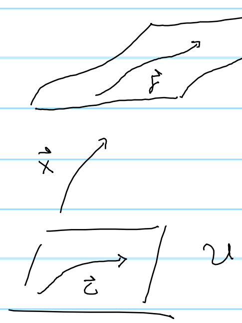
10/12/12

Test Notes

#3

$T_p(M)$ is the set of all tangent vectors (not just unit tangents) to M at point p.

$\vec{c}(t)$ is a curve down on $\mathcal{U}$.



$\vec{\gamma} = \vec{x} \circ \vec{c}$

Assume that the directional derivatives are well-defined.

#1.h

We need to express $\vec{x}_{11}$ in terms of $\vec{x}_1$, $\vec{x}_2$, and $\vec{n}$.

Write

$\vec{x}_{11} = \_\_\,\vec{x}_1 + \_\_\,\vec{x}_2 + \_\_\,\vec{n}$

and find the pieces!

#3 again

$\vec{c} : \mathbb{R} \to \mathbb{R}^2$

$\vec{x} : \mathbb{R}^2 \to \mathbb{R}^3$

$\vec{\gamma} = \vec{x} \circ \vec{c} : \mathbb{R} \to \mathbb{R}^3$

Remember that $\vec{\gamma}\,' = D\vec{x}\,\dot{\vec{c}}$

Also remember that $\vec{x}_i$, $\vec{x}_{ij}$, & $\vec{n}$ are all also functions from $\mathcal{U}$ to $\mathbb{R}^3$. We can therefore write something like $D\vec{n}$.

Let $\vec{\gamma}(0) = p$ and $\vec{\gamma} = \vec{x} \circ \vec{c}$. There is some corresponding $q = \vec{c}(0) \in \mathcal{U}$, such that $\vec{x}(q) = p$.

$$\vec{x}_1(q) = \vec{x}_1\left(c^1(0), c^2(0)\right)$$

gives one particular tangent vector at p. $\vec{x}_1$ is also a tangent vector for the u^1 curves.

#2 We are allowed to use k_n and k_g as we know them by definition.

Parallelism

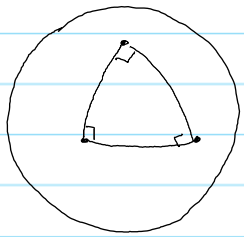
"Intrinsic ~~bey~~ beings" can measure the metric tensor. With (g_{ij}) we can measure length and angle.

Take the surface

$z = x^2 + y^2$

$2\pi R >$ Circumference

$2\pi R <$ Circ.



This triangle has $270°$!

We could also look at tangent vectors.

Let $V \subset M$. A <u>vector field</u> on V is a function $\mathbb{X} : V \to \mathbb{R}^3$

<u>tangent vector field</u> : $\mathbb{X} : V \to T(M)$. So for each p on V, $\mathbb{X}(p) \to T_p(M)$.

<u>curve vector field</u> : A tangent vector field along a curve $\gamma \subset M$ is just a tangent vector field on $\gamma[a,b]$.

<u>Theorem</u> :

A tangent vector field $\mathbb{X}$ is parallel (to itself) along γ iff $d\vec{\mathbb{X}}/dt$ is parallel to $\vec{n}$.

This means that intrinsic beings couldn't measure the ~~st~~ change.

Equivalently, $\vec{\mathbb{X}}'$ always normal to the surface, or $\mathbb{X}' \perp \vec{x}_i$.

Proposition :

If $\vec{\gamma} = \vec{x} \circ \vec{c}$ is a regular curve and $\vec{X}(t)$ is a v.f. on $\vec{\gamma}$ with $\vec{X}(t) = \sum_i X^i(t)\, \vec{x}_i(c(t))$. We are simply expressing $\vec{X}$ in terms of $\vec{x}_1$ and $\vec{x}_2$, the basis vectors for the tangent plane. Then $\vec{X}$ is parallel iff

$$0 = \dot{X}^k + \sum_{i,j} \Gamma_{ij}^k X^i \dot{c}^j$$

for $k = 1, 2$.

Proof

$$\vec{X} = \sum X^i \vec{x}_i$$

$$\dot{\vec{X}} = \sum \left(\dot{X}^i \vec{x}_i + X^i \frac{d}{dt} \vec{x}_i \right)$$

Note that

$$\frac{d}{dt} \vec{x}_i = \frac{d}{dt} \vec{x}_i \left(u^1(t), u^2(t) \right)$$

$$= \frac{\partial \vec{x}_i}{\partial u^1} \frac{du^1}{dt} + \frac{\partial \vec{x}_i}{\partial u^2} \frac{du^2}{dt}$$

$$= \cancel{\frac{\partial \vec{x}_i}{\partial u^1} \vec{c}}$$

$$= \frac{\partial \vec{x}_i}{\partial u^1} \dot{c}^1 + \frac{\partial \vec{x}_i}{\partial u^2} \dot{c}^2$$

So

$$\dot{\vec{X}} = \sum_i \left(\dot{X}^i \vec{x}_i + X^i \sum_j \overset{\vec{x}_{ij}}{\cancel{\vec{x}_{ij}}} \dot{c}^j \right)$$

$\vec{X}$ is parallel iff $\langle \dot{\vec{X}} | \vec{x}_j \rangle = 0$, for $j = 1, 2$.

But

$$\langle \dot{\vec{X}} | \vec{x}_\ell \rangle = \sum_i \left(\dot{X}^i g_{i\ell} + X^i \sum_j \dot{c}^j \Gamma_{ij\ell} \right) = 0$$

Sum against $g^{\ell k}$.

$$\sum_\ell \left(\sum_i \left(\dot{X}^i g_{i\ell} g^{\ell k} + X^i \sum_j \dot{c}^j \Gamma_{ij\ell} g^{\ell k} \right) \right)$$

$$= \sum_i \left(\dot{X}^i \delta_i^{\,k} + X^i \sum_j \dot{c}^j \Gamma_{ij}^{\,k} \right)$$

$$= \boxed{\sum_i \left(\dot{X}^k + \sum_j \Gamma_{ij}^{\,k} \dot{c}^j X^i \right)}$$

Theorem 6.7:

Let $\vec{\gamma}$ be a curve (regular) on M and let $\tilde{X} \in T_{\vec{\gamma}(t_0)} M$.
Then there exists only one vector field $\vec{X}$ such that $\vec{X}$ is parallel along $\vec{\gamma}$ and $\vec{X}(t_0) = \tilde{X}$.