Draft of some problems

Problems are sorted into six categories: Algebra, Geometry, Number Theory, Combinatorics / Probability, Sequences / Series / Polynomials, and Complex Numbers. They are currently rated on a [1] to [5] rating scale, with [1] being easy and [5] being hard.

Algebra

Problem 1 [2]: An number n is constructed by concatenating consecutive numbers: $n = 12345678910111213...979899100$. What is the difference between the sum of the odd digits of n (referring to the value of the digits not their index) and the sum of the even digits of n?

Answer: $\boxed{101}$ If we ignore the 100 at the end, we will notice that every digit (besides 0, which doesn't have an effect on the answer) occurs exactly ten times in all of the "ones" digit of numbers between and exactly ten times in all the "tens" digit place. So this means that every digit that will add to the total sum (in either a positive or negative way) will do so exactly 20 times. So if we know what the difference between all the odd single-digit numbers and the even single-digit numbers, the total difference will be twenty times that. So we have $20 \cdot (1 + 3 + 5 + 7 + 9 - 2 - 4 - 6 - 8) = 20 \cdot 5 = 100$ But recall this number was only for the double or less digit numbers, we still have an extra odd digit from the 100 (remember the 0s don't do anything). This brings the grand sum (and answer) to $100 + 1 = 101$.

Problem 2 [3]: Find the sum of the solutions to the equation $(x + 6)^4 + (x + 8)^4 = 81$.

Answer: $\boxed{-28}$ A cool trick to notice is that a substitution: $u = x + 7$ will make this problem far nicer to deal with. After performing the substitution we get $(u - 1)^4 + (u + 1)^4 = 81$ which we then expand: $u^4 - 4u^3 + 6u^2 - 4u + 1 + u^4 + 4u^3 + 6u^2 + 4u + 1 = 81$ and simplify to: $u^4 + 6u^2 - \frac{79}{2} = 0$ Now if we have four roots to this equation, then it is equivalent to the equation $(u - u_0)(u - u_1)(u - u_2)(u - u_3) = 0$. Expanding this we find $u^4 - (u_0 + u_1 + u_2 + u_3)u^3 + ... = 0$. Notice how the u^3 coefficient is the sum of the roots? (this is a specific case of Vieta's Theorem) This means that the

actual sum of the roots will be 0. But remember these are the u values, the x values are all 7 less, and because there are 4 roots, the sum of all the roots of the original equation will be $4 \cdot 7$ less than the sum of the roots on the u expression. So out answer is $0 - 28 = -28$.

Problem 3 [3]: Evaluate the expression $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$ to a real, rational number.

Answer: $\boxed{2}$ Start by setting $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$ to be x. then cube both sides. This results in the equality:
$(\sqrt{108} + 10) - 3((\sqrt{108} + 10)^{\frac{2}{3}}(\sqrt{108} - 10)^{\frac{1}{3}}) + 3((\sqrt{108} + 10)^{\frac{1}{3}}(\sqrt{108} - 10)^{\frac{2}{3}}) - (\sqrt{108} - 10) = x^3$ Which can factor to:
$(\sqrt{108} + 10) - 3\left(((\sqrt{108} + 10)(\sqrt{108} - 10))^{\frac{1}{3}}\right)\left((\sqrt{108} + 10)^{\frac{1}{3}} - (\sqrt{108} - 10)^{\frac{1}{3}}\right) - (\sqrt{108} - 10) = x^3$ Completing the square and substituting:
$(\sqrt{108} + 10) - 3\left((108 - 100)^{\frac{1}{3}}\right)\left(x\right) - (\sqrt{108} - 10) = x^3$ Simplifying:
$20 - 6 * x = x^3$
This is just a cubic and can be solved using the rational root theorem and substitution. Which give the answer $x = 2$

Geometry

Problem 1 [2]: A square is placed on top of a circle such that two vertices of the square lie on the circle and one-sixth of the circumference of the circle is contained within the square, however, the center of the circle is not contained within the square. (points on the edge or vertex of a square are not considered to be contained within the square) What is the sum of all possible values of $\frac{r}{s}$, where r is the radius of the circle and s is the side length of the square?

Answer: $\boxed{1 + \sqrt{2}}$ there are only two different ways for the the square to be placed on top of the circle, one with two adjacent vertices being placed on the square and one with two opposite vertices being placed on the square. In the former case the length of the cord by adjoining the two points of intersection is s and in the latter case it is $\sqrt{2} \cdot s$. In either case the two radii from the center of the circle to the endpoints of the cord along with the chord make an equilateral triangle (because the angle between the radii is 60^{deg}).

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Therefore in either case the length of the chord is the same as the length of the radii. So, in the former case $r = s$ and in the latter case $r = \sqrt{2} * s$. So, $\frac{r}{s} = 1$ in the former case, and $\frac{r}{s} = \sqrt{2}$ in the latter case.

Number Theory

Problem 1 [1]: Find the smallest number n such that $n!$ is divisible by 2016.

Answer: $\boxed{8}$ The easiest way to do this problem is to factor $2016 = 2^5 \cdot 3^2 \cdot 7$ So because of the twos requirement, $n!$ must contain at least 5 factors of 2, $n = 7$ has only 4 factors and $n = 8$ has 7 factors. So $n \geq 8$. Applying a similar argument for 3 and 7 we find $n \geq 6$ and $n \geq 7$. Therefore the smallest bound for n is 8.

Problem 2 [3]: How many zeros are there at the end of the base six expansion of 333! (note the 333 is a decimal number and not a base six number)

Answer: $\boxed{165}$ The number of zeros that appear at the end of a base six number is the same as the number of sixes that go into that number. ($6^3 \cdot 5^2 \cdot 3^2$ would have 3 zeros at the end of its base six expansion) Because $6 = 3 \cdot 2$ the number of sixes is going to be determined by the number of threes and the number of twos. Two is a smaller number so there are going to be far more twos in 333! than threes, so to calculate the number of zeros all we need to do is calculate the number of threes that go into 333!. There are $\lfloor \frac{333}{3} \rfloor$ numbers less that 333 that are divisible by three, $\lfloor \frac{333}{9} \rfloor$ numbers that are divisible by three twice, $\lfloor \frac{333}{27} \rfloor$ numbers that are divisible by three three times, and so forth. Because the factorial is just the product of all the numbers before it, adding $\lfloor \frac{333}{3} \rfloor + \lfloor \frac{333}{9} \rfloor + \lfloor \frac{333}{27} \rfloor + ... = 165$ will determine the total number of threes that go into 333!.

Problem 3 [4]: A young child wants to write down all the positive integers in increasing order. However, this child does not like to write the numbers 4 or 7, so this child skips every number that has a 4 or 7 in it. (after writing 69, the child would write 80). What is the 2016th number the child would write?

Answer: $\boxed{3950}$ The child is counting with the normal rules of counting,

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except they are only using 8 digits instead of the normal 10. So the child is writing base 8 number, with the restrictions that some numbers are representing other numbers. For example when the child writes "9" they mean "7", "8" means "6", "6" means "5" and so on. Converting the base ten number 2016_{10} to base 8 results in the number 3740_8. but the child is writing some numbers in place of others so this becomes 3950.

Combinatorics / Probability

Problem 1 [4]: How many ways can a room that is 3 units by 15 units be tiled with blocks that are 1 unit by 3 units? (The room has an orientation so tilings that are the same under rotation / refection are still distinct).

Answer: $\boxed{189}$ There are two different ways to tile one of the short edges of a $3xN$ room: either using one bock being placed perpendicular to the long edge, or with three blocks parallel to the long edge stacked on to of each other. In the first case, a $3xN - 1$ room is left; in the second case, a $3xN - 3$ room is left. Call $t(n)$ the number of ways to tile a $3xn$ room. This means $t(n) = t(n - 1) + t(n - 3)$ Given that, and the fact that $t(1) = t(2) = 1$ and $t(3) = 2$ (We know this because they are small enough cases we can just try all possible combinations and count them). Therefore $t(4) = t(3) + t(1) = 2 + 1 = 3$, $t(5) = 3 + 1 = 4$, $t(6) = 6$, $t(7) = 9$, $t(8) = 13$, $t(9) = 19$, $t(10) = 28$, $t(11) = 41$, $t(12) = 60$, $t(13) = 88$, $t(14) = 129$, , $t(15) = 189$

Sequences / Series / Polynomials

Problem 1 [2]: Find the sum: $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{n}{2^n} + \ldots$

Answer: $\boxed{3}$ Call the sum S. This means

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\begin{aligned}
S - \tfrac{1}{2}S &= \tfrac{1}{2} + \tfrac{2}{4} + \tfrac{3}{8} + \tfrac{4}{16} + \ldots + \tfrac{n}{2^n} + \ldots - \tfrac{1}{4} - \tfrac{2}{8} - \tfrac{3}{16} - \tfrac{4}{32} + \ldots + \tfrac{n}{2^{n+1}} + \ldots \\
\tfrac{1}{2}S &= \tfrac{1}{2} + \left(\tfrac{2}{4} - \tfrac{1}{4}\right) + \left(\tfrac{3}{8} - \tfrac{2}{8}\right) + \ldots + \left(\tfrac{n}{2^n} - \tfrac{n-1}{2^n}\right) + \ldots \\
\tfrac{1}{2}S &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots + \tfrac{1}{2^n} + \ldots
\end{aligned}
$$

Which is just a geometric series, and we get $\frac{1}{2}S = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}}$ which simplifies to $S = 2$ So the sum of the original sequence is just S but we just showed $S = 2$

Problem 2 [3]: Find the number of distinct real (rational and irrational), (not necessarily irreducible,) polynomial factors of $x^9 - x^8 - x + 1$ (including 1).

Answer: $\boxed{48}$ The first step to solving this is to factor the expression:

$$x^9 - x^8 - x + 1$$

$$x^8(x-1) - (x-1)$$

$$(x^8 - 1)(x-1)$$

$$(x^4 + 1)(x^4 - 1)(x-1)$$

$$(x^4 + 2x^2 + 1 - 2x^2)(x^2 + 1)(x^2 - 1)(x-1)$$

$$((x^2 + 1)^2 - 2x^2)(x^2 + 1)(x+1)(x-1)(x-1)$$

$$(x^2 + 1 + \sqrt{2}x)(x^2 + 1 - \sqrt{2}x)(x^2 + 1)(x+1)(x-1)^2$$

These are the irreducible factors of the original expression. To find all the factors we must figure out the number of ways to recombine these factors. There are 4 irreducible factors that can either be combined with the rest or not, and there is 1 irreducible factor that can be combined once, twice or not at all. So, there are $2^4 * 3^1$ different ways to recombine these factors. Therefore there are 48 real factors of the original expression.

Complex Numbers

Problem 1 [2]: Express $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)^{501}$ in the for $a + bi$.

Answer: $\boxed{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i}$ If you are familiar with polar notation for complex numbers then you can write this as $cis^{501}(\frac{\pi}{4}) = cis(501 \cdot \frac{\pi}{4}) = cis(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ If not than this can be solved by repeated squaring:

$$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = i \therefore \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^8 = i^4 = 1$$

But this means that every multiple of 8 that goes into the exponent just becomes a one. And one times any number is just that number so we can

pull out all the multiples of 8 from 501.

$$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{501} = \left(\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{8}\right)^{62} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{4} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{1}$$

Simplifying we get:

$$(1)^{62} (i)^{2} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 1 \cdot (-1) \cdot \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$