Power Round

Power Series and Generating Functions

Carleton Mathematics Competition 2017

1 Rules

- This exam consists of 14 problems, with values from 2 to 15 points, for a total value of 100 points.
- All the necessary work and corresponding explanations or proofs to justify an answer must be provided to receive full credit on a problem.
- Partial credit will be awarded, so even if a solution is incomplete points may still be granted. Accordingly, if you only have a partial solution for a problem it may still be worth submitting.
- The use of calculators will be permitted, but all other outside resources are strictly prohibited.
- Your solution for each problem should be written on a separate sheets of paper, clearly labeled at the top of each sheet with both team name and problem number.
- Your solutions should be submitted at the time of your registration on the morning of April 1.
- If you have any questions about this exam, send them to matsonb@carleton.edu. Besides that, you can only discuss these problems with other people on your team until you have handed in your solutions.

2 Works Referenced

Egge, Eric. *Introductory Topics in Combinatorics.* Northfield, MN: Carleton College, 2014.
Wilf, Herbert. *Generatingfunctionology.* Boston: Academic Press, 1990.

3 Notation

For nonnegative integers $n \geq k$, n **choose** k, written $\binom{n}{k}$, is equal to the expression $\frac{n!}{(n-k)!k!}$ where for any positive integer j, $j! = j(j-1)(j-2)...(2)(1)$ and $0!$ is defined to equal 1.

[2] **Problem 1.** Calculate $\binom{7}{0}$, $\binom{7}{1}$, $\binom{7}{2}$, and $\binom{7}{3}$.

[2] **Problem 2.** Prove $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

A **sequence** of numbers, written $\{a_i\}_{i=n}^{m}$, is a collection of numbers $a_n, a_{n+1}, a_{n+2}, ..., a_{m-1}, a_m$. Much of the time on this test, we will use $n = 0$ and $m = \infty$, giving us $\{a_i\}_{i=0}^{\infty}$, in which case for every positive integer i, there is a number that we will represent as a_i.

For integers $n \geq m$ and some sequence $\{a_i\}_{i=m}^{n}$, we define $\displaystyle\sum_{j=m}^{n} a_j = a_m + a_{m+1} + a_{m+2} + ... + a_{n-1} + a_n$. Note: in a sum such as this, we can (and often will) let $n = \infty$, in which case a_i is added for all $i \geq m$.

4 Power Series

Given a sequence a_n of real numbers and some real number c, a *Power Series* is an infinite series in one variable, here denoted x, of the form $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$.

As that definition would suggest, power series are extremely varied, so general statements about them are difficult to prove, but there are some specific types of power series that are much easier to work with than the general case. One example of those, and one that will be very relevant to generating functions is a *geometric series.*

Given some initial term a and ratio r, the resulting geometric series is a power series of the form $\displaystyle\sum_{n=0}^{\infty} ar^k$.

[5] **Problem 3.** Prove that if $|r| < 1$, $\displaystyle\sum_{n=0}^{\infty} r^k = \frac{1}{1-r}$.

4.1 Induction

A standard proof technique for sequences is *induction*, where some claim is made and then proven by showing it is true for some number of base cases, after which that is assumed to be true for the first $n-1$ terms and using that it can be proven for the n^{th}.

Proposition $\displaystyle\sum_{i=1}^{n} 1 = n$.

Proof Base case ($n = 1$): Here, we have $\displaystyle\sum_{i=1}^{1} 1$, which clearly equals 1. Inductive step: Assuming that $\displaystyle\sum_{i=1}^{k} 1 = k$ for all $1 \leq k < n$, we want to show that this holds for n. Through algebraic manipulation of the sum, we can see: $\displaystyle\sum_{i=1}^{n} 1 = 1 + \sum_{i=1}^{n-1} 1$, and we know $\displaystyle\sum_{i=1}^{n-1} 1 = n-1$, so $\displaystyle\sum_{i=1}^{n} 1 = 1 + n - 1 = n$.

[3] **Problem 4.** Prove that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

[10] **Problem 5.**

part a. Calculate $\displaystyle\sum_{i=1}^{n} 3i^2 - 3i + 1$ for the values of n in the range $1 \leq n \leq 5$.

part b. Conjecture a formula for $\sum_{i=1}^{n} 3i^2 - 3i + 1$.

part c. Prove the formula you conjectured in part b correct.

4.2 Recurrence

Another common way to generate the terms of a power series is through what is called a *recurrence relation*, in which the n^{th} term of a sequence is determined by some formula involving some subset of the first $n-1$ terms and then the first small number of terms in the sequence are defined. An example of this is the Fibonacci numbers, where we have F_0 and F_1 defined both to be zero, and then for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

[3] **Problem 6.** Given the following series defined by a recurrence relation: $a_0 = 0$, $a_1 = 2$, and for $i \geq 2$, $a_i = 2a_{i-1} - 4a_{i-2}$, find and prove the value of a_n for every $n \geq 2$.

[7] **Problem 7.** Come up with a recurrence relation with no constant terms (i.e. all of the terms in the recurrence relation must be a constant times a_{n-k} for some k) for the sequence of square integers ($\{a_i\}_{i=0}^{\infty}$ with $a_i = i^2$).

4.3 Binomial Theorem

Another important result from power series is the *Binomial Theorem*, which states: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, or in its single variable form: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

This can be used not only for evaluating powers of numbers but also for proving identities, such as $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, since the left hand side of that equality can be interpreted as the binomial expansion using $x = y = 1$.

[7] **Problem 8.** Show that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

[15] **Problem 9.** Prove that for all $n \geq 0$ we have: $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{k}$.

One final note for this section: finite sums, which describes many of the ones explored in this section, can be viewed as power series by assuming that $a_n = 0$ for every n beyond the last term of the finite sum.

5 Generating Functions

Given a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers, the *Generating Function* for a_n is the power series $G(x) = \sum_{n=0}^{\infty} a_n x^n$. While the definitions of generating functions and power series may seem extremely similar, the primary difference is that, in general, generating functions are never evaluated for x while one of the main areas of study for power series is their radii of convergence, that being the interval or intervals of values of x for which the series converges.

Due to identities such as $\sum_{n=0}^{\infty} r^k = \frac{1}{1-r}$, generating functions can often be used as extremely concise ways of expressing a sequence of numbers from which retrieving an individual term becomes a relatively simple task.

[6] Problem 10. Given an unlimited number of identical pennies, find a simple expression (that is, not involving a sum) for the generating function of the number of ways to get x cents. Then do the same inserting dimes for pennies. Then, do this a third time for quarters.

[8] Problem 11. Show that if $A(x)$ is the ordinary generating function for the sequence $\{a_n\}_{n=0}^{\infty}$, then $\frac{A(x)}{1-x}$ is the ordinary generating function for the sequence $\{s_n\}_{n=0}^{\infty}$ where $s_n = \sum_{i=0}^{n} a_i$.

Before delving into problems related to generating functions, we first need one theorem that tells us how generating functions interact. Instead of giving a formal statement of the theorem, we will explain it first in an unrigorous but ideally unambiguous terms and then give an example of an application of it. Basically, when there are two independent events, such as two different strings of letters, that are combined in such a way that they are both completely preserved, such as concatenating them, then the generating function for this new event, the number of these (presumably) longer strings, is the product of the generating functions for the events that created it. Think of it this way: in generating functions, what matters is the power on x for a given term to determine which member of the sequence it is, so by multiplying two terms together what is happening is that the exponents, just like the lengths of the strings, are being added.

[9] Problem 12. Given an unlimited number of pennies, dimes, and quarters, find a simple expression for the generating function whose terms are the number of ways you can get x cents (up to reordering the coins, so 1 dime and 1 quarter is the same as 1 quarter and 1 dime).

Another way we can construct generating functions, besides taking combinations of ones that are simple to figure out, is through recurrence relations.

For a simple example of this, take the recurrence relation $a_0 = 1$ and for $n \geq 1$, $a_n = 2a_{n-1}$. Based on the fact that this is a geometric series with starting term 1 and ratio 2, the generating function should be $\sum_{n=0}^{\infty} 2^n$.

We can find this using the recurrence relation too, though. We know that, for $n \geq 1$, $a_n = 2a_{n-1}$, so we can take the sum of that for n from 1 to ∞. That gives us $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 2a_{n-1}x^n$. We define $G(x) = \sum_{n=0}^{\infty} a_n x^n$, so we want to see what needs to be added to both sides in order to get this. The left side is simply $G(x) - 1$, but the right side requires a little more work since it includes the a_{n-1}. To take care of that, we need to re-index the sum so that we have a_n. That will give us $\sum_{n=1}^{\infty} 2a_{n-1}x^n = \sum_{n=0}^{\infty} 2a_n x^{n+1}$, which can easily be verified. Then, $\sum_{n=0}^{\infty} 2a_n x^{n+1} = 2xG(x)$. This gives us $G(x) - 1 = 2xG(x)$, and solving for $G(x)$ gives us $G(x) = \frac{1}{1-2x}$, which can be shown to be the same as $\sum_{n=0}^{\infty} 2^n x^n$ using what we know about geometric series.

[10] Problem 13. Come up with a simple expression for the generating function of the Fibonacci numbers (as they are defined above).

[13] Problem 14. Come up with a simple expression of the generating function for a_n if $a_0 = 1$, $a_1 = 3$, and for all $n \geq 2$ $a_n = 7a_{n-1} - 10a_{n-2}$. Then use that to come up with a formula for a_n.

Carleton College Mathematics Competition

Welcome Coaches and Students! We are excited that you will be participating in Carleton College's Mathematics Competition Saturday April 1st. Attached you will find a map of Carleton's Campus.

Check-In

Most likely your bus will drop you off in front of Sayles or Willis Hall. Please make your way to Olin Hall for check-in, located in the direction of the lakes. Check-in will be between 0800-0845. The first test will begin promptly at 0915.
IMPORTANT: Each team must submit the power round at check-in.

Testing Locations

Testing will take place the the following buildings: Leighton Hall, Center for Mathematics and Computing (CMC), and Olin Hall of Science. The room is dependent on the individual topic exam chosen by the student. Teams are assigned a room for the team round. This information will be given at check in.

Itinerary

Below is an outline of the schedule you can expect for Saturday. More information will be provided at check-in

Start Time	End Time	Event Description	Location
0800	0845	Check-in (Teams submit Power Round)	Olin Lobby
0845	0900	Introduction / Announcements	Olin 149
0915	1000	Individual Topic Test 1	Testing Room
1015	1100	Individual Topic Test 2	Testing Room
1100	1200	Lunch	Olin 149
1215	1315	Team Round	Testing Room
1320	1420	Carleton Special	Testing Room
1430	1530	Professor Lecture	Olin 149
1545	1630	Awards Ceremony	Olin 149

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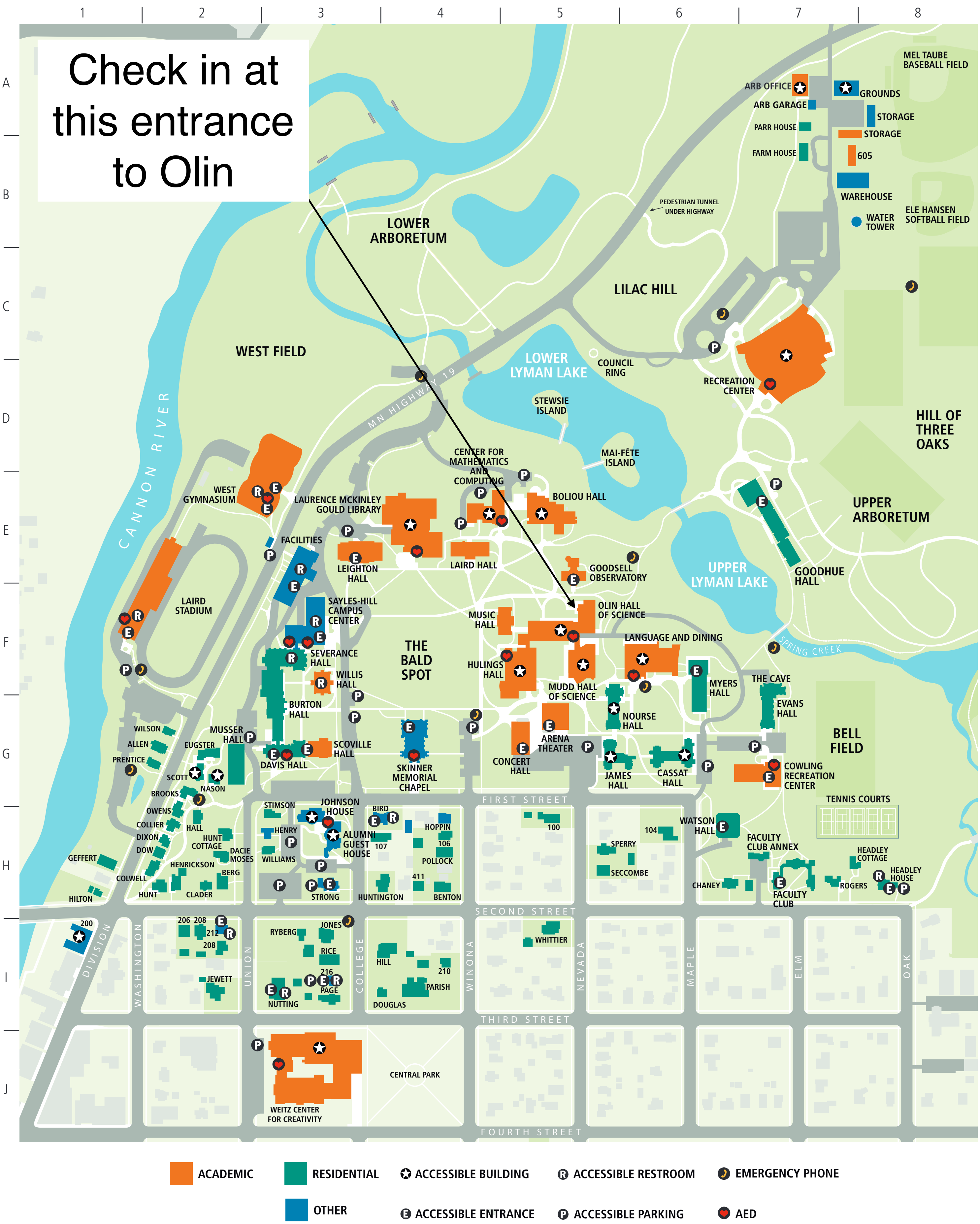

Check in at this entrance to Olin
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PARR HOUSE
STORAGE
FARM HOUSE
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WAREHOUSE
PEDESTRIAN TUNNEL UNDER HIGHWAY
WATER TOWER
ELE HANSEN SOFTBALL FIELD
LOWER ARBORETUM
LILAC HILL
WEST FIELD
LOWER LYMAN LAKE
COUNCIL RING
RECREATION CENTER
STEWSIE ISLAND
MN HIGHWAY 19
CANNON RIVER
HILL OF THREE OAKS
CENTER FOR MATHEMATICS AND COMPUTING
MAI-FÊTE ISLAND
WEST GYMNASIUM
LAURENCE MCKINLEY GOULD LIBRARY
BOLIOU HALL
UPPER ARBORETUM
FACILITIES
LEIGHTON HALL
LAIRD HALL
GOODSELL OBSERVATORY
UPPER LYMAN LAKE
GOODHUE HALL
LAIRD STADIUM
SAYLES-HILL CAMPUS CENTER
MUSIC HALL
OLIN HALL OF SCIENCE
THE BALD SPOT
LANGUAGE AND DINING
SPRING CREEK
SEVERANCE HALL
HULINGS HALL
WILLIS HALL
MUDD HALL OF SCIENCE
MYERS HALL
THE CAVE
BURTON HALL
NOURSE HALL
EVANS HALL
WILSON
MUSSER HALL
ALLEN
EUGSTER
SCOVILLE HALL
ARENA THEATER
BELL FIELD
PRENTICE
DAVIS HALL
SKINNER MEMORIAL CHAPEL
CONCERT HALL
JAMES HALL
CASSAT HALL
COWLING RECREATION CENTER
SCOTT
NASON
BROOKS
TENNIS COURTS
OWENS
STIMSON
JOHNSON HOUSE
BIRD
FIRST STREET
COLLIER
HALL
HENRY
HOPPIN
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WATSON HALL
DIXON
HUNT COTTAGE
ALUMNI GUEST HOUSE
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SPERRY
FACULTY CLUB ANNEX
DOW
DACIE MOSES
WILLIAMS
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POLLOCK
HEADLEY COTTAGE
GEFFERT
HENRICKSON
BERG
SECCOMBE
HEADLEY HOUSE
COLWELL
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MAPLE
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PARISH
NUTTING
DOUGLAS
THIRD STREET
CENTRAL PARK
WEITZ CENTER FOR CREATIVITY
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OTHER
ACCESSIBLE BUILDING
ACCESSIBLE ENTRANCE
ACCESSIBLE RESTROOM
ACCESSIBLE PARKING
EMERGENCY PHONE
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