answers

1. To fit all six of you in the car, you must share seats. There aren't enough for everybody to have their own.
2. We would need two more rackets in order for everyone to have their own.
3. Yes because there are more people than months.
4. Yes, because exactly two people in each month is 24 people and the last (25^{th}) must have been born in a month with at least two other people.
5. The pigeonhole principle doesn't say anything about which people or months are repeated.
6. No, all we know is at least one.
7. Answers vary.
8. Answers vary.
9. Answers vary.
10. We let the residents be the objects, and sort them into the category of how many hairs they have on their head. There are 1,000,000 categories, and 3.28 million residents, so we clearly have two with the same number of hairs.
11. The limitation of the Pigeonhole Principle is that it doesn't give us any idea of where to start looking. It is definitely true (observable in the world) that there are two bald people, but the pigeonhole principle cannot help us prove it.

12. The pigeonholes or categories are the sock colors. So we need the number of socks to be greater than the number of different colors. If you pull out three socks, then either two of them are violet or two are black.

13. The pigeonhole principle doesn't tell us anything about which categories are full. The worst case scenario is where you grab all of the violet socks before you even get a single black sock. Thus you'd need 4+2=6 socks to be sure to have a pair of black ones.

14. The pigeonholes or categories are the patterns of sock. So we need the number of socks to be greater than the number of different patterns. So if she takes out 4 socks then she will have a matching pair.

15. If the socks don't have to match, then any two socks that she takes out of the drawer are an acceptable pair. This means there is only one category or pigeonhole, so to get an acceptable pair of socks Zinnia can take only 2 socks.

16. In this case, the pigeonhole categories are argyle and non-argyle. Any two non-argyle socks form an acceptable pair, and any two argyle socks also form a pair. So there are two pigeonholes, and so we need three socks, just as in question 12.

17. The minimum number is 0, the maximum number is 49 (since you can't get your own phone number!). The list $0, 1, 2, \dots, 49$ has 50 entries in total.

18. No, in this case there are 50 objects being placed into 50 categories, so there is no guarantee that the numbers chosen are all different.

19. The difference is that in the mutual phone number exchange, one guest's choices affect the possibilities for the other guests. If one person decides not to share their phone number with anyone, then no one at the party can exchange phone numbers with them. Then the maximum number of phone number exchanges for any of these guests is 48. The argument also works in reverse– if one guest got the phone numbers of all of the other 49 people, then all of those guests shared their numbers with at least one person. On the other hand, if each guest chooses individually, it is possible that one guest would choose 0 and another would choose 49.

20. By the previous question, either each guest's number of phone number exchanges is between 0 and 48, or each guest's number is between 1 and 49. In each interval there are only 49 possible categories, and 50 guests, so the Pigeonhole Principle shows there must be two guests in the same category.

21. Yes, this statement is true for a party with any number of guests. There will always be the same number of possible phone number exchanges as there are guests, and it will always be impossible to have a guest who exchanges numbers with everyone and a guest who exchanges numbers with no one.

Logic

Formal Logic

Goals

Gain an understanding of the logical system used in mathematics.

Supplies

N/A

Prior Knowledge

N/A

Answers

1. **Logical Implications**

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

2. **Logical Disjunction**

    - True
    - True
    - True
    - False

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

3. **Logical Conjunction**

- True
- False
- False
- True

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	T

4. **Logical Negation**

P	∼P	Q	∼Q	P ⇒ Q	∼P ⇒ Q	∼Q ⇒ ∼P
T	F	T	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	T	T
F	T	F	T	T	F	T

5. **Extensions**

P	∼P	Q	∼Q	P ∧ Q	∼(P ∧ Q)	∼P ∧ Q	P ∧ ∼Q	∼P ∧ ∼Q
T	F	T	F	T	F	F	F	T
T	F	F	T	F	T	T	T	F
F	T	T	F	F	T	T	T	F
F	T	F	T	T	F	F	F	T

P	∼P	Q	∼Q	P ∨ Q	∼(P ∨ Q)	∼P ∨ Q	P ∨ ∼Q	∼P ∨ ∼Q
T	F	T	F	T	F	T	T	F
T	F	F	T	T	F	F	T	T
F	T	T	F	T	F	T	F	T
F	T	F	T	F	T	T	T	T

Logic

Logical Paradoxes

Goals

To learn about the definition and examples of paradoxes, and understand the mathematical applications of paradox in mathematics

Supplies

N/A

Prior Knowledge

Truth Tables activity or other familiarity with formal logic.

Hints

1. Even numbers are multiples of 2.

Answers

1. No, this is not possible. Then the fraction would not be fully reduced, because the 2 in the numerator and the 2 in the denominator could be canceled.
2. Now a^2 must be even by definition, because it is $2b^2$, which is a multiple of 2. We know that b^2 is a whole number because b is a whole number.
3. The product of two odd numbers is odd, and the product of two even numbers is even. So since a^2 is even, a must be even as well.
4. We see $2b^2 = 4c^2$ so $b^2 = 2c^2$.
5. Since it is a multiple of 2, b^2 must be even.
6. By the same reasoning as part 3, b is even as well.
7. We have shown that a is even and b is even. However, we also know that this statement must be false because we assumed that the fraction representation of $\sqrt{2}$ was fully reduced and therefore a and b both can't be even.

Logic

Logic Puzzles

Goals

- Practice logical thinking and creative problem solving.

Supplies

- N/A

Prior Knowledge

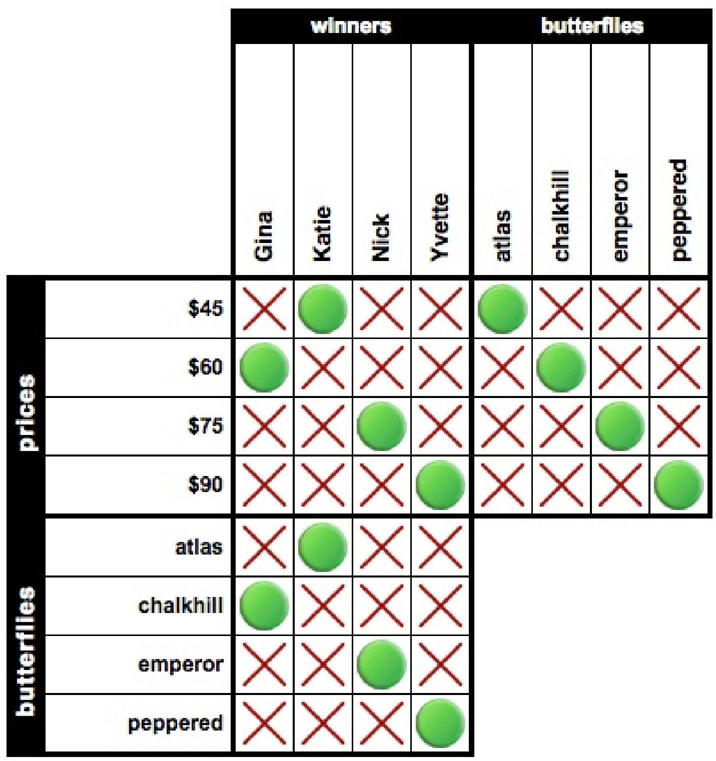
N/A

Puzzle #1

Answers

This grid will auto-populate with all the true relationships you've created on the top 4 rows on the grid. Once this table is fully populated you will be able to submit your solution.

Prices	Winners	Butterflies
$45	Katie	atlas
$60	Gina	chalkhill
$75	Nick	emperor
$90	Yvette	peppered

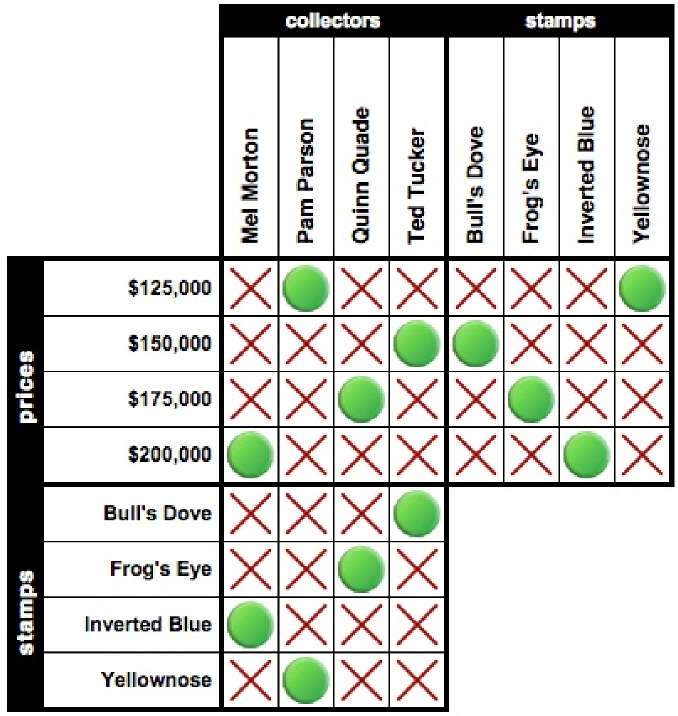


Puzzle #2

Answers

This grid will auto-populate with all the true relationships you've created on the top 4 rows on the grid. Once this table is fully populated you will be able to submit your solution.

Prices	Collectors	Stamps
$125,000	Pam Parson	Yellownose
$150,000	Ted Tucker	Bull's Dove
$175,000	Quinn Quade	Frog's Eye
$200,000	Mel Morton	Inverted Blue

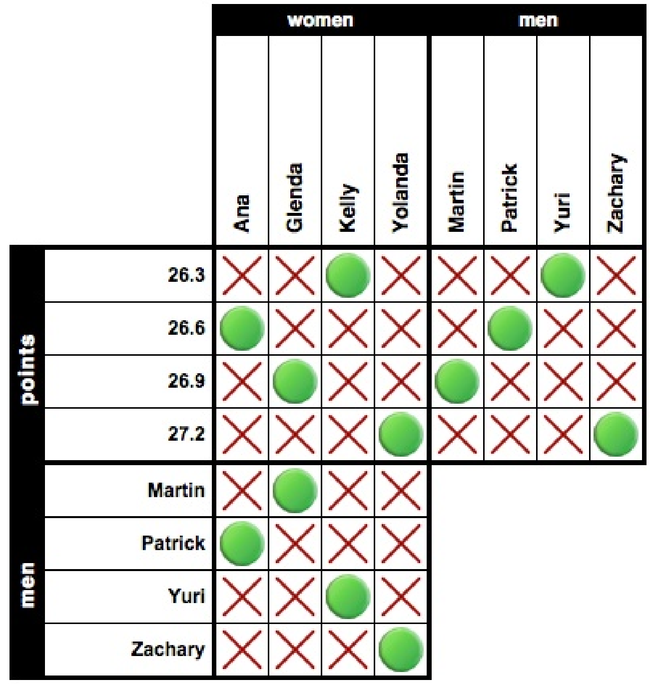


Puzzle #3

Answers

This grid will auto-populate with all the true relationships you've created on the top 4 rows on the grid. Once this table is fully populated you will be able to submit your solution.

Points	Women	Men
26.3	Kelly	Yuri
26.6	Ana	Patrick
26.9	Glenda	Martin
27.2	Yolanda	Zachary

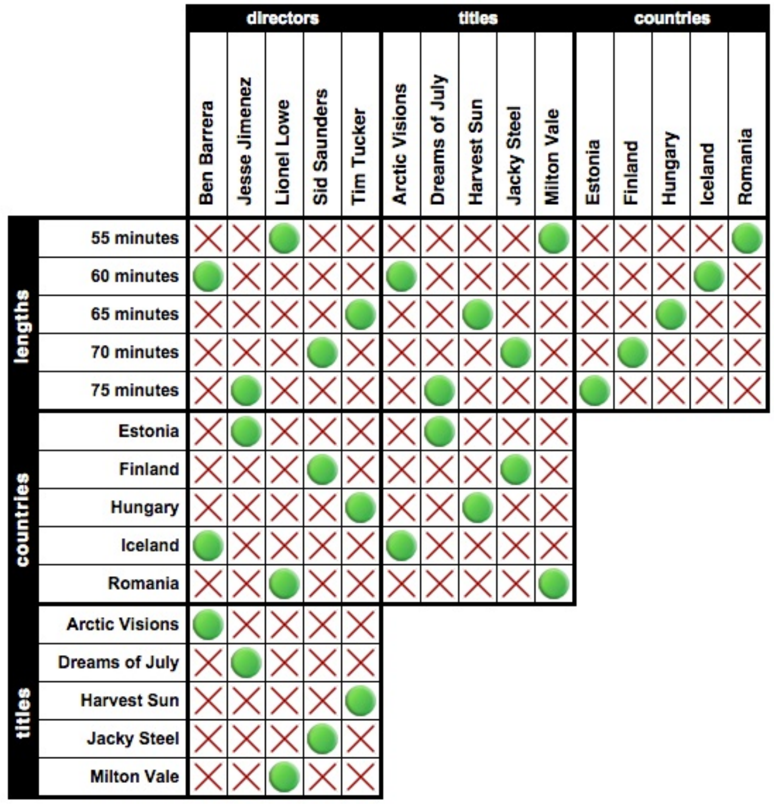


Puzzle #4

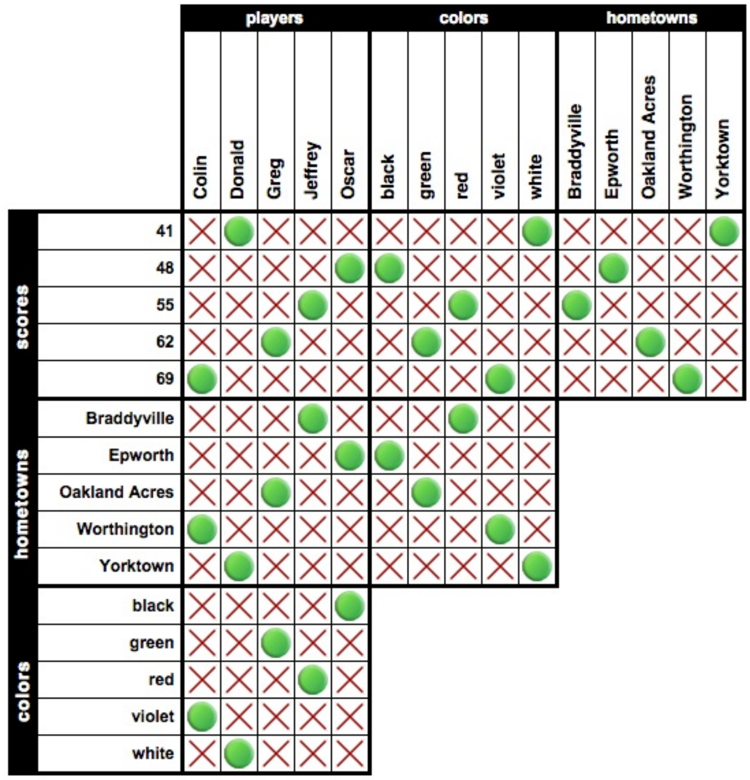
Answers

This grid will auto-populate with all the true relationships you've created on the top 5 rows on the grid. Once this table is fully populated you will be able to submit your solution.

Lengths	Directors	Titles	Countries
55 minutes	Lionel Lowe	Milton Vale	Romania
60 minutes	Ben Barrera	Arctic Visions	Iceland
65 minutes	Tim Tucker	Harvest Sun	Hungary
70 minutes	Sid Saunders	Jacky Steel	Finland
75 minutes	Jesse Jimenez	Dreams of July	Estonia



Puzzle #5

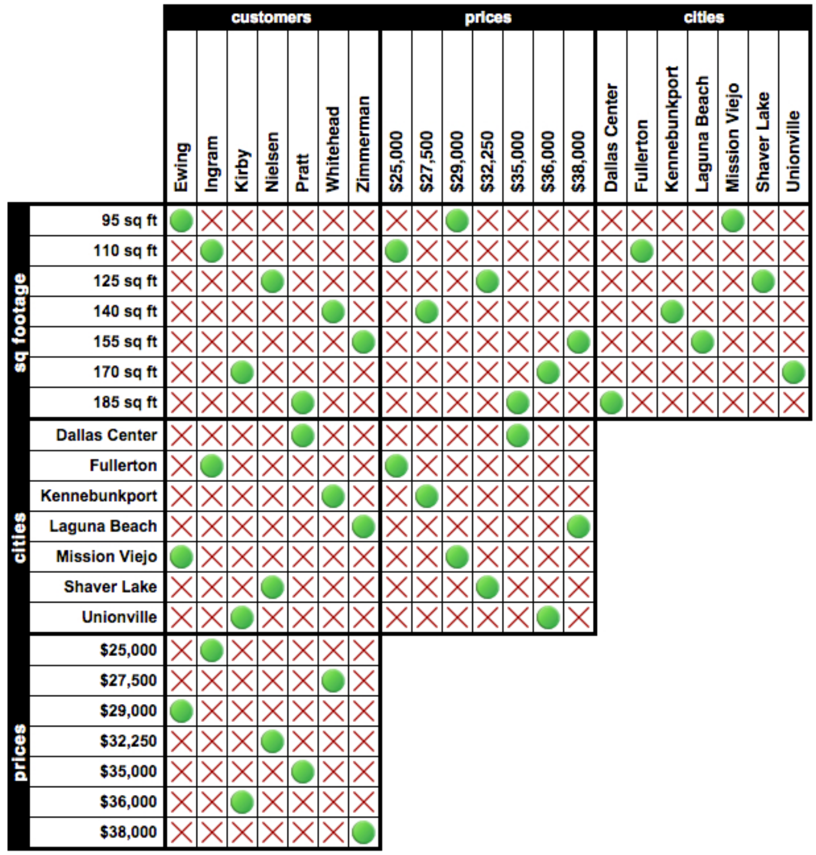


Answers

This grid will auto-populate with all the true relationships you've created on the top 5 rows on the grid. Once this table is fully populated you will be able to submit your solution.

Scores	Players	Colors	Hometowns
41	Donald	white	Yorktown
48	Oscar	black	Epworth
55	Jeffrey	red	Braddyville
62	Greg	green	Oakland Acres
69	Colin	violet	Worthington

Puzzle #6



Answers

This grid will auto-populate with all the true relationships you've created on the top 7 rows on the grid. Once this table is fully populated you will be able to submit your solution.

Sq Footage	Customers	Prices	Cities
95 sq ft	Ewing	$29,000	Mission Viejo
110 sq ft	Ingram	$25,000	Fullerton
125 sq ft	Nielsen	$32,250	Shaver Lake
140 sq ft	Whitehead	$27,500	Kennebunkport
155 sq ft	Zimmerman	$38,000	Laguna Beach
170 sq ft	Kirby	$36,000	Unionville
185 sq ft	Pratt	$35,000	Dallas Center

Part VIII

Miscellaneous

Miscellaneous

Map Coloring

Goals

Discover rules of map coloring and basic graph theory.

Supplies

Colored pencils (or other coloring supplies).

Prior Knowledge

N/A

Answers

1. Since they are not connected to the rest of the United States, they can be any color.
2. Answers vary.
3. Yes, it should be possible.
4. Yes, it is possible - if you didn't make it, think about what states you could have colored differently.
5. No, it is not possible. For example, Vermont, New Hampshire, and Massachusetts must be three different colors.
6. No, at least four colors are needed. Look at Nevada and the states around it.
7. 48 colors, one for each state!
8. Answers vary; may include ideas of simplification and clarity of which states are connected.
9. Here is a legend that corresponds to each vertex.

   A - Brazil, B - French Guiana, C - Suriname, D - Venezuela, E - Columbia, F - Ecuador, G - Peru, H - Bolivia, I - Chile, J - Argentina, K - Paraguay, L - Uruguay, M - Guyana

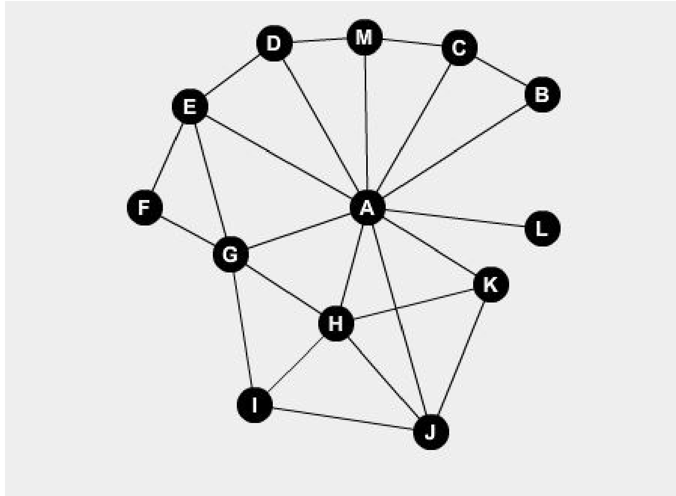


Figure 1: An example of a map that diagrams South America

10. You need at least four colors.

Miscellaneous

Shortest Path Problem

Goals

To learn Dijkstra's Algorithm and experience algorithmic thinking. Also to get a basic understanding of one the most popular math problems, The Traveling Salesman.

Supplies

N/A

Prior Knowledge

N/A

Answers

1. The possible paths are {1,2,5}, {1,4,5}, {1,2,3,5}, {1,2,4,5}, {1,4,2,5}, {1,4,2,3,5}
2. The lengths of each path are {1,2,5} = 6, {1,4,5} = 63, {1,2,3,5} = 50, {1,2,4,5} = 65, {1,4,2,5} = 14, {1,4,2,3,5} = 58
3. The shortest path is {1,2,5}.
4. The shortest path using Dijkstra's algorithm is {A,B,H} and has a length of 56.

	A	B	C	D	E	F	G	H	I
A	0	22_A	9_A	12_A	∞	∞	∞	∞	∞
C	0	22_A	9_A	12_A	74_C	51_C	∞	∞	∞
D	0	22_A	9_A	12_A	45_D	51_C	∞	∞	42_D
B	0	22_A	9_A	12_A	45_D	51_C	∞	56_B	42_D
E	0	22_A	9_A	12_A	45_D	51_C	68_E	56_B	42_D
I	0	22_A	9_A	12_A	45_D	51_C	63_I	56_B	42_D
F	0	22_A	9_A	12_A	45_D	51_C	63_I	56_B	42_D

5. The possible paths are {A,C,E,D,F}, {A,B,D,F}, {A,B,C,E,D,F}
6. The lengths of each path are {A,C,E,D,F} = 20, {A,B,D,F} = 25, {A,B,C,E,D,F} = 27.
7. Using Dijkstra's Algorithm the shortest path is {A,C,E,D,F} with length 20.

8. This table gives the proper completion for the directed weighted graph.

	A	B	C	D	E	F
A	0	4_A	2_A	∞	∞	∞
C	0	4_A	2_A	∞	3_C	∞
E	0	4_A	2_A	4_E	3_C	∞
D	0	4_A	2_A	4_E	3_C	11_D
B	0	4_A	2_A	4_E	3_C	11_D

Miscellaneous

Flexagons

Goals

Provide a basis for understanding phase diagrams, learning how to investigate unknown structures.

Supplies

Copies of the flexagon templates, scissors and glue or paste.

Prior Knowledge

N/A

Hints/Answers

Your First Flexagon

1. There are three faces.
2. Answers to this question will vary– Depending on which face starts up. It will either be repeating through the cycle 1-2-3-1, or 3-2-1-3.
3. As alluded to above, the order reverses when the flexagon is flipped upside down.

HexaHexaFlexagon

4. Since hexa- means "six", we might expect to see six different faces.
5. Answers will vary– there are 6 in all, but some faces are harder to find than others.
6. Again, answers will vary. It's possible to find cycles of 3 like with the trihexaflexagon. However, these cycles can be rearranged with careful flexing: see below.
7. There is not a fixed order for the faces to appear in, not all of the faces show up in a complete cycle. We can find simple cycles of three faces, but we can also deviate from those cycles.
8. This pattern should reveal the same three faces, cycling in order.

9. This is reminiscent of the trihexaflexagon, where you cannot flex on the same corner more than once.

10. If you cannot fold on this corner, there will turn out to only be one way to flex the flexagon. However, there are other faces which can be flexed in two different ways.

11. There are some faces which are always flexible in only one way, and some faces which can sometimes be flexed in two different ways.

12. Faces 4, 5, and 6 can always be flexed in only one way, but faces 1, 2, and 3 will end up on both lists.

13. There must be two different faces, made of the same 6 triangles, which behave in different ways. Maybe this is caused by the way the triangles are arranged relative to each other .

14. If you follow the above procedure, folding at a single corner until you can no longer flex at that corner, and then moving onto an adjacent corner, and continuing in the same direction, you will see the star which used to mark the inside corner now split up, with the marks on outer corners. This means that even though this face has the same triangles, their orientation is different. This difference will correspond to whether the face can be flexed in two different ways, or only one way.

Mapping The Flexagon

15. This represents the order in which the faces of the trihexaflexagon appear when flexed.

16. In step 8, where we moved to adjacent corners of the flexagon, we see a repeating cycle of three faces.

17. There should be 4 different simple cycles. Although order changes (depending on which side is face up, just like with the trihexaflexagon) these cycles will be $(1 \to 2 \to 3 \to 1\ldots)$; $(1 \to 4 \to 3 \to 1\ldots)$, $(2 \to 5 \to 1 \to 2)$, and $(2 \to 3 \to 6 \to 2\ldots)$;

18. The cycles are $(1^* \to 2^* \to 3^* \to 1^*\ldots)$; $(1^* \to 4 \to 3 \to 1^*\ldots)$, $(2^* \to 5 \to 1 \to 2^*)$, and $(2 \to 3^* \to 6 \to 2\ldots)$;

19. To combine the diagrams, you'd simply want to make sure that each of the "starred" multiflexible faces are connected to two different cycles. There is a "main" 1-2-3 cycle, and each of those three faces has an additional cycle of three faces attached to it.

Miscellaneous

Math in Literature

Goals

Explore familiar and new mathematical concepts through the lens of literature. Lots of projects for students to choose.

Supplies

Flatland by Edwin A. Abbott

Prior Knowledge

N/A

Answers

Flatland

1. Playing on a donut creates four new possible winning configurations.
2. All of them are on new diagonals - the horizontal and vertical moves are unchanged.
3. Answers vary; may be easier with the donut to stick your opponent in a "trap" where the first player wins regardless of the other's move. There are also likely fewer ties.
4. The four diagonals added by the normal donut still win - but there are also four more winning diagonals created by the twist! Did you notice that the middle square is completely unaffected by the changes?
5. Answers again vary; did you notice that it's actually impossible to tie?

The Number Devil

- Triangles will be formed!
- Again, we see triangles, but now they are oriented and sized differently.
- The sum of all the entries in a row is a power of 2. In fact, each row sums to double what the row above it sums to.